ON THE CHARACTERISTIC FUNCTIONS OF EXTREMA OF CERTAIN FAMILIES OF DISTRIBUTIONS

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by

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under the guidance of

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I hereby certify that, this thesis entitled **"On the Characteristic Functions of Extrema of Certain Families of Distributions"** is a bonafide record of research work carried out by **Ms. Aparna Aravindakshan M.**, Research Scholar, Department of Statistics, University of Calicut, under my supervision and guidance for the award of the Degree of Doctor of Philosophy in Statistics, of the University of Calicut. The work reported herein does not form part of any other thesis or dissertation submitted previously for the award of any degree or diploma of any other university or institution. Also certified that the contents of the thesis have been checked using anti-plagiarism data base and no unacceptable similarity was found through the software check.

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DECLARATION

I, Aparna Aravindakshan M., hereby declare that this thesis entitled **"On the Characteristic Functions of Extrema of Certain Families of Distributions"** submitted to the University of Calicut for the award of the degree of Doctor of Philosophy in Statistics is a bonafide record of the work done by me under the guidance and supervision of Dr. C. Chandran, Professor and Head, Department of Statistics, University of Calicut. This thesis contains no material which has been accepted for the award of any degree or diploma of any university or institution and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference was made.

10^{th} August 2018 **Aparna Aravindkshan M.**

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10^{th} *August 2018* **Aparna Aravindkshan M.**

CONTENTS

ABBREVIATIONS

$B(n, p)$	Binomial distribution with parameters n and p
c.f.	characteristic function
$\chi^2(1)$	Chi-square distribution with 1 degree of freedom
d.f.	distribution function
$\exp(\theta)$	exponential distribution with parameter θ
i.i.d.	independent and identically distributed
p.d.f.	probability density function
p.m.f.	probability mass function
r.v.	random variable
R.V.	random vector
s.f.	survival function

CHAPTER

ONE

Introduction

The crux of the problems of probability theory is random variables (r.v.s) and their distribution functions (d.f.s) or their asymptotic distributions. Since every Borel measurable function of a r.v. is again a r.v. (Laha and Rohatgi, 1979, Page 5, Remark 1.1.1.), there are several relationships among r.v.s. A r.v. may be expressed as a function of one or more r.v.s. Some of the examples are: if $X \sim N(0,1)$, then $Y = X^2 \sim \chi^2(1)$. If $X_1 \sim$ Poisson (λ_1), $X_2 \sim$ Poisson (λ_2) and if X_1 and X_2 are independent, then their sum $X_1 + X_2 \sim$ Poisson $(\lambda_1 + \lambda_2)$. A negative binomial (n, p) r.v. with $n = 1$ is the geometric (p) r.v. If $X \sim B(n, p)$, and if n is large and np approaches to $\lambda > 0$, then the d.f. of X can be approximated by Poisson distribution. If $\{X_n\}$, is a sequence of i.i.d. binomial r.v.s, then for $S_n = \sum_{j=1}^{n} X_j$, the distribution of normalized sequence of partial sums, approaches to Normal distribution. Some of the relationships among the r.v.s which we discuss in this thesis are the partial maxima and partial minima of sequence of independent r.v.s, $\{X_n\}$, defined by $M_n = \max(X_1, X_2, \ldots, X_n)$ and $m_n = \min(X_1, X_2, \ldots, X_n)$ respectively.

Another measurable function of r.v.s $X_1, X_2, \ldots, X_n$, we discuss in this thesis are their corresponding ordered r.v.s, $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$, known as order statistics. Note that $X_{1:n} = m_n$ and $X_{n:n} = M_n$. We also consider the componentwise maxima and componentwise minima of bivariate sequence of independent r.v.s $\{(X_n, Y_n)\}$.

1.1 Objective and Summary of the Thesis

Tools for studying r.v.s are selected based on their compactness and ease of use. The d.f., probability mass function (p.m.f.), probability density function (p.d.f.), integral transforms like characteristic function (c.f.), moment generating function and probability generating function are some of the most commonly used tools in the study of r.v.s. When a sequence of r.v.s is independently distributed, the d.f.s of partial maxima, M_n and partial minima, m_n are in a compact and explicit form, see for example David (1970). However, the c.f.s of partial maxima and partial minima of a sequence of independent r.v.s, $\{X_n\}$, do not have a compact and explicit form in the literature. The main objective of the thesis is to investigate the conditions under which the c.f.s of partial maxima and partial minima of a sequence of independent r.v.s to be written in a compact form. The restricted families of distributions for which the c.f.s can be derived in compact form are discussed giving enough illustrative examples. Sufficient conditions for d.f.s of r.v.s to belong to such classes of distributions are obtained. Some important properties of these classes of distributions are also discussed. The compact forms of the c.f.s of partial maxima and partial minima of a sequence of independent r.v.s are also derived under the conditions identified and the results corresponding to independent and identically distributed (i.i.d.) sequence of r.v.s are derived as special cases. The results discussed for partial maxima and partial minima are then extended to r^{th} order statistic of a sequence of independent r.v.s for every fixed n and the results corresponding to i.i.d. sequence is deduced as a particular case. An attempt is also made to extend these results to the bivariate set up. Necessary and sufficient conditions for a family of bivariate distributions to have closure property under the minima or maxima are also obtained.

1.2 Organization of the Thesis

The thesis is organized into seven chapters. As discussed above, Chapter 1 of the thesis presents the problem addressed in the work in a gentle manner. Section 1.2 describes the organization of other chapters of the thesis. The rest of the thesis is organized as follows.

Chapter 2 introduces the basic concepts in probability theory which are required to understand the later developments of the thesis. The measurable functions of a sequence of r.v.s on a probability space, like partial sums, partial maxima and partial minima are discussed in this chapter. We verify that, in general the c.f.s of partial minima and partial maxima can not be written in a compact form and identify some special families of distributions for which one can write the c.f.s of partial maxima or partial minima in a compact form. Section 2.1 reviews the basic concepts of probability theory required. This includes the probability space induced by a r.v., the d.f. of a r.v. and the c.f. of a r.v., and the existence and interrelationship between d.f. of a r.v. and the corresponding c.f. Section 2.2 is on the partial sums of independent r.v.s which describes the representation of the exact distribution of partial sums as convolutions and the c.f.s of partial sums as the product of the c.f.s of the underlying distributions. This section also discusses the stability properties of a sequence of partial sums of independent r.v.s like the weak and strong laws of large numbers and the central limit theorem. These discussions are based on Billingsley (1995) and Laha and Rohatgi (1979). Section 2.3 gives a review on the partial minima and partial maxima. The representation of the d.f.s of partial maxima and partial minima in terms of the d.f. of the underlying distribution for both i.i.d. and independent non-identical r.v.s are discussed. Then the discussion move on to the limiting distribution of a sequence of linearly normalized maxima (Embrechts et. al., 1997, Leadbetter et. al., 1983 and Resnick, 1987) and non-linearly normalized maxima (Pancheva, 2010). The corresponding results for the minima follow from the simple relationship between the maxima and minima. In Section 2.3, we also observe that, in general, the c.f.s of minima and maxima can not be expressed in terms of the c.f.s of the underlying distribution in a simple form.

Chapter 3 of the thesis is on families of distributions closed under the minima or maxima. These concepts are defined and illustrated with suitable examples. Some important properties of these families are also obtained in this chapter. The results corresponding to sequences of i.i.d. r.v.s is based on Aparna and Chandran (2017) and those corresponding to independent non-identically distributed r.v.s is based on Aparna and Chandran (2018a). Section 3.1 discusses various types of closure properties of sets. Section 3.2 provides the definitions of the concepts of closure under the minima and maxima of r.v.s or corresponding families of distributions. This section gives suitable examples to illustrate these concepts. Sufficient conditions for families of distributions to have closure property under the minima or maxima are also derived in this section. Section 3.3 is on how the closure property under the minima or maxima changes under monotone transformations of the r.v.s closed under the minima or maxima. Section 3.4 discusses how the closure property under the minima and maxima holds for truncated r.v.s.

Chapter 4 obtains the representations for the c.f.s of partial minima and partial maxima of a sequence of independent r.v.s in terms of the c.f. of the underlying distribution. Section 4.1 derives the representation for the d.f.s of partial maxima in terms of the d.f.s of partial minima and vice-versa for a sequence of independent r.v.s. From these representations, when the underlying family of distributions is closed under the minima or maxima, we deduce the representations for the d.f.s of partial minima and partial maxima in terms of the d.f. of the underlying distribution. By the one to one correspondence between the d.f.s and their integral transforms, we have similar representations for the c.f.s and are discussed in Section 4.2. All other integral transforms like moments, probability generating function etc. have similar representations, whenever they exist. Representations for the moments are given in Section 4.3 and an application of this result is also discussed. The results corresponding to a sequence of independent non-identical r.v.s is based on the discussions in Aparna and Chandran (2018a) and the results deduced as a particular case, corresponding to i.i.d. sequence, is discussed based on Aparna and Chandran (2017).

Another measurable function of $\{X_n\}$, interested in mathematical statistics, is the r^{th} order statistic for a fixed integer n and any integer r between 1 and n. The minima and maxima are special cases of this r^{th} order statistic when r takes the values 1 and n respectively. Chapter 5 extends the results derived in Chapter 4 to other order statistics and is based on Aparna and Chandran (2018b). Section 5.1 gives a brief introduction on order statistics. In Section 5.2, we see that, for every fixed n the d.f.s of order statistics of a sequence of independent r.v.s also have representation in terms of the d.f.s of partial maxima and partial minima. Furthermore, when the r.v.s are closed under the minima or maxima, the d.f.s of r^{th} order statistic can be expressed in terms of the d.f. of the underlying distribution and are discussed in this section. Section 5.3 introduces similar representations for the c.f.s of order statistics. Section 5.4 provides similar representations for the moments, whenever they exist, and an application of this result is also discussed.

Chapter 6 extends the closure property under the minima and maxima discussed in Chapter 3 to bivariate case. Even though, our discussions are on the two-dimensional random vectors (R.V.s), the results can be extended to higher dimensional R.V.s. The chapter is based on Aparna and Chandran (2018c). Section 6.1 reviews the basic probability theory of R.V.s. Section 6.2 introduces the notions of the componentwise minima and componentwise maxima. The function which uniquely identifies the joint d.f. with the marginal d.f.s are called copulas, and is discussed in Section 6.3 with the help of Nelson (1999). Section 6.4 defines families of bivariate distributions closed under the minima and maxima. A necessary and sufficient condition for a family of bivariate distributions to have closure property under the minima or maxima is obtained. In this section, copulas closed under extrema are defined and some examples are given. Sections 6.5 deals with the changes in bivariate closure property under the minima and maxima on the monotone transformations of the marginal r.v.s. Section 6.6 describes how the bivariate closure property under the minima and maxima changes, on the truncations of the marginal r.v.s. In Section 6.7 we see that the representations for the d.f.s of partial minima in terms of the d.f.s of partial maxima and vice-versa can not be extended to the two dimensional case.

Chapter 7 summarises the overall content of this thesis work. Cited references of the thesis are given at the end of the thesis.

CHAPTER

TWO

Preliminaries

In this chapter, we review some basic concepts in probability theory required to understand this thesis. Measurable functions of a sequence of r.v.s like the partial sums, partial minima and partial maxima are described. The exact and limiting distributions of these measurable functions are also discussed. The chapter also introduces the main problem addressed in this thesis.

The chapter is organized as follows: Section 2.1 introduces basic concepts in probability theory. Section 2.2 is on partial sums. The exact distributions of partial sums and their stability properties are discussed. Section 2.3 reviews partial minima and partial maxima. Their exact distributions and the stability property under linear and non-linear normalizations are discussed. In this section, it is observed that the c.f.s of partial minima and partial maxima can not be expressed in terms of the c.f. of the underlying distribution unless some restrictions are imposed.

2.1 Introduction

Probability theory, which deals with any bounded and normed measure is mainly devoted to the study of random variables (r.v.s). To each outcome of an experiment if we assign a number, then we define a measurable function on the sample space called the r.v. Mathematically, if $(\Omega, \mathscr{A}, P)$ is a probability space, a real valued $\mathscr{A}$-measurable function on Ω is called a r.v. on $(\Omega, \mathscr{A}, P)$. That is, if for every $B \in \mathscr{B}$, the Borel σ-field in $\mathbb{R}$,

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{A}, \tag{2.1.1}$$

then X defines a r.v. on $(\Omega, \mathscr{A}, P)$. The r.v. X induces a probability measure P_X on the measurable space $(\mathbb{R}, \mathscr{B})$ given by

$$P_X(B) = P(X^{-1}(B)), \quad \forall \quad B \in \mathscr{B}, \tag{2.1.2}$$

called the probability distribution of X. Hence, $(\mathbb{R}, \mathscr{B}, P_X)$ is a new probability space called the probability space induced by the r.v. X on its range space. The distribution function (d.f.) of a r.v. X is a mapping $F_X : \mathbb{R} \to \mathbb{R}$, which can be derived from (2.1.2) as follows

$$\begin{aligned} F_X(x) &= P_X((-\infty, x]), \quad \forall \quad x \in \mathbb{R} \\ &= P(X^{-1}(-\infty, x]), \quad \forall \quad x \in \mathbb{R} \\ &= P(\omega \in \Omega : X(\omega) \leq x), \quad \forall \quad x \in \mathbb{R} \\ &= P(X \leq x), \quad \forall \quad x \in \mathbb{R}. \end{aligned} \tag{2.1.3}$$

That is, corresponding to every r.v. X there is a d.f. F_X. The d.f. is nondecreasing, right-continuous and attains the value zero as $x \to -\infty$ and one as $x \to +\infty$. Hence, more precisely we can say that, the d.f. of a r.v. is a mapping from $\mathbb{R}$ to $[0,1]$ with these properties. Conversely, if F is a nondecreasing, right-continuous function on $\mathbb{R}$ satisfying $\lim_{x\to-\infty} F(x) = F(-\infty) = 0$ and $\lim_{x\to+\infty} F(x) = F(+\infty) = 1$, then by Theorem (1.1.2) of Laha and Rohatgi (1979), there exists a probability measure P_F on $(\mathbb{R}, \mathscr{B})$ determined uniquely by the relation $P_F((-\infty, x]) = F(x)$, for all $x \in \mathbb{R}$. Hence, by Remark (1.1.3) of Laha and Rohatgi (1979), there exist a r.v. X on some probability space such that F is the distribution function of X. That is, consider the probability space $(\mathbb{R}, \mathscr{B}, P_F)$ and let $X(\omega) = \omega$ for all $\omega \in \mathbb{R}$. Now, this X induces a probability measure P_X on $(\mathbb{R}, \mathscr{B})$ given by $P_X(B) = P_F(X^{-1}(B)) = P_F(B)$. In particular, if $B = (-\infty, x]$, then $F_X(x) = F(x)$ for all $x \in \mathbb{R}$. That is, F is the d.f. of the r.v. X. Hence, corresponding to any d.f. F, there exists a r.v. on some probability space with F as its d.f.

If X is a r.v. on some probability space, then the transformation given by

$$\phi_X(t) = E(e^{itX}), \quad t \in \mathbb{R} \tag{2.1.4}$$

$$= \int_{\mathbb{R}} e^{itx} dF_X(x), \tag{2.1.5}$$

is known as the characteristic function (c.f.) of the d.f. F_X or the r.v. X. If $\phi(t)$ is a c.f., it is a complex valued function and it satisfies the following properties:

i) $\phi(0) = 1$.

ii) $\phi(-t) = \bar{\phi}(t)$, the complex conjugate of $\phi(t)$.

iii) $|\phi(t)| \leq 1$.

iv) $\phi(t)$ is uniformly continuous on $\mathbb{R}$.

By Cramér (1946), (page 84, Section 7.4), any bounded and measurable function is integrable with respect to any d.f. over $\mathbb{R}$, which assures the existence of the c.f. for every d.f. There is a one to one correspondence between the d.f. of a r.v. and its c.f. That is, two d.f.s F_1 and F_2 are identical if, and only if, their c.f.s ϕ_1 and ϕ_2 are identical. This is the content of the uniqueness theorem of the c.f. For an absolutely continuous r.v. X with p.d.f. f_X, the c.f. is given by

$$\phi_X(t) = \int_x e^{itx} f_X(x)dx \tag{2.1.6}$$

and for a discrete r.v. X with p.m.f. p_X it can be expressed as

$$\phi_X(t) = \sum_x e^{itx} p_X(x), \tag{2.1.7}$$

where $p_X(x) = P(X = x)$. For more details of the above discussions see Laha and Rohatgi (1979) and Lukacs (1960).

Suppose there are n r.v.s say $X_1, X_2, \ldots, X_n$ on $(\Omega, \mathscr{A}, P)$. Then the Borel measurable functions of these r.v.s, like their sum, maximum and minimum are defined by

$$S_n = \sum_{j=1}^{n} X_j \tag{2.1.8}$$

$$M_n = \max(X_1, X_2, \ldots, X_n) \tag{2.1.9}$$

$$m_n = \min(X_1, X_2, \ldots, X_n) \tag{2.1.10}$$

and are also r.v.s on $(\Omega, \mathscr{A}, P)$ whose d.f.s are determined by the d.f. of

$X_1, X_2, \ldots, X_n$. Hence, the exact distributions of these functions are dealt in probability theory. In many situations, the exact distributions of these statistics may not be easy to handle or useful for further statistical treatments. In such situations, generally the asymptotic distribution of the above mentioned statistics are derived and applied in the statistics literature. The following section provides a description of the d.f.s and c.f.s of partial sums of a sequence of independent r.v.s.

2.2 Partial Sums

Let $\{X_n, n \geq 1\}$ be a sequence of independent r.v.s with X_j having d.f. F_{X_j} and c.f. ϕ_{X_j}. Then $\{S_n\}$ is the sequence of partial sums of $\{X_n\}$, where S_n is as defined in (2.1.8) for every $n \geq 1$. The d.f.s of S_n of independent r.v.s are usually expressed as convolution of individual d.f.s. The convolution of two d.f.s, say F_1 and F_2 is given by

$$\begin{aligned} G(z) &= \int_{-\infty}^{\infty} F_1(z-x)dF_2(x) \\ &= \int_{-\infty}^{\infty} F_2(z-x)dF_1(x) \\ &= F_1 * F_2. \end{aligned} \tag{2.2.1}$$

If the d.f. G is the convolution of 3 d.f.s F_1, F_2, and F_3, then

$$\begin{aligned} G &= (F_1 * F_2) * F_3 = F_1 * (F_2 * F_3) \\ &= F_1 * F_2 * F_3. \end{aligned} \tag{2.2.2}$$

Similarly, if the d.f. G is the convolution of n d.f.s $F_1, F_2, \dots, F_n$, then

$$G = F_1 * F_2 * \dots * F_n. \tag{2.2.3}$$

When these independent r.v.s are identically distributed as F, the representation in (2.2.3) reduces to $G = F^{*n}$, the n-fold convolution of F. For details see Feller (1966) and Lukacs (1960). Hence,

$$F_{S_n} = F_{X_1} * F_{X_2} * \dots * F_{X_n} \tag{2.2.4}$$

and for the i.i.d. sequence $\{X_n\}$

$$F_{S_n} = F_{X_1}^{*n}. \tag{2.2.5}$$

From equation (2.1.4), we have for $t \in \mathbb{R}$

$$\phi_{S_n}(t) = \prod_{j=1}^{n} \phi_{X_j}(t), \quad n \geq 1, \tag{2.2.6}$$

and when the r.v.s are i.i.d. as X with c.f. ϕ_X, (2.2.6) reduces to

$$\phi_{S_n}(t) = (\phi_X(t))^n, \quad n \geq 1. \tag{2.2.7}$$

We can see that the c.f. of S_n has more compact and useful representation compared to that of its d.f. It is these explicit forms of $\phi_{S_n}(t)$ that helps to derive the stability properties of the sequence $\{X_n\}$ of i.i.d. or independent r.v.s in terms of the partial sum sequence $\{S_n\}$. For example, the central limit theorem, the laws of large numbers and the laws of iterated logarithm of the sequence $\{X_n\}$ of r.v.s are derived in the probability literature with the help of (2.2.6) and (2.2.7). Even though, the expressions (2.2.4) and (2.2.5) appear simple, it is not so. In general, the d.f. of the partial sum, S_n, does not have an explicit form for either the i.i.d. or independent non-identical case. In such situations, generally the asymptotic distribution of S_n is used in the literature. If for a sequence of independent r.v.s, $\{X_n\}$, $E(X_n^2) < \infty$, then

$$\frac{S_n - E(S_n)}{n} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty$$

and if they have common mean μ, then

$$\frac{S_n}{n} \xrightarrow{P} \mu \quad \text{as} \quad n \to \infty.$$

The only condition for this convergence is that the variance, $V\left(\frac{S_n}{n}\right) \to 0$ as $n \to \infty$ and the condition of independence is not necessary and this is the content of Chebychev's weak law of large numbers. Khintchine's weak law of large numbers is a stronger result which says, if $\{X_n\}$ is a sequence of i.i.d. r.v.s with common mean μ,

$$\frac{S_n}{n} \xrightarrow{P} \mu \quad \text{as} \quad n \to \infty.$$

We can see that, the weak laws of large numbers discuss the convergence in probability of the sequence of partial sums. When the mode of convergence changes to almost sure convergence, we have the strong laws of large numbers. The Kolmogorov's strong law of large numbers says,

$$\frac{S_n}{n} \xrightarrow{a.s.} \mu \quad \text{as} \quad n \to \infty$$

if, and only if, $E|X_n| < \infty$. For details, see Billingsley (1995) and Laha and Rohatgi (1979).

Since convergence in probability and almost sure convergence imply convergence in distribution, when normalized with n, the sequence of partial sums converges to the mean in distribution. That is, $\frac{S_n}{n}$ converges to a d.f. degenerated at the mean, which does not provide much information. But, if we normalize S_n with $\sqrt{n}$ and change to convergence in distribution, we get more information about the stability of $\frac{S_n}{\sqrt{n}}$. This is the main idea used in the central limit theorems. The first attempt in this direction is on the convergence in distribution of a sequence of i.i.d. Bernoulli r.v.s normalized with $\sqrt{n}$. If $\{X_n\}$ is a sequence of i.i.d. Bernoulli r.v.s with $P(X_j = 1) = p$, $0 < p < 1$, and $P(X_j = 0) = 1 - p = q$, then for every $x \in \mathbb{R}$,

$$P\left(\frac{S_n - np}{\sqrt{npq}} \le x\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \quad \text{as} \quad n \to \infty.$$

This is the content of the Bernoulli's central limit theorem, by DeMoivre and Laplace. Later Lévy identified that this result holds not only for i.i.d. Bernoulli r.v.s but also for any sequence of i.i.d. r.v.s. The Lévy central limit theorem says that for a sequence $\{X_n\}$ of i.i.d. r.v.s, if $0 < V(X_n) = \sigma^2 < \infty$, for every $x \in \mathbb{R}$,

$$P\left(\frac{S_n - E(S_n)}{\sigma\sqrt{n}} \le x\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \quad \text{as} \quad n \to \infty.$$

This is the most useful version of the celebrated central limit theorem in the i.i.d. case. By relaxing the identically distributed condition, Lindeberg obtained a set of sufficient conditions for the convergence of suitably centered and normalized S_n to the Normal r.v. These conditions were later proved to be necessary by Feller.

Theorem 2.2.1 (Lindeberg-Feller Central Limit Theorem). *Let* $\{X_n\}$ *be a sequence of independent but not necessarily identically distributed r.v.s with* $V(X_n) = \sigma_n^2 < \infty$, $n = 1, 2, \ldots$. *Let* $E(X_n) = \alpha_n$ *and* $S_n = \sum_{j=1}^{n} X_j$. *Set* $V(S_n) = B_n^2$. *Let* F_n *be the d.f. of* X_n. *Then the following two conditions:*

i) $\lim_{n\to\infty} \max_{1\le k\le n} (\sigma_k^2/B_n^2) = 0$,

ii) $\lim_{n\to\infty} P\{[S_n - E(S_n)]/B_n \le x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$,

for every $x \in \mathbb{R}$, *hold if, and only if, for every* $\epsilon > 0$ *the condition*

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{k=1}^{n} \int_{|x-\alpha_k|\ge\epsilon B_n} (x - \alpha_k)^2 dF_k(x) = 0 \qquad (2.2.8)$$

is satisfied.

Lindeberg showed that (2.2.8) implies (ii) and Feller showed that (i) and (ii) imply (2.2.8). For proof see Laha and Rohatgi (1979). In general, the more precise content of the central limit theorem is, if there exists sequence of constants $\{a_n\}$ and $\{b_n\}$, with $b_n > 0$, such that the normalized sequence of partial sums $\{\frac{S_n - a_n}{b_n}\}$ converges in law to some non-degenerate r.v. Z whose d.f. is G, then G is an α-stable distribution for some $\alpha > 0$. A non-degenerate r.v. X is stable if, and only if, for all $n > 1$, there exist constants $a_n \in \mathbb{R}$ and $b_n > 0$ such that

$$S_n \stackrel{d}{=} b_n X + a_n \qquad (2.2.9)$$

where $X_1, X_2, \ldots, X_n$ are independent, identical copies of X. A r.v. X is strictly stable if, and only if, $a_n = 0$ for all n. Stable laws are also known as α- stable or Lévy stable. The α- stable distributions require four parameters for complete description: an index of stability also known as the tail index, tail exponent or characteristic exponent $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\sigma > 0$ and a location parameter $\mu \in \mathbb{R}$. The normal distribution arises as a special case of this when $\alpha = 2$. This approximation is extensively used in the statistics literature when the exact distribution of S_n cannot be computed. For example, if the second moment of X_n exists for a sequence of i.i.d. r.v.s, $\{X_n\}$, S_n has an asymptotic normal distribution. For details, see Billingsley (1995), Feller (1966) and Laha and Rohatgi (1979).

2.3 Partial Minima and Partial Maxima

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. r.v.s. Then $\{M_n\}$ and $\{m_n\}$ are the corresponding sequences of partial maxima and partial minima, where M_n and m_n are as defined in (2.1.9) and (2.1.10) respectively. Unlike the d.f. of S_n, the d.f. of M_n and m_n of an i.i.d sequence $\{X_n\}$ are expressible in explicit form given by

$$F_{M_n}(x) = (F_X(x))^n \tag{2.3.1}$$

$$F_{m_n}(x) = 1 - (1 - F_X(x))^n \tag{2.3.2}$$

where $\bar{F}_X(x)$ is the survival function of $F_X(x)$. When the sequence of r.v.s are independent but not identically distributed, the d.f. of M_n and m_n are given by

$$F_{M_n}(x) = \prod_{j=1}^{n} F_{X_j}(x) \tag{2.3.3}$$

$$F_{m_n}(x) = 1 - \prod_{j=1}^{n} (1 - F_{X_j}(x)) \tag{2.3.4}$$

For convenience, the expressions in (2.3.2) and (2.3.4) can be rewritten using the survival function $\bar{F}$ of F as

$$\bar{F}_{m_n}(x) = (\bar{F}_X(x))^n \quad (2.3.5)$$

$$\bar{F}_{m_n}(x) = \prod_{j=1}^{n} \bar{F}_{X_j}(x) \quad (2.3.6)$$

respectively. These exact distributions are extensively used in the statistics literature. For detailed account of the exact distributions of M_n, m_n and other Borel measurable functions of M_n and m_n see David (1970).

As in the case of partial sums discussed in Section 2.2, in many situations the limiting distribution of the normalized maxima will be more useful. Let us look at the behavior of M_n as $n \to \infty$ for a sequence of i.i.d. r.v.s. Obviously, as n increases, M_n increases. Hence, asymptotic behavior of M_n is related to the d.f. of X at its right tail. Let x_F be the right end point of the d.f. F. That is,

$$x_F = \sup\{x \in \mathbb{R} : F_X(x) < 1\}.$$

Now,

$$F_{M_n}(x) = \begin{cases} F^n(x), & x < x_F \\ 1, & x \geq x_F. \end{cases}$$

Hence, as $n \to \infty$

$$F_{M_n}(x) \xrightarrow{d} G(x) = \begin{cases} 0, & x < x_F \\ 1, & x \geq x_F. \end{cases}$$

i.e., M_n converges to a r.v., degenerated at x_F.

Example 2.3.1. *Let $\{X_n\}$ be a sequence of i.i.d. $U(a, b)$ r.v.s. Then $x_F = b$*

and $F_X(x) = \frac{x-a}{b-a}$, $a < x < b$. *Now, for* $x < b$, $0 \leq F_X(x) < 1$ *and hence* $F_{M_n}(x) \to 0$ *as* $n \to \infty$. *For* $x \geq b$, $b < \infty$, $F_X(x) = 1$ *and hence* $F_{M_n}(x) \to 1$. *i.e.,*

$$F_{M_n}(x) \xrightarrow{d} G(x) = \begin{cases} 0, & x < b \\ 1, & x \geq b. \end{cases}$$

That is, sequence $\{M_n\}$ *of a sequence of i.i.d.* $U(a, b)$ *r.v.s converges to a r.v. degenerated at* b.

When the distribution does not have a finite right end point, $\{M_n\}$ converges to a r.v. degenerated at a point mass x_F near $+\infty$. Convergence in distribution to a degenerate r.v. implies convergence in probability and hence,

$$M_n \xrightarrow{P} x_F \quad \text{as} \quad n \to \infty.$$

Since M_n is monotone increasing, the above convergence implies convergence almost sure. That is,

$$M_n \xrightarrow{a.s.} x_F \quad \text{as} \quad n \to \infty.$$

These convergence concepts are not much useful as they do not provide much information. As discussed for the partial sum sequence $\{S_n\}$, the weak convergence results of the centered and normalized maxima provides more insight into the order of magnitude of the maxima. But, this need not exist always. One needs certain continuity conditions at the right end point x_F for the existence of the limit of $P\left(\frac{M_n - a_n}{b_n} \leq x\right) = P(M_n \leq u_n)$ as $n \to \infty$, where $u_n = b_n x + a_n$. For example, if X follows Poisson distribution, $P(M_n \leq u_n)$ will never have a limit in $(0, 1)$ for any sequence $\{u_n\}$. That is, the normalized maxima of i.i.d. Poisson r.v.s do not have a non-degenerate limit distribution. Here comes the crucial difference between the sums and maxima. In the case of sums, the condition of the existence of the second moment assures the convergence to Normal distribution and its failure leads to the class of other α-stable distributions (see, Section 2.2). The following theorem provides the limit distribution of normalized maxima, whenever it exists.

Theorem 2.3.1. *Let* $\{X_n\}$ *be a sequence of i.i.d. r.v.s on* $(\Omega, \mathscr{A}, P)$ *having d.f.* F*. If for some constants* $\{a_n\}$ *and* $\{b_n > 0\}$ $P\left(\frac{M_n - a_n}{b_n} \leq x\right) \to H_\xi(x)$*, then*

$$H_\xi(x) = \begin{cases} e^{-(1+\xi x)^{-1/\xi}}, & \xi \neq 0 \\ e^{-e^{-x}}, & \xi = 0 \end{cases}$$

where $(1 + \xi x) \geq 0$.

$H_\xi(x)$ is known as the generalized extreme value (GEV) distribution. This is the Jenkinson-Von Mises representation of the standard extreme value distributions, where $\xi = 0$ corresponds to the standard Gumbel distribution (Type I), $\xi = \alpha^{-1} > 0$ corresponds to the standard Fréchet distribution (Type II) and $\xi = -\alpha^{-1} > 0$ corresponds to the standard Weibull distribution (Type III) given by

$$\text{Gumbel:} \quad \Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}, \tag{2.3.7}$$

$$\text{Fréchet:} \quad \Phi_\alpha(x) = \begin{cases} 0, & x \leq 0 \\ e^{-x^{-\alpha}}, & x > 0 \end{cases}, \alpha > 0 \tag{2.3.8}$$

$$\text{Weibull:} \quad \Psi_\alpha(x) = \begin{cases} e^{-(-x)^{\alpha}}, & x \leq 0 \\ 1, & x > 0 \end{cases}, \alpha > 0, \tag{2.3.9}$$

which are the max-stable or standard extreme value distributions referred in the fundamental Fisher-Tippett theorem. The three standard extreme value distributions are very different for the purpose of modeling, but in the mathematical point of view they are closely linked. Suppose X is a positive valued r.v., then

$$X \sim \Phi_{\alpha} \quad \Leftrightarrow \quad \ln X \sim \Lambda \quad \Leftrightarrow \quad -X^{-1} \sim \Psi_{\alpha}.$$

The r.v.s corresponding to extreme value distributions are called extremal r.v.s. For details, see Embrechts et. al. (1997), Leadbetter et. al. (1983) and Resnick (1987).

There is a very nice mathematical relation between m_n and M_n. We can rewrite expression (2.1.10) as

$$\begin{aligned} m_n &= -\max\{-X_1, -X_2, \ldots, -X_n\} &\quad (2.3.10) \\ &= -M_n{}', &\quad (2.3.11) \end{aligned}$$

where $M_n{}' = \max\{-X_1, -X_2, \ldots, -X_n\}$. By this relation, the GEV distribution, G corresponding to the minima and H corresponding to the maxima are related by

$$G(x) = 1 - H(-x)$$

and hence, G is given by

$$G_{\xi}(x) = \begin{cases} 1 - e^{-(1-\xi x)^{-1/\xi}}, & \xi \neq 0 \\ 1 - e^{-e^{x}}, & \xi = 0 \end{cases}$$

the GEV distribution corresponding to minima, where $(1 - \xi x) \geq 0$ and the standard extremal types distributions for minima are given by

$$\text{Gumbel:} \quad \Lambda(x) = 1 - e^{-e^{x}}, \quad x \in \mathbb{R}. \tag{2.3.12}$$

$$\text{Fréchet:} \quad \Phi_{\alpha}(x) = \begin{cases} 1 - e^{-(-x)^{-\alpha}}, & x < 0 \\ 1, & x \geq 0 \end{cases}, \alpha > 0 \tag{2.3.13}$$

$$\text{Weibull:} \quad \Psi_{\alpha}(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{(-x)^{\alpha}}, & x \geq 0 \end{cases}, \alpha > 0, \tag{2.3.14}$$

For more details, see Embrechts et. al. (1997), Leadbetter et. al. (1983) and Resnick (1987).

Max-stability we discussed above is under affine transformations. But, do the norming mappings have to be linear? The answer is 'not' and Theorem 2.1 of Pancheva (2010) gives three equivalent conditions which characterize the generalized max-stable distributions on $\mathbb{R}$ under a more general normalization. According to Pancheva (2010), any continuous strictly increasing d.f. is max-stable and the corresponding version of the min-stability under non-linear normalization is obtained by the relation (2.3.10). For details see Pancheva (2010).

Does there exist any situation in which one can express the c.f.s of partial maxima or partial minima or both of a sequence of independent r.v.s in an explicit form? This natural curiosity is the motivation behind this work. From equations (2.1.6) and (2.1.7), for $t \in \mathbb{R}$, the c.f.s of M_n and m_n can be expressed as

$$\phi_{M_n}(t) = \begin{cases} \int_x e^{itx} f_{M_n}(x)dx, & \text{if F is absolutely continuous} \\ \sum_x e^{itx} P_{M_n}(x), & \text{if F is discrete} \end{cases} \tag{2.3.15}$$

and

$$\phi_{m_n}(t) = \begin{cases} \int_x e^{itx} f_{m_n}(x)dx, & \text{if F is absolutely continuous} \\ \sum_x e^{itx} P_{m_n}(x), & \text{if F is discrete} \end{cases} \tag{2.3.16}$$

respectively. If the sequence of r.v.s are i.i.d., then (2.3.15) becomes

$$\phi_{M_n}(t) = \begin{cases} \int_x e^{itx} n[F_X(x)]^{(n-1)} f_X(x)dx, & \text{if F is absolutely continuous} \\ \sum_x e^{itx}[[F_X(x)]^n - [F_X(x-)]^n], & \text{if F is discrete.} \end{cases} \tag{2.3.17}$$

and (2.3.16) becomes

$$\phi_{m_n}(t) = \begin{cases} \int_x e^{itx} n[1 - F_X(x)]^{(n-1)} f_X(x)dx, & \text{if F is absolutely continuous} \\ \sum_x e^{itx}[[\bar{F}_X(x-)]^n - [\bar{F}_X(x)]^n], & \text{if F is discrete.} \end{cases} \tag{2.3.18}$$

No further simplifications to the above integrals or sums in equations (2.3.15) and (2.3.16) are possible unless or until we impose some restrictions over the d.f. F. Let us try to evaluate these through some specific situations.

Example 2.3.2. *Suppose $\{X_n\}$ is a sequence of independent r.v.s such that $X_j \sim \exp(\theta_j)$. Therefore, $F_{X;\theta_j}(x) = 1 - e^{-\theta_j x}$, $x > 0$, $\theta_j > 0$. Then for all $n \geq 1$, the d.f.s of partial maxima, M_n are given by*

$$F_{M_n;\theta_1,\theta_2,\ldots,\theta_n}(x) = \prod_{j=1}^{n}(1 - e^{-\theta_j x})$$

and the d.f.s of partial minima, m_n are given by

$$F_{m_n;\theta_1,\theta_2,\ldots,\theta_n}(x) = 1 - e^{-\sum_{j=1}^{n}\theta_j x}.$$

It is to be noted that minima of n independent exponential r.v.s with parameter

θ_j, $j = 1, 2, \ldots, n$ has an exponential distribution with parameter $\sum_{j=1}^{n} \theta_j$. Let $\phi_{X;\theta_j}$ denote the c.f. of X_j. Then $\phi_{X;\theta_j}(t) = \frac{\theta_j}{\theta_j - it}$. Now, due to the one to one correspondence between the d.f. and the c.f. of a r.v., the c.f.s of partial minima, m_n, are given by

$$\phi_{m_n;\theta_1,\theta_2,\ldots,\theta_n}(t) = \frac{\sum_{j=1}^{n} \theta_j}{\sum_{j=1}^{n} \theta_j - it} = \phi_{X;\sum_{j=1}^{n} \theta_j}(t). \qquad (2.3.19)$$

In the above example, if the r.v.s are identically distributed with parameter θ, then

$$\phi_{m_n;\theta}(t) = \frac{n\theta}{n\theta - it} = \phi_{X;n\theta}(t).$$

That is, the c.f. of m_n of i.i.d. exponential r.v.s can be expressed in a very compact form. Even for independent non-identical exponential r.v.s the c.f. of m_n is in a compact form. Does there exist a class of distributions for which the c.f.s of partial minima or partial maxima can be expressed in a simple form? The next chapter introduces such a new class of distributions and study their properties.

CHAPTER

THREE

Families of Univariate Distributions Closed under the Minima or Maxima

The discussions in this chapter is on families of distributions for which the d.f.s of either the maxima or minima belong to the same family. The concept is defined and described providing various examples. Sufficient conditions for a family of distributions to have this property are obtained. Some properties of such families are also discussed. The results corresponding to i.i.d. sequence of r.v.s is based on Aparna and Chandran (2017) and those corresponding to independent non-identical case is based on Aparna and Chandran (2018a).

The chapter is organized as follows: Section 3.1 discusses various types of closure property of sets. Section 3.2 provides the definitions of the concepts of closure under the minima and maxima providing suitable examples to illustrate these concepts. Sufficient conditions for families of distributions to have closure property under the minima or maxima are also obtained in this section. Section 3.3 is on how the closure property under the minima and maxima changes on monotone transformations of the r.v.s. Section 3.4 discusses how the closure property under the minima and maxima changes on truncation of r.v.s.

3.1 Introduction

Let A be a set and $*$ a binary operation defined on that set. Let a_1 and a_2 be any two elements of the set A. If $a_1 * a_2$ is also in A, then we say that A is closed under the operation $*$. That is, a set is said to be closed under an operation if, and only if, on applying that operation on any two or more elements of that set provides another element of the same set. The basic operations in elementary algebra are addition, subtraction, multiplication and division. Some examples of sets having closure property with respect to these elementary operations are:

Example 3.1.1. *The set of positive integers is closed under addition and multiplication and is not closed under subtraction and division.*

Example 3.1.2. *The set of integers is closed under addition, subtraction and multiplication and is not closed under division.*

Example 3.1.3. *The set of real numbers is closed under addition, subtraction, multiplication and division.*

From the above examples one can see that, the set of positive integers, the set of integers and the set of real numbers are all closed with respect to the operations of addition and multiplication. The set of integers and the set of real numbers are closed with respect to subtraction, while the set of positive integers is not closed with respect to subtraction. Similarly, the set of real numbers is closed with respect to division, while the set of integers and the set of positive integers are not closed with respect to division. Closure property of a set with respect to an operation does not necessarily imply closure on all subsets and supersets.

In some situations, the elements of a set will be again sets. Such sets are called a class of sets. Some examples of the operations on these sets are; union, intersection and complementation.

Example 3.1.4. *A σ-field is a non-empty class of sets which is closed under countable union, countable intersection and complementation. That is, if $\mathscr{A}$ is a σ-field and A_j, $j \geq 1$ integer, are elements of $\mathscr{A}$, then $\bigcup_{j=1}^{n} A_j$ and $\bigcap_{j=1}^{n} A_j$ are in $\mathscr{A}$, and for any element A of $\mathscr{A}$, A^c, the complement of A is also in $\mathscr{A}$.*

Let $\mathscr{F}$ be a family of distributions of independent r.v.s with same functional form except for a parameter. One of the most common operations on such a set is convolution of d.f.s. Such a family of distributions is said to be closed under convolution, if the convolution of any two or more members of that family belongs to the same family of distributions. A set of independent r.v.s are said to be closed under addition if the corresponding family of distributions are closed under convolution.

Example 3.1.5. *The family of Binomial distributions with common probability of success is closed under convolution.*

Example 3.1.6. *The family of Normal distributions is closed under convolution.*

We have discussed different types of closure properties. In the following section we are going to define families of distributions for which the d.f.s of either the maxima or minima belong to the same family of distributions. Sufficient conditions for a family of distributions to have these properties are also obtained.

3.2 Families of Distributions Closed under the Minima or Maxima

In this section we first define the closure property under the minima as well as maxima of a sequence of r.v.s or a family of distributions and provide suitable examples. Let $\{X_n\}$ be a sequence of independent r.v.s such that $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$. Let us denote the collection of parameters $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, corresponding to $X_1, X_2, \ldots, X_n$ whose d.f.s belong to $\mathscr{F}_{X;\boldsymbol{\alpha}}$ by $\underset{\sim}{\alpha}_n$, $n \geq 1$.

Definition 3.2.1. *A family of distributions, $\mathscr{F}_{X;\boldsymbol{\alpha}}$, is said to be closed under the minima with respect to $\boldsymbol{\alpha}$, if for all $n \geq 1$ and $F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$, $j = 1, 2, \ldots, n$, the d.f.s of minima, $F_{m_n;\underset{\sim}{\alpha}_n} = F_{X;g(\underset{\sim}{\alpha}_n)} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$ with parameter $g(\underset{\sim}{\alpha}_n)$ depending on $\underset{\sim}{\alpha}_n$.*

Remark 3.2.1. *A set of r.v.s is said to be closed under the minima, if the corresponding family of distributions is closed under the minima.*

For a sequence of i.i.d. r.v.s, we can restate the above definition as follows.

Definition 3.2.2. *Let X be a r.v. with d.f. $F_{X;\alpha}$ involving a parameter α and $(X_1, X_2, \ldots, X_n)$ be i.i.d. copies of X. Then $F_{X;\alpha}$ is said to be closed under the minima, with respect to the parameter α, if for every $n \geq 1$, the d.f. of m_n is $F_{X;g_n(\alpha)}$, for some $g_n(\alpha)$ which is a function of n and the parameter α.*

Remark 3.2.2. *When the r.v.s are i.i.d. $g(\underset{\sim}{\alpha}_n)$ coincides with $g_n(\alpha)$ of Definition 3.2.2.*

Example 3.2.1. *Let X_j, $j = 1, 2, \ldots, n$ be independent r.v.s such that, $X_j \sim F_{X;\theta_j}$ and $F_{X;\theta_j}(x) = 1 - e^{-\theta_j x}$, $x > 0$, $\theta_j > 0$. Then $F_{m_n;\underset{\sim}{\theta}_n}(x) = 1 - e^{-\sum_{j=1}^{n} \theta_j x} = F_{X;g(\underset{\sim}{\theta}_n)}(x)$, where $g(\underset{\sim}{\theta}_n) = \sum_{j=1}^{n} \theta_j$. If X_js are i.i.d. with parameter θ, then $g_n(\theta) = n\theta$.*

Example 3.2.2. *Suppose* X_j, $j = 1, 2, \ldots, n$ *follows Pareto distribution with d.f.* $F_{X;\alpha_j}(x) = 1 - (\frac{p}{x})^{\alpha_j}$, $x > p$, $p > 0$, $\alpha_j > 0$, *then* $F_{m_n;\alpha}(x) = 1 - (\frac{p}{x})^{\sum_{j=1}^{n}\alpha_j} = F_{X;g_n(\underset{\sim}{\alpha}_n)}(x)$, *where* $g(\underset{\sim}{\alpha}_n) = \sum_{j=1}^{n}\alpha_j$ *When the r.v.s are i.i.d.,* $g_n(\alpha) = n\alpha$. *i.e., the family of independent Pareto distributions is closed under the minima.*

Example 3.2.3. *Let* X_j, $j = 1, 2, \ldots, n$ *be independent r.v.s which follows the Geometric distribution with parameter* $0 < p_j < 1$, *i.e.,*

$$F_{X;p_j}(x) = \begin{cases} 0, & x < 0 \\ 1 - (1 - p_j)^{[x]+1}, & x \geq 0, \end{cases}$$

where $[x]$ *denote the greatest integer less than or equal to* x. *Then the d.f.s of the minima is given by*

$$F_{m_n;\underset{\sim}{p}}(x) = \begin{cases} 0, & x < 0 \\ 1 - \prod_{j=1}^{n}(1 - p_j)^{[x]+1}, & x \geq 0. \end{cases}$$

So, m_n *has Geometric distribution with parameter* $0 < 1 - \prod_{j=1}^{n}(1 - p_j) < 1$. *Here,* $g(\underset{\sim}{p}_n) = 1 - \prod_{j=1}^{n}(1 - p_j)$. *Therefore, the family of geometric distributions is closed under the minima. If* $X_j s$ *are i.i.d. Geometric with parameter* p, *then* $g_n(p) = 1 - (1 - p)^n$.

Remark 3.2.3. *The Geometric distribution with support on* $1, 2, \ldots$ *is also closed under the minima with* $g(\underset{\sim}{p}_n) = 1 - \prod_{j=1}^{n}(1 - p_j)$.

Example 3.2.4. *Let* $F_{X;\theta_j}(x) = 1 - e^{-\left(\frac{x}{\theta_j}\right)^{\alpha}}$, $x > 0$, $\theta_j > 0$, $\alpha > 0$ *and* X_j, $j = 1, 2, \ldots, n$ *be independent r.v.s such that* $X_j \sim F_{X;\theta_j}$. *Then* $F_{m_n;\underset{\sim}{\theta}_n}(x) = 1 - e^{-\sum_{j=1}^{n}\left(\frac{x}{\theta_j}\right)^{\alpha}} = 1 - e^{-\left(x\left(\sum_{j=1}^{n}\left(\frac{1}{\theta_j}\right)^{\alpha}\right)^{1/\alpha}\right)^{\alpha}}$. *Therefore, the family of Weibull* *distributions is closed under the minima and* $g(\underset{\sim}{\theta}_n) = \frac{1}{\left(\sum_{j=1}^{n}\left(\frac{1}{\theta_j}\right)^{\alpha}\right)^{1/\alpha}}$. *If* $X_j s$ *are i.i.d. with parameter* θ, *then* $g_n(\theta) = \frac{\theta}{n^{1/\alpha}}$.

Definition 3.2.3. *A family of distributions,* $\mathscr{F}_{X;\boldsymbol{\alpha}}$, *is said to be closed under the maxima with respect to* $\boldsymbol{\alpha}$, *if for all* $n \geq 1$ *and* $F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$, $j = 1, 2, \ldots, n$, *the d.f.s of maxima,* $F_{M_n;\underset{\sim}{\alpha}_n} = F_{X;h(\underset{\sim}{\alpha}_n)} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$ *with parameter* $h(\underset{\sim}{\alpha}_n)$ *depending on* $\underset{\sim}{\alpha}_n$.

Remark 3.2.4. *A set of r.v.s is said to be closed under the maxima, if the corresponding family of distributions is closed under the maxima.*

When the sequence of r.v.s are i.i.d. we can restate the above definition as follows.

Definition 3.2.4. *Let* X *be a r.v. with d.f.* $F_{X;\alpha}$ *involving a parameter* α *and* $(X_1, X_2, \ldots, X_n)$ *be a random sample of size* n *from* $F_{X;\alpha}$. *Then* $F_{X;\alpha}$ *is said to be closed under the maxima, with respect to the parameter* α, *if for every* $n \geq 1$, *the d.f. of* M_n *is* $F_{X;h_n(\alpha)}$, *for some* $h_n(\alpha)$ *which is a function of* n *and the parameter* α.

Remark 3.2.5. *When the r.v.s are i.i.d.* $h(\underset{\sim}{\alpha}_n)$ *coincides with* $h_n(\alpha)$ *of Definition 3.2.4.*

Example 3.2.5. *Suppose* X_j, $j = 1, 2, \ldots, n$ *are independent observations having the power distribution with d.f.* $F_{X;\alpha_j}(x) = \left(\frac{x}{\theta}\right)^{\alpha_j}$, $0 < x < \theta$, $\alpha_j > 0$. *Then* $F_{M_n;\underset{\sim}{\alpha}_n}(x) = \left(\frac{x}{\theta}\right)^{\sum_{j=1}^{n}\alpha_j} = F_{X;h(\underset{\sim}{\alpha}_n)}(x)$. *i.e., the family of power distributions is closed under the maxima with* $h(\underset{\sim}{\alpha}_n) = \sum_{j=1}^{n}\alpha_j$. *In the i.i.d. case with parameter* α, $h_n(\alpha) = n\alpha$.

Example 3.2.6. *If* X_j, $j = 1, 2, \ldots, n$ *are independent inverse Weibull r.v.s with corresponding d.f.* $F_{X;\theta_j}(x) = e^{-\left(\frac{\theta_j}{x}\right)^{\alpha}}$, $x > 0, \theta > 0, \alpha > 0$, *then*

$F_{M_n;\underset{\sim}{\theta}_n}(x) = e^{-\sum_{j=1}^{n}\left(\frac{\theta_j}{x}\right)^{\alpha}} = F_{X;h_n(\theta)}(x)$. *Hence, inverse Weibull distribution is closed under the maxima. In this case* $h_n(\theta) = n^{1/\alpha}\theta$.

Example 3.2.7. *Suppose* X_j, $j = 1, 2, \ldots, n$ *are independent r.v.s having the generalized exponential distribution with d.f.* $F_{X;\alpha_j}(x) = (1 - e^{-\lambda x})^{\alpha_j}$, $x > 0$, $\lambda, \alpha_j > 0$. *Then* $F_{M_n;\underset{\sim}{\alpha}_n}(x) = (1 - e^{-\lambda x})^{\sum_{j=1}^{n}\alpha_j}$. *Hence, the family of generalized exponential distributions is closed under the maxima with* $h(\underset{\sim}{\alpha}_n) = \sum_{j=1}^{n}\alpha_j$. *If* $X_j s$ *are i.i.d. with parameter* α, *then* $g_n(\alpha) = n\alpha$.

The next example describes a family of distributions which is closed under both the minima and maxima.

Example 3.2.8. *If* X_j $j = 1, 2, \ldots, n$ *has independent Bernoulli r.v.s with d.f.*

$$F_{X;p_j}(x) = \begin{cases} 0, & x < 0 \\ 1 - p_j, & 0 \le x < 1, \quad 0 < p_j < 1. \\ 1, & x \ge 1 \end{cases}$$

Then

$$F_{m_n;\underset{\sim}{p}_n}(x) = \begin{cases} 0, & x < 0 \\ 1 - \prod_{j=1}^{n} p_j, & 0 \le x < 1, \quad 0 < p_j < 1 \\ 1, & x \ge 1 \end{cases} = F_{X;g(\underset{\sim}{p}_n)}(x),$$

where $g(\underset{\sim}{p}_n) = \prod_{j=1}^{n} p_j$ *and*

$$F_{M_n;\underset{\sim}{p}_n}(x) = \begin{cases} 0, & x < 0 \\ \prod_{j=1}^{n}(1 - p_j), & 0 \le x < 1, \quad 0 < p_j < 1 \\ 1, & x \ge 1 \end{cases} = F_{X;h(\underset{\sim}{p}_n)}(x).$$

where $h(\underset{\sim}{p}_n) = 1 - \prod_{j=1}^{n}(1 - p_j)$. *Hence, the family of Bernoulli distributions is closed under both the minima and maxima. When the r.v.s are i.i.d.* $g_n(p) = p^n$ *and* $h_n(p) = 1 - (1 - p)^n$.

Remark 3.2.6. *Every two point distribution is closed under both the minima and maxima.*

Remark 3.2.7. *Families of max-stable (min-stable) distributions are closed under the maxima (minima).*

Remark 3.2.8. *Exponential distribution is closed under minima, but is not closed under maxima in the sense of Definition 3.2.4. However, exponential distribution is max-stable and min-stable in the sense of Pancheva (2010).*

Note 3.2.1. *If a family of distributions is either closed under the maxima or closed under the minima, we say that the family of distributions is closed under extrema.*

From the examples discussed above we can see that in most of the cases $g(\underset{\sim}{\alpha}_n)$ or $h(\underset{\sim}{\alpha}_n)$ is equal to $\sum_{j=1}^{n} \alpha_j$ and it is not so always. In the following we will be looking into how the functional form of $g(\underset{\sim}{\alpha}_n)$ or $h(\underset{\sim}{\alpha}_n)$ is related to the functional form of the s.f. or d.f. respectively. We can see that except in Example (3.2.3) and Example (3.2.8), the parameter with respect to which the family is closed is an exponent parameter, while in these two cases it is the base parameter. The following theorem gives a sufficient condition for families of distributions to have closure property under the minima with respect to the exponent parameter.

Theorem 3.2.1. *Let* $\{X_j\}$ *be a sequence of independent r.v.s such that* $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$ *and* χ *be a one-one onto function from* $\mathbb{R}^+$ *to* $\mathbb{R}^+$. *The fam-*

ily $\mathscr{F}_{X;\alpha}$ *is closed under the minima with respect to the exponent parameter if for every* j *and all* $x \in \mathbb{R}$, $\bar{F}_{X;\alpha_j}(x) = (\bar{F}_{X;1}(x))^{\chi(\alpha_j)}$, *then* $g(\underset{\sim}{\alpha}_n) = \chi^{-1}\{\sum_{j=1}^{n} \chi(\alpha_j)\}$.

Proof. Let $\bar{F}_{X;\alpha_j}(x) = (\bar{F}_{X;1}(x))^{\chi(\alpha_j)}$ for every j and all $x \in \mathbb{R}$. Then

$$
\begin{aligned}
\bar{F}_{m_n;\underset{\sim}{\alpha}_n}(x) &= \prod_{j=1}^{n}(\bar{F}_{X;\alpha_j}(x)) \\
&= (\bar{F}_{X;1}(x))^{\sum_{j=1}^{n} \chi(\alpha_j)} \\
&= (\bar{F}_{X;1}(x))^{\chi(\chi^{-1}\{\sum_{j=1}^{n} \chi(\alpha_j)\})} \\
&= \bar{F}_{X;\chi^{-1}\{\sum_{j=1}^{n} \chi(\alpha_j)\}}(x) \\
&= \bar{F}_{X;g(\underset{\sim}{\alpha}_n)}(x),
\end{aligned}
$$

where $g(\underset{\sim}{\alpha}_n) = \chi^{-1}\{\sum_{j=1}^{n} \chi(\alpha_j)\}$. □

Corollary 3.2.1. *If* X_j, $j = 1, 2, \ldots, n$ *are i.i.d. as* X *with* $\bar{F}_{X;\alpha}(x) = (\bar{F}_{X;1}(x))^{\chi(\alpha)}$, *then* $g(\underset{\sim}{\alpha}_n) = \chi^{-1}\{n\chi(\alpha)\}$.

Example 3.2.9. *Consider Example 3.2.1, since* $\bar{F}_{X;\theta_j}(x) = e^{-\theta_j x} = (e^{-x})^{\theta_j}$ *Theorem 3.2.1 can be applied. Therefore,* $\chi(\theta_j) = \theta_j$ *and* $\chi^{-1}(\theta_j) = \theta_j$. *Hence,* $g(\underset{\sim}{\theta}_n) = \chi^{-1}\left(\sum_{j=1}^{n} \chi(\theta_j)\right) = \sum_{j=1}^{n} \theta_j$.

Example 3.2.10. *From Example 3.2.4,* $\bar{F}_{X;\theta_j}(x) = e^{-\left(\frac{x}{\theta_j}\right)^{\alpha}}$. *Hence, by Theorem 3.2.1,* $g(\underset{\sim}{\theta}_n) = \left(\frac{1}{\sum_{j=1}^{n}\left(\frac{1}{\theta_j}\right)^{\alpha}}\right)^{1/\alpha}$.

The next theorem, similar to the previous one, gives a sufficient condition for families of distributions to be closed under the maxima with respect to the exponent parameter.

Theorem 3.2.2. *Let* $\{X_j\}$ *be a sequence of independent r.v.s such that* $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$ *and* χ *be a one-one onto function from* $\mathbb{R}^+$ *to* $\mathbb{R}^+$. *The family* $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *is closed under the maxima with respect to the exponent parameter, if for every* j *and all* $x \in \mathbb{R}$, $F_{X;\alpha_j}(x) = (F_{X;1}(x))^{\chi(\alpha_j)}$, *then* $h(\underset{\sim}{\alpha}_n) = \chi^{-1}\{\sum_{j=1}^{n} \chi(\alpha_j)\}$.

Proof. The proof is similar to that of Theorem 3.2.1. □

Corollary 3.2.2. *If* X_j, $j = 1, 2, \ldots, n$ *are i.i.d. as* X *with* $F_{X;\alpha}(x) = (F_{X;1}(x))^{\chi(\alpha)}$, *then* $h(\underset{\sim}{\alpha}_n) = \chi^{-1}\{n\chi(\alpha)\}$.

The following theorem gives a sufficient condition for a family of distributions to be closed under the minima with respect to the base parameter.

Theorem 3.2.3. *Let* $\{X_j\}$ *be a sequence of independent r.v.s such that* $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$ *and* χ *be a one-one onto function from* $(0,1)$ *to* $(0,1)$. *The family* $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *is closed under the minima with respect to the base parameter, if for every* j *and all* $x \in \mathbb{R}$, $\bar{F}_{X;p_j}(x) = (\chi(p_j))^{\omega(x)}$, *then* $g(\underset{\sim}{p}_n) = \chi^{-1}(\prod_{j=1}^{n} \chi(p_j))$.

Proof. Let $\bar{F}_{X;p_j}(x) = (\chi(p_j))^{\omega(x)}$ for every j and all $x \in \mathbb{R}$. Then

$$\begin{aligned}
\bar{F}_{m_n;\underset{\sim}{p}_n}(x) &= \prod_{j=1}^{n} (\bar{F}_{X;p_j}(x)) \\
&= (\prod_{j=1}^{n} \chi(p_j))^{\omega(x)} \\
&= (\chi\{\chi^{-1}(\prod_{j=1}^{n} \chi(p_j))\})^{\omega(x)} \\
&= \bar{F}_{X;\chi^{-1}(\prod_{j=1}^{n} \chi(p_j))}(x) \\
&= \bar{F}_{X;g(\underset{\sim}{p}_n)}(x),
\end{aligned}$$

where $g(\underset{\sim}{p}_n) = \chi^{-1}(\prod_{j=1}^{n} \chi(p_j))$. □

Corollary 3.2.3. *If* X_j, $j = 1, 2, \ldots, n$ *are i.i.d. as* X *with* $\bar{F}_{X;p}(x) = (\chi(p))^{\omega(x)}$, *then* $g(\underset{\sim}{p}_n) = \chi^{-1}(\chi^n(p))$.

Example 3.2.11. *Consider Example 3.2.3 where* $\bar{F}_{X;p_j}(x) = (1-p_j)^{[x]+1}$. *This is of the form* $(\chi(p_j))^{\omega(x)}$ *with* $\omega(x) = [x]+1$, $\chi(p_j) = 1-p_j$ *and* $\chi^{-1}(p_j) = 1-p_j$. *Hence, by Theorem 3.2.3,* $g(\underset{\sim}{p}_n) = \chi^{-1}(\prod_{j=1}^{n} \chi(p_j)) = 1 - \prod_{j=1}^{n}(1 - p_j)$.

Sufficient conditions for a family of distributions to be closed under the maxima with respect to the base parameter is given in the following theorem.

Theorem 3.2.4. *Let* $\{X_j\}$ *be a sequence of independent r.v.s such that* $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$ *and* χ *be a one-one onto function from* $(0,1)$ *to* $(0,1)$. *The family* $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *is closed under the maxima with respect to the base parameter, if for every* j *and all* $x \in \mathbb{R}$, $F_{X;p_j}(x) = (\chi(p_j))^{\omega(x)}$, *then* $g(\underset{\sim}{p}_n) = \chi^{-1}(\prod_{j=1}^{n} \chi(p_j))$.

Proof. The proof is similar to that of Theorem 3.2.3. □

Corollary 3.2.4. *If* X_j, $j = 1, 2, \ldots, n$ *are i.i.d. as* X *with* $F_{X;p}(x) = (\chi(p))^{\omega(x)}$, *then* $h(\underset{\sim}{p}_n) = \chi^{-1}(\chi^n(p))$.

From Example 3.2.4 we can see that the Weibull distribution is closed under the minima with respect to the scale parameter. Dose it have closure property under extrema with respect the shape parameter? The answer is 'no' and is illustrated in the following example.

Example 3.2.12. *Let* $\bar{F}_{X_1}(x) = e^{-\left(\frac{x}{\theta_1}\right)^{\alpha_1}}$ *and* $\bar{F}_{X_2}(x) = e^{-\left(\frac{x}{\theta_2}\right)^{\alpha_2}}$. *Then the d.f.s of their minima, given by* $\bar{F}_{m_2}(x) = e^{-\left(\frac{x}{\theta_1}\right)^{\alpha_1} - \left(\frac{x}{\theta_2}\right)^{\alpha_2}}$ *is not Weibull.*

Remark 3.2.9. *The Family of Weibull distributions does not have closure property under the minima with respect to the shape parameter.*

The above examples of families of distributions or corresponding set of r.v.s which are closed under the minima or maxima are of standard distributions. Given an arbitrary distribution F, can one construct families of distributions having closure property under the maxima or minima? The answer is YES, and is as follows: consider an arbitrary distribution F, the family of distributions $\mathscr{F}_{\boldsymbol{\alpha}} = \{F^{\alpha}; \alpha > 0\}$ is closed under the maxima with respect to $\boldsymbol{\alpha}$ with $h(\underset{\sim}{\alpha}_n) = \sum_{j=1}^{n} \alpha_j$ and the family of distributions $\mathscr{F}_{\boldsymbol{\alpha}} = \{1 - (1 - F)^{\alpha}; \alpha > 0\}$ is closed under the minima with respect to $\boldsymbol{\alpha}$ with $g(\underset{\sim}{\alpha}_n) = \sum_{j=1}^{n} \alpha_j$.

So we have gone through some sufficient conditions for the families of distributions to be closed under the minima or maxima. The next section is on strictly monotone Borel measurable transformations of a set of r.v.s closed under extrema.

3.3 Monotone Transformations and Closure Property under Extrema

Let X be a r.v. on a probability space $(\Omega, \mathscr{A}, P)$ and $\xi(X)$ be any Borel measurable function of X. Then $\xi(X)$ is also a r.v. on $(\Omega, \mathscr{A}, P)$ and the d.f. of $\xi(X)$ is determined by that of X (Laha and Rohatgi (1979), Page 5, Remark 1.1.1.). In this section, we discuss the closure property under extrema of strictly increasing or strictly decreasing Borel measurable transformations of the elements of a set of r.v.s closed under extrema. Some examples of such transformations are; $\xi(X) = \ln X$, $\xi(X) = -X$ and $\xi(X) = \frac{1}{X}$, with some restrictions on the support of X. We observe that closure property under extrema is invariant under strictly increasing transformations and they interchange under strictly decreasing transformations. Let us denote the range of a r.v. X by

$\mathsf{Ran}X$. Then we have the following theorem for strictly increasing measurable transformations.

Theorem 3.3.1. *Let the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ be closed under extrema with respect to $\boldsymbol{\alpha}$ and ξ be a Borel measurable and strictly increasing function on $\mathsf{Ran}X$. Let $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$ is the family of distributions of $\xi(X)$. Then the following results hold.*

i) If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima, then so is $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$.

ii) If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima, then so is $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$.

Furthermore, the corresponding $g(\underset{\sim}{\alpha}_n)$ or $h(\underset{\sim}{\alpha}_n)$ for both the families will be same.

Proof. Let $Y_j = \xi(X_j)$ and $G_{Y;\alpha_j}$ be the d.f. of Y_j. Since ξ is strictly increasing, we have

$$\begin{aligned} G_{Y;\alpha_j}(x) &= P_{\alpha_j}(\xi(X) \leq x) \\ &= P_{\alpha_j}(X \leq \xi^{-1}(x)) \\ &= F_{X;\alpha_j}(\xi^{-1}(x)). \end{aligned}$$

Let $\mathscr{F}_{X;\boldsymbol{\alpha}}$ be closed under the maxima. Then for all $F_{X;\alpha_j}$, $j = 1, 2, \ldots, n$ belonging to $\mathscr{F}_{X;\boldsymbol{\alpha}}$;

$$F_{M_n;\underset{\sim}{\alpha}_n}(x) = F_{X;h(\underset{\sim}{\alpha}_n)}(x).$$

Hence,

$$\begin{aligned} G_{M_n;\underset{\sim}{\alpha}_n}(x) &= \prod_{j=1}^{n} G_{Y;\alpha_j}(x) \\ &= \prod_{j=1}^{n} F_{X;\alpha_j}(\xi^{-1}(x)) \end{aligned}$$

$$= F_{X;h(\underset{\sim}{\alpha}_n)}(\xi^{-1}(x))$$
$$= G_{Y;h(\underset{\sim}{\alpha}_n)}(x).$$

i.e., $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$ is closed under the maxima with $h(\underset{\sim}{\alpha}_n)$ of $\mathscr{F}_{X;\boldsymbol{\alpha}}$. Now, let $\mathscr{F}_{X;\boldsymbol{\alpha}}$ be closed under the minima. Then for all $F_{X;\alpha_j}$, $j = 1, 2, \ldots, n$, belonging to $\mathscr{F}_{X;\boldsymbol{\alpha}}$;

$$F_{m_n;\underset{\sim}{\alpha}_n}(x) = F_{X;g(\underset{\sim}{\alpha}_n)}(x).$$

Hence,

$$\begin{aligned}\bar{G}_{m_n;\underset{\sim}{\alpha}_n}(x) &= \prod_{j=1}^{n} \bar{G}_{Y;\alpha_j}(x) \\ &= \prod_{j=1}^{n} \bar{F}_{X;\alpha_j}(\xi^{-1}(x)) \\ &= \bar{F}_{X;g(\underset{\sim}{\alpha}_n)}(\xi^{-1}(x)) \\ &= \bar{G}_{Y;g(\underset{\sim}{\alpha}_n)}(x).\end{aligned}$$

i.e., $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$ is closed under the minima with $g(\underset{\sim}{\alpha}_n)$ of $\mathscr{F}_{X;\boldsymbol{\alpha}}$. Hence the proof. □

The following results provide particular cases of strictly increasing Borel measurable transformations of X.

Result 3.3.1. *Let the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$, $X > 0$ a.s., be closed under the minima (maxima) with respect to $\boldsymbol{\alpha}$ and $\xi(X) = \ln X$. Then the family $\mathscr{G}_{\ln X;\boldsymbol{\alpha}}$ is closed under the minima (maxima) with respect to $\boldsymbol{\alpha}$.*

Result 3.3.2. *Closure property under extrema is invariant under change of scale and origin.*

The next theorem is on strictly decreasing measurable transformations.

Theorem 3.3.2. *Let the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ be closed under extrema with respect to $\boldsymbol{\alpha}$ and ξ be Borel measurable and strictly decreasing on $\mathsf{Ran}X$. Let $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$ is the family of distributions of $\xi(X)$. Then the following results hold.*

i) If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima, then the family $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$ is closed under the minima with corresponding $g(\underset{\sim}{\alpha}_n) = h(\underset{\sim}{\alpha}_n)$ of $\mathscr{F}_{X;\boldsymbol{\alpha}}$.

ii) If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima, then $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$ is closed under the maxima with corresponding $h(\underset{\sim}{\alpha}_n) = g(\underset{\sim}{\alpha}_n)$ of $\mathscr{F}_{X;\boldsymbol{\alpha}}$.

Proof. Let $G_{Y;\alpha_j}$ be the d.f. of $Y_j = \xi(X_j)$. Since ξ is strictly decreasing, $-\xi$ is strictly increasing. Hence, by Theorem 3.3.1 and the relation (2.3.10),

$$\begin{aligned}
\bar{G}_{m_n;\underset{\sim}{\alpha}_n}(x) &= P_{\underset{\sim}{\alpha}_n}(\min(\xi(X_1), \xi(X_2), \ldots, \xi(X_n)) > x) \\
&= P_{\underset{\sim}{\alpha}_n}(-\max(-\xi(X_1), -\xi(X_2), \ldots, -\xi(X_n)) > x) \\
&= P_{\underset{\sim}{\alpha}_n}(\max(-\xi(X_1), -\xi(X_2), \ldots, -\xi(X_n)) < -x) \\
&= P_{h(\underset{\sim}{\alpha}_n)}(-\xi(X) < -x) \\
&= P_{h(\underset{\sim}{\alpha}_n)}(\xi(X) > x) \\
&= \bar{G}_{Y;h(\underset{\sim}{\alpha}_n)}(x).
\end{aligned}$$

i.e., $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$ is closed under the minima with $g(\underset{\sim}{\alpha}_n) = h(\underset{\sim}{\alpha}_n)$ of $\mathscr{F}_{X;\boldsymbol{\alpha}}$. Similarly, by Theorem 3.3.1 and the relation (2.3.10), we have

$$\begin{aligned}
G_{M_n;\underset{\sim}{\alpha}_n}(x) &= P_{\underset{\sim}{\alpha}_n}(\max(\xi(X_1), \xi(X_2), \ldots, \xi(X_n)) \le x) \\
&= P_{\underset{\sim}{\alpha}_n}(-\min(-\xi(X_1), -\xi(X_2), \ldots, -\xi(X_n)) \ge x) \\
&= P_{\underset{\sim}{\alpha}_n}(\min(-\xi(X_1), -\xi(X_2), \ldots, -\xi(X_n)) \le -x) \\
&= P_{g(\underset{\sim}{\alpha}_n)}(-\xi(X) \le -x)
\end{aligned}$$

$$= P_{g(\underset{\sim}{\alpha}_n)}(\xi(X) \geq x)$$
$$= G_{Y;g(\underset{\sim}{\alpha}_n)}(x).$$

i.e., $\mathscr{G}_{\xi(X);\boldsymbol{\alpha}}$ is closed under the maxima with $h(\underset{\sim}{\alpha}_n) = g(\underset{\sim}{\alpha}_n)$ of $\mathscr{F}_{X;\boldsymbol{\alpha}}$. Hence the proof. □

The following results are particular cases of strictly decreasing Borel measurable transformations of X.

Result 3.3.3. *Let the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ be closed under the minima (maxima) with respect to $\boldsymbol{\alpha}$ and $\xi(X) = -X$. Then $\mathscr{G}_{-X,\alpha}$ is closed under the maxima (minima) with respect to $\boldsymbol{\alpha}$.*

Result 3.3.4. *Let the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$, $P(X \neq 0) = 1$, be closed under the minima (maxima) with respect to $\boldsymbol{\alpha}$ and $\xi(X) = \frac{1}{X}$. Then the family $\mathscr{G}_{\frac{1}{X},\alpha}$ is closed under the maxima (minima) with respect to $\boldsymbol{\alpha}$.*

Result 3.3.5. *Let the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ be closed under the minima (maxima) with respect to $\boldsymbol{\alpha}$ and $\xi(X) = X^\gamma$. Then*

1. *the family $\mathscr{G}_{X^\gamma,\alpha}$, $X \geq 0$, a.s., is closed under the minima (maxima) with respect to $\boldsymbol{\alpha}$, for $\gamma > 0$, a constant.*

2. *the family $\mathscr{G}_{X^\gamma,\alpha}$, $X > 0$, a.s., is closed under the maxima (minima) with respect to $\boldsymbol{\alpha}$, for $\gamma < 0$, a constant.*

3.4 Truncation and Closure Property under Extrema

Truncated distributions are obtained by restricting the domain of a probability distribution. The restriction can be either on the left side of the domain or the right side of the domain or both. The probability distribution of X, conditional on $X > a$, is called "the left-truncated distribution of X, truncated at a" and the probability distribution of X, conditional on $X < b$, is called "the right-truncated distribution of X, truncated at b". If X has a p.d.f. or p.m.f., then the truncated distribution also has a p.d.f. or p.m.f. respectively and is equal to that of X restricted to $x > a$ or $X < b$ or $a < X < b$ and normalized to have total mass 1. The d.f. of X truncated at both ends, or doubly truncated, is denoted by $F_{X_{T(a,b)}}$ and is given by

$$F_{X_{T(a,b)}}(x) = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} \tag{3.4.1}$$

with corresponding s.f.

$$\bar{F}_{X_{T(a,b)}}(x) = \frac{\bar{F}_X(x) - \bar{F}_X(b)}{\bar{F}_X(a) - \bar{F}_X(b)} \tag{3.4.2}$$

for $x_a \leq a < x < b \leq x_b$, where x_a and x_b are respectively the left and the right end points of the support of X. If $a = x_a$ and $b < x_b$, we get a right truncated distribution. Similarly, if $a > x_a$ and $b = x_b$ we get a left truncated distribution. Hence, the d.f. of a right truncated distribution truncated at b is given by

$$F_{X_{T(b)}}(x) = \frac{F_X(x)}{F_X(b)} \tag{3.4.3}$$

and the s.f. of a left truncated distribution truncated at a is given by

$$\bar{F}_{X_{T(a)}}(x) = \frac{\bar{F}_X(x)}{\bar{F}_X(a)}. \tag{3.4.4}$$

For details, see page 62 of Johnson et. al. (2005).

In this section, we see how truncation affects the closure property under extrema of a set of r.v.s closed under extrema. From (3.4.1) and (3.4.2) we can easily say that the closure under extrema is not preserved under truncation at both the ends. Let us denote the family of distributions truncated at b on the right side of the support of r.v.s whose d.f.s belong to $\mathscr{F}_{X;\boldsymbol{\alpha}}$, a family of distributions closed under extrema, by $\mathscr{F}_{X_{T(b)};\alpha}$. The next theorem is on closure property under the maxima of $\mathscr{F}_{X_{T(b)};\alpha}$.

Theorem 3.4.1. *Let the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ be closed under the maxima. Then the right truncated family, $\mathscr{F}_{X_{T(b)};\alpha}$, is also closed under the maxima.*

Proof. Since $\mathscr{F}_{X;\boldsymbol{\alpha}}$ closed under minima, we have $F_{M_n;\underset{\sim}{\alpha}_n}(x) = F_{X;h(\underset{\sim}{\alpha}_n)}$. Then

$$
\begin{aligned}
F_{M_{n_{T(b)}};\underset{\sim}{\alpha}_n}(x) &= \prod_{j=1}^{n} F_{X_{T(b)};\alpha_j}(x) \\
&= \prod_{j=1}^{n} \frac{F_{X;\alpha_j}(x)}{F_{X;\alpha_j}(b)} \\
&= \frac{F_{X;h(\underset{\sim}{\alpha}_n)}(x)}{F_{X;h(\underset{\sim}{\alpha}_n)}(b)} \\
&= F_{X_{T(b)};h(\underset{\sim}{\alpha}_n)}(x).
\end{aligned}
$$

Hence the proof. □

We have seen that closure property under the maxima is invariant under the truncation at the right side of the support of the family of distributions. Let us denote the family of distributions truncated at a on the left side of the support of r.v.s having d.f.s belonging to $\mathscr{F}_{X;\boldsymbol{\alpha}}$, a family of distributions closed under extrema, by $\mathscr{F}_{X_{T(a)};\alpha}$. The next theorem is on closure property under the minima of $\mathscr{F}_{X_{T(a)};\alpha}$.

Theorem 3.4.2. *Let the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ be closed under the minima. Then the left truncated family $\mathscr{F}_{X_{T(a)};\alpha}$ is also closed under the minima.*

Proof. Since $\mathscr{F}_{X;\boldsymbol{\alpha}}$ closed under minima, we have $\bar{F}_{m_n;\underset{\sim}{\alpha}_n}(x) = F_{X;g(\underset{\sim}{\alpha}_n)}$. Then in the same line of the proof of Theorem 3.4.1, we get $\bar{F}_{{m_n}_{T(b)};\underset{\sim}{\alpha}_n}(x) = \bar{F}_{X_{T(a)};g(\underset{\sim}{\alpha}_n)}(x)$. Hence, $\mathscr{F}_{X_{T(a)};\alpha}$ is closed under the minima. □

The theorem says that closure under the minima is invariant under the left truncation of r.v.s closed under the minima.

Example 3.4.1. *Let X_j, $j = 1, 2, \ldots, n$ be distributed as in Example 3.2.1. Then the s.f. of m_n is given by $\bar{F}_{m_n;\underset{\sim}{\theta}_n}(x) = e^{-\sum_{j=1}^{n}\theta_j x}$. The s.f. of X_j left truncated at 'a' is given by $\bar{F}_{X_{T(a)};\theta_j} = e^{-\theta_j(x-a)}$, $j = 1, 2, \ldots, n$. Hence the s.f. corresponding to the minimum of ${X_j}_{T(a)}$, $j = 1, 2, \ldots, n$ is given by $\bar{F}_{{m_n}_{T(b)};\underset{\sim}{\theta}_n}(x) = e^{-\sum_{j=1}^{n}\theta_j(x-a)} = F_{X_{T(a)};\sum_{j=1}^{n}\theta_j}(x)$. Therefore, the left truncated family of exponential distributions is also closed under minima.*

In this chapter, we introduced the concepts of a family of distributions or corresponding set of r.v.s being closed under extrema. Sufficient conditions for families of distributions to have closure under extrema are discussed. After that we discussed how closure property under extrema changes under strictly monotone transformations and truncations, of the r.v.s closed under extrema.

CHAPTER

FOUR

Characteristic Functions of Extrema

In Example 2.3.2, we have seen that partial minima of a sequence of independent exponential r.v.s are also exponential and hence we can express the c.f.s of these partial minima in terms of the c.f. of the underlying distribution, which is exponential. In Chapter 3, we saw that there exist a class of distributions having the property, closure under extrema. Hence, similar result hold for all other members of this class and is stated as a theorem later. What about the c.f.s of partial maxima in Example 2.3.2? In this Chapter, we raise the question; whether one can represent the c.f.s of partial maxima of a sequence of independent r.v.s in terms of the c.f.s of the underlying distribution, closed under the minima? Similarly, is it possible to represent the c.f.s of partial minima of a sequence of independent r.v.s in terms of the c.f.s of the underlying distribution, closed under the maxima? These problems are addressed in this chapter. The results corresponding to a sequence of independent non-identical r.v.s are based on Aparna and Chandran (2018a) and that corresponding to a sequence of i.i.d. r.v.s are based on Aparna and Chandran (2017).

The chapter is organized as follows: Section 4.1 obtains the representations for the d.f.s of partial maxima in terms of the d.f.s of partial minima for a sequence of independent r.v.s. From these representations, when the sequence of r.v.s is closed under the minima, one can represent the d.f.s of partial maxima in terms of the d.f. of the underlying distribution. Similarly, when the sequence of r.v.s is closed under the maxima, one can represent the d.f.s of partial minima in terms of the d.f. of the underlying distribution. The results corresponding to i.i.d. sequence of r.v.s are deduced as particular cases. By the one to one correspondence between the d.f.s and their integral transforms, one can obtain similar representations for the c.f. and other integral transforms, whenever they exist and is discussed in Section 4.2. Section 4.3 provides similar representations for the moments of partial extrema of r.v.s closed under extrema. An application of this result is also discussed.

4.1 Distributions of Extrema

In this section we derive the relationships between the d.f.s of partial maxima and partial minima, of a sequence of independent r.v.s. The d.f.s of partial maxima can be represented in terms of the d.f.s of partial minima and the d.f.s of partial minima can be represented in terms of the d.f.s of partial maxima. This has special significance when the sequence of r.v.s is closed under extrema. The p.d.f. and p.m.f. also have similar representations, whenever they exist. Let us first introduce some notations.

Let $\{X_n\}$ be a sequence of independent non-identically distributed r.v.s such that $X_j \sim F_{X_j}$. Let $\mathcal{D}_k$, $1 \leq k \leq n$, represents the set of all subsets of $\{1, 2, \ldots, n\}$ having cardinality k. We denote an element of $\mathcal{D}_k$ by $d_k = (j_1, j_2, \ldots, j_k)$. Let $m_{d_k} = \min(X_{j_1}, X_{j_2}, \ldots, X_{j_k})$ and

$M_{d_k} = \max(X_{j_1}, X_{j_2}, \ldots, X_{j_k})$ (i.e., the minimum and maximum of r.v.s corresponding to d_k) with corresponding d.f.s $F_{m_{d_k}}(x)$ and $F_{M_{d_k}}(x)$. We denote the collection of parameters $(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_k})$ of $(X_{j_1}, X_{j_2}, \ldots, X_{j_k})$ corresponding to an d_k by $\underset{\sim}{\alpha}_{d_k}$. i.e., $\underset{\sim}{\alpha}_{d_k} = (\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_k})$. Note that $\underset{\sim}{\alpha}_n = \underset{\sim}{\alpha}_{d_n}$.

Lemma 4.1.1. *Let $\{X_n\}$ be a sequence of independent r.v.s and $X_j \sim F_{X_j}$. Then the d.f.s of partial maxima, M_n and partial minima, m_n have the representations given by*

$$F_{M_n}(x) = \sum_{k=1}^{n}(-1)^{k-1} \sum_{d_k \in \mathcal{D}_k} F_{m_{d_k}}(x) \qquad (4.1.1)$$

and

$$F_{m_n}(x) = \sum_{k=1}^{n}(-1)^{k-1} \sum_{d_k \in \mathcal{D}_k} F_{M_{d_k}}(x). \qquad (4.1.2)$$

Proof. From (2.3.3), we have the d.f. of M_n given by

$$\begin{aligned}
F_{M_n}(x) &= \prod_{j=1}^{n} F_{X_j}(x) \\
&= \prod_{j=1}^{n}(1 - \bar{F}_{X_j}(x)) \\
&= 1 - \sum_{1 \le j_1 \le n} \bar{F}_{X_{j_1}}(x) + \sum_{1 \le j_1 < j_2 \le n} \bar{F}_{X_{j_1}}(x)\bar{F}_{X_{j_2}}(x) - \ldots \\
&\quad + (-1)^k \sum_{1 \le j_1 < \ldots < j_k \le n} \bar{F}_{X_{j_1}}(x) \ldots \bar{F}_{X_{j_k}}(x) + \ldots \\
&\quad + (-1)^n \sum_{1 \le j_1 < \ldots < j_n \le n} \bar{F}_{X_{j_1}}(x) \ldots \bar{F}_{X_{j_n}}(x)
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \binom{n}{1} + \ldots + (-1)^{(k)}\binom{n}{k} + \ldots + (-1)^{(n)}\binom{n}{n} \\
&\quad + \sum_{1 \le j_1 \le n} (1 - \bar{F}_{X_{j_1}}(x)) - \sum_{1 \le j_1 < j_2 \le n} (1 - \bar{F}_{X_{j_1}}(x)\bar{F}_{X_{j_2}}(x)) + \ldots \\
&\quad +(-1)^{k-1} \sum_{1 \le j_1 < \ldots < j_k \le n} (1 - \bar{F}_{X_{j_1}}(x) \ldots \bar{F}_{X_{j_k}}(x)) + \ldots \\
&\quad +(-1)^{n-1} \sum_{1 \le j_1 < \ldots < j_n \le n} (1 - \bar{F}_{X_{j_1}}(x) \ldots \bar{F}_{X_{j_n}}(x)) \\
&= \sum_{1 \le j_1 \le n} (1 - \bar{F}_{X_{j_1}}(x)) - \sum_{1 \le j_1 < j_2 \le n} (1 - \bar{F}_{X_{j_1}}(x)\bar{F}_{X_{j_2}}(x)) + \ldots \\
&\quad +(-1)^{k-1} \sum_{1 \le j_1 < \ldots < j_k \le n} (1 - \bar{F}_{X_{j_1}}(x) \ldots \bar{F}_{X_{j_k}}(x)) + \ldots \\
&\quad +(-1)^{n-1} \sum_{1 \le j_1 < \ldots < j_n \le n} (1 - \bar{F}_{X_{j_1}}(x) \ldots \bar{F}_{X_{j_n}}(x)) \\
&= \sum_{1 \le j_1 \le n} (1 - (1 - F_{X_{j_1}}(x))) - \sum_{1 \le j_1 < j_2 \le n} (1 - \prod_{l=1}^{2}(1 - F_{X_{j_l}}(x))) + \ldots \\
&\quad +(-1)^{k-1} \sum_{1 \le j_1 < \ldots < j_k \le n} (1 - \prod_{l=1}^{k}(1 - F_{X_{j_l}}(x))) + \ldots \\
&\quad +(-1)^{n-1} \sum_{1 \le j_1 < \ldots < j_n \le n} (1 - \prod_{l=1}^{n}(1 - F_{X_{j_l}}(x))) \\
&= \sum_{d_1 \in \mathcal{D}_1} F_{m_{d_1}}(x) - \sum_{d_2 \in \mathcal{D}_2} F_{m_{d_2}}(x) + \ldots + (-1)^{k-1} \sum_{d_k \in \mathcal{D}_k} F_{m_{d_k}}(x) \\
&\quad + \ldots + (-1)^{n-1} \sum_{d_n \in \mathcal{D}_n} F_{m_{d_n}}(x) \\
&= \sum_{k=1}^{n} (-1)^{k-1} \sum_{d_k \in \mathcal{D}_k} F_{m_{d_k}}(x).
\end{aligned}
$$

Similarly, from (2.3.4), we have the d.f. of m_n as:

$$
F_{m_n}(x) = 1 - \prod_{j=1}^{n}(1 - F_{X_j}(x)).
$$

On expanding and simplifying, we get

$$F_{m_n}(x) \;=\; \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k}F_{M_{d_k}}(x).$$

Hence the proof. □

When the sequence of r.v.s are i.i.d., we have $F_{X_j}(x) = F_X(x), \quad \forall \quad j.$ Hence, for all $d_k \in \mathcal{D}$

$$F_{m_{d_k}}(x) = F_{m_k}(x).$$

and

$$F_{M_{d_k}}(x) = F_{M_k}(x).$$

Number of elements of $\mathcal{D}_k = \binom{n}{k}$, which is the number of subsets of $\{1, 2, \ldots, n\}$ having cardinality k. Therefore,

$$\sum_{d_k\in\mathcal{D}_k}F_{m_{d_k}}(x) = \binom{n}{k}F_{m_k}(x).$$

Hence, from (4.1.1), we have

$$F_{M_n}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}F_{m_k}(x). \tag{4.1.3}$$

Similarly,

$$\sum_{d_k\in\mathcal{D}_k}F_{M_{d_k}}(x) = \binom{n}{k}F_{M_k}(x).$$

Hence, from (4.1.2), we have

$$F_{m_n}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}F_{M_k}(x). \tag{4.1.4}$$

When the r.v.s are absolutely continuous with respect to the Lebesgue measure, we have the next corollary.

Corollary 4.1.1. *Let* $\{X_n\}$ *be a sequence of independent r.v.s with* $X_j \sim F_{X_j}$ *having p.d.f.* f_{X_j}*. Then the p.d.f.s of partial maxima,* M_n *and partial minima,* m_n *are given by*

$$f_{M_n}(x) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k \in \mathcal{D}_k} f_{m_{d_k}}(x) \tag{4.1.5}$$

and

$$f_{m_n}(x) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k \in \mathcal{D}_k} f_{M_{d_k}}(x). \tag{4.1.6}$$

Proof. Since the p.d.f. exists, the proof follows from Lemma 4.1.1. □

When $\{X_n\}$ is a sequence of i.i.d. r.v.s, the expression (4.1.5) reduces to

$$f_{M_n}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k} f_{m_k}(x)$$

and (4.1.6) reduces to

$$f_{m_n}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k} f_{M_k}(x).$$

When the r.v.s are absolutely continuous with respect to the counting measure, the next corollary follows.

Corollary 4.1.2. *Let* $\{X_n\}$ *be a sequence of independent r.v.s with* $X_j \sim F_{X_j}$ *having p.m.f.* p_{X_j}*. Then the p.m.f.s of partial maxima,* M_n *and partial minima,*

m_n are given by

$$p_{M_n}(x) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} p_{m_{d_k}}(x) \tag{4.1.7}$$

and

$$p_{m_n}(x) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} p_{M_{d_k}}(x). \tag{4.1.8}$$

Proof. Since the p.m.f. exists, the proof follows from Lemma 4.1.1. □

When $\{X_n\}$ is a sequence of i.i.d. r.v.s, the expression (4.1.7) reduces to

$$p_{M_n}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k} p_{m_k}(x)$$

and (4.1.8) reduces to

$$p_{m_n}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k} p_{M_k}(x).$$

Now, if $\{X_n\}$ is a sequence of independent r.v.s closed under extrema, we have the following theorem.

Theorem 4.1.1. *Let $\{X_n\}$ be a sequence of independent r.v.s having d.f. belonging to $\mathscr{F}_{X;\boldsymbol{\alpha}}$. If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima with respect to $\boldsymbol{\alpha}$, then*

$$F_{M_n;\underset{\sim}{\alpha}_n}(x) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} F_{X;g(\underset{\sim}{\alpha}_{d_k})}(x) \tag{4.1.9}$$

and if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to $\boldsymbol{\alpha}$, then

$$F_{m_n;\underset{\sim}{\alpha}_n}(x) \;=\; \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} F_{X;h(\underset{\sim}{\alpha}_{d_k})}(x). \tag{4.1.10}$$

Proof. Suppose $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima with respect to $\boldsymbol{\alpha}$. Then

$$F_{m_{d_k}}(x) = F_{X;g(\underset{\sim}{\alpha}_{d_k})}(x) \quad \forall \quad d_k \in \mathcal{D}_k.$$

Hence, we have the representation given in equation (4.1.9). Similarly, if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to $\boldsymbol{\alpha}$, then

$$F_{M_{d_k}}(x) = F_{X;h(\underset{\sim}{\alpha}_{d_k})}(x) \quad \forall \quad d_k \in \mathcal{D}_k.$$

Hence, the representation given in equation (4.1.10) follows. □

If $\{X_n\}$ is a sequence of i.i.d. r.v.s whose d.f. belongs to a family of distributions having closure property under the minima, then (4.1.9) becomes

$$F_{M_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}F_{X;g_k(\alpha)}(x) \tag{4.1.11}$$

and if the family is closed under the maxima, then (4.1.10) becomes

$$F_{m_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}F_{X;h_k(\alpha)}(x). \tag{4.1.12}$$

If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is a family of absolutely continuous distributions closed under extrema, then we have the following corollary.

Corollary 4.1.3. *Let $\{X_n\}$ be a sequence of independent r.v.s having d.f. belonging to $\mathscr{F}_{X;\boldsymbol{\alpha}}$, with X_j having p.d.f. $f_{X;\alpha_j}$. If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima with respect to $\boldsymbol{\alpha}$, then*

$$f_{M_n;\underset{\sim}{\alpha}_n}(x) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} f_{X;g(\underset{\sim}{\alpha}_{d_k})}(x) \tag{4.1.13}$$

and if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to $\boldsymbol{\alpha}$, then

$$f_{m_n;\underset{\sim}{\alpha}_n}(x) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} f_{X;h(\underset{\sim}{\alpha}_{d_k})}(x). \tag{4.1.14}$$

Proof. Since the p.d.f. exists, the proof follows from Theorem 4.1.1. □

Suppose $\{X_n\}$ is a sequence of i.i.d. continuous r.v.s having d.f. belonging to $\mathscr{F}_{X;\boldsymbol{\alpha}}$. Then, if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under minima, (4.1.13) reduces to

$$f_{M_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k} f_{X;g_k(\alpha)}(x)$$

and if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under maxima, from (4.1.14), we have

$$f_{m_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k} f_{X;h_k(\alpha)}(x).$$

When $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is a family of discrete distributions closed under extrema, we have the next corollary.

Corollary 4.1.4. *Let $\{X_n\}$ be a sequence of independent r.v.s having d.f. belonging to $\mathscr{F}_{X;\boldsymbol{\alpha}}$, with X_j having p.m.f. $p_{X;\alpha_j}$. If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the* *minima with respect to* $\boldsymbol{\alpha}$*, then*

$$p_{M_n;\underset{\sim}{\alpha}_n}(x) \;=\; \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} p_{X;g(\underset{\sim}{\alpha}_{d_k})}(x) \tag{4.1.15}$$

and if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *is closed under the maxima with respect to* $\boldsymbol{\alpha}$*, then*

$$p_{m_n;\underset{\sim}{\alpha}_n}(x) \;=\; \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} p_{X;h(\underset{\sim}{\alpha}_{d_k})}(x). \tag{4.1.16}$$

Proof. Since the p.m.f. exists, the proof follows from Theorem 4.1.1. □

Suppose $\{X_n\}$ is a sequence of i.i.d. discrete r.v.s having d.f. belonging to $\mathscr{F}_{X;\boldsymbol{\alpha}}$. Then, if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima, (4.1.15) reduces to

$$p_{M_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k} p_{X;g_k(\alpha)}(x)$$

and if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima, from (4.1.16), we have

$$p_{m_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k} p_{X;h_k(\alpha)}(x).$$

Now, we have derived the expressions for the d.f.s of partial maxima in terms of the d.f. of the underlying distribution, closed under the minima and the d.f.s of partial minima in terms of the d.f. of the underlying distribution, closed under the maxima. In Chapter 2, we have discussed the one to one correspondence between the d.f. of a r.v. and its c.f. Due to this correspondence, we have similar representations for the c.f.s. of extrema and are discussed in the next section.

4.2 Characteristic Functions of Extrema

We have seen from equations (2.3.15) and (2.3.16) that, in general, the c.f.s of partial maxima and partial minima do not have a compact form in terms of the c.f.s of the underlying distribution. However, in Example 2.3.2, we have seen that the c.f.s of partial minima of a sequence of independent exponential r.v.s, with X_j, having parameter θ_j, $j = 1, 2, \ldots, n$, is same as the c.f. of an exponential r.v. with parameter $\sum_{j=1}^{n} \theta_j$, which is in compact form. In this section, we derive the c.f.s of partial minima in terms of the c.f. of the underlying distribution, when the r.v.s are closed under the maxima and the c.f.s of partial maxima in terms of the c.f. of the underlying distribution, when the r.v.s are closed under the minima. If $\{X_n\}$ is a sequence of independent r.v.s closed under extrema, we have the following theorem for the c.f.s of partial extrema.

Theorem 4.2.1. *Let $\{X_n\}$ be a sequence of independent r.v.s with $X_j \sim F_{X_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$ and let the c.f. of X_j be $\phi_{X;\alpha_j}$. If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima with respect to $\boldsymbol{\alpha}$, then*

$$\phi_{m_n;\underset{\sim}{\alpha}_n}(t) = \phi_{X;g(\underset{\sim}{\alpha}_n)}(t) \tag{4.2.1}$$

and

$$\phi_{M_n;\underset{\sim}{\alpha}_n}(t) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k}\phi_{X;g(\underset{\sim}{\alpha}_{d_k})}(x), \tag{4.2.2}$$

If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to $\boldsymbol{\alpha}$, then

$$\phi_{M_n;\underset{\sim}{\alpha}_n}(t) = \phi_{X;h(\underset{\sim}{\alpha}_n)}(t) \tag{4.2.3}$$

and

$$\phi_{m_n;\underset{\sim}{\alpha}_n}(t) \quad = \quad \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k}\phi_{X;h(\underset{\sim}{\alpha}_{d_k})}(x). \tag{4.2.4}$$

Proof. Suppose $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima. Then

$$F_{m_n;\underset{\sim}{\alpha}_n}(X) = F_{X;g(\underset{\sim}{\alpha}_n)}(x).$$

Hence, by the one to one correspondence between the d.f. and the c.f. of a r.v., we have (4.2.1). Now, from Theorem 4.1.1 and the one to one correspondence between the d.f. and the c.f. of a r.v., we have(4.2.2). Similarly, if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima similar arguments give (4.2.3) and (4.2.4). □

Example 4.2.1. *In Example 3.2.1,* $\phi_{X;\theta_j}(t) = \left(1 - \frac{it}{\theta_j}\right)^{-1}$, *then*

$$\phi_{m_n;\underset{\sim}{\theta}}(t) = \left(1 - \frac{it}{\sum_{j=1}^{n}\theta_j}\right)^{-1}$$

and

$$\phi_{M_n;\underset{\sim}{\theta}}(t) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k}\left(1 - \frac{it}{\sum_{l=1}^{k}\theta_{j_l}}\right)^{-1}.$$

Example 4.2.2. *If* X_j *are distributed as in Example 3.2.5, then* $\phi_{X;\alpha_j}(t) = \alpha_j\sum_{s=0}^{\infty}\frac{(it\theta)^s}{s!(\alpha_j+s)}$. *Hence, the c.f.s of partial maxima is given by*

$$\phi_{M_n;\underset{\sim}{\alpha}_n}(t) = \sum_{j=1}^{n}\alpha_j\sum_{s=0}^{\infty}\frac{(it\theta)^s}{s!(\sum_{j=1}^{n}\alpha_j + s)}$$

and that of partial minima by

$$\phi_{m_n;\underline{\alpha}_n}(t) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k}\sum_{l=1}^{k}\alpha_{j_l}\sum_{s=0}^{\infty}\frac{(it\theta)^s}{s!(\sum_{l=1}^{k}\alpha_{j_l}+s)}.$$

If $\{X_n\}$ is a sequence of i.i.d. r.v.s whose d.f. belongs to a family of distributions closed under minima, (4.2.1) becomes

$$\phi_{m_n;\alpha}(t) = \phi_{X;g_n(\alpha)}(t) \tag{4.2.5}$$

and (4.2.2) becomes

$$\phi_{M_n;\alpha}(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\phi_{X;g_k(\alpha)}(t). \tag{4.2.6}$$

Example 4.2.3. *In Example 4.2.1 if the r.v.s are i.i.d. with* $F_{X;\theta}(x) = 1-e^{-\theta x}$, *then* $\phi_{X;\theta}(t) = \frac{\theta}{\theta-it}$. *Hence, the c.f.s of minima,*

$$\phi_{m_n;\theta}(t) = \frac{n\theta}{n\theta-it}$$

and the c.f.s of maxima is

$$\phi_{M_n;\theta}(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{k\theta}{k\theta-it}.$$

Example 4.2.4. *Suppose X has Weibull distribution with d.f.* $F_{X;\theta}(x) = 1 - e^{-(\frac{x}{\theta})^{\alpha}}$, $x > 0,\ \theta > 0,\ \alpha > 0$ *and c.f.* $\phi_{X;\theta}(t) = \sum_{s=0}^{\infty}\frac{(it\theta)^s}{s!}\Gamma(1+\frac{s}{\alpha})$. *Then the c.f.s of* m_n *is*

$$\phi_{m_n;\theta}(t) = \sum_{s=0}^{\infty}\frac{(it\theta)^s}{n^{s/\alpha}s!}\Gamma(1+\frac{s}{\alpha})$$

and hence, that of M_n is

$$\phi_{M_n;\theta}(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{s=0}^{\infty}\frac{(it\theta)^s}{k^{s/\alpha}s!}\Gamma(1+\frac{s}{\alpha}).$$

If $\{X_n\}$ is a sequence of i.i.d. r.v.s whose d.f. belongs to a family of distributions closed under maxima, (4.2.4) becomes

$$\phi_{M_n;\alpha}(t) = \phi_{X;h_n(\alpha)}(t) \tag{4.2.7}$$

and (4.2.3) becomes

$$\phi_{m_n;\alpha}(t) \;=\; \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\phi_{X;h_k(\alpha)}(t). \tag{4.2.8}$$

Example 4.2.5. *If Example 4.2.2, if the r.v.s are i.i.d. as* $F_{X;\alpha}(x) = \left(\frac{x}{\theta}\right)^{\alpha}$, $0 < x < \theta$, $\alpha > 0$, *then* $\phi_{X;\alpha}(t) = \alpha\sum_{s=0}^{\infty}\frac{(it\theta)^s}{s!(\alpha+s)}$. *Hence,*

$$\phi_{M_n;\alpha}(t) = n\alpha\sum_{s=0}^{\infty}\frac{(it\theta)^s}{s!(n\alpha+s)}$$

and

$$\phi_{m_n;\alpha}(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}k\alpha\sum_{s=0}^{\infty}\frac{(it\theta)^s}{s!(k\alpha+s)}.$$

The c.f. is a particular case of integral transform with kernal e^{itx}. The kernel of an integral transform $k(t,x)$ is a function of the variable x and also have a parameter t. The parameter can be either discrete or continuous. The integral transform is given by

$$\int_{-\infty}^{\infty} k(t,x)dF(x), \tag{4.2.9}$$

provided the integral (4.2.9) exist. The kernels which are useful in the study of d.f.s are:

i) $K(t,x) = x^t$

ii) $K(t,x) = |x|^t$

iii) $K(t,x) = x^{(t)} = x(x-1)(x-2)\dots(x-t+1), \quad \text{where} \quad x^{(0)} = 1$

iv) $K(t,x) = e^{tx}$

v) $K(t,x) = t^x$

vi) $K(t,x) = e^{itx}$ where $i = \sqrt{-1}$.

In (i), (ii) and (iii) the parameter t can take values on non-negative integers. To emphasis this discrete character we replace t by k. And hence by substituting (i), (ii) and (iii) in (4.2.9) we will obtain the k^{th} moment denoted by $E(X^k)$, k^{th} absolute moment and k^{th} factorial moment of $F(x)$ respectively. In (iv), (v) and (vi) the parameter t takes real values. By putting (iv) in (4.2.9) we get the moment generating function of $F(x)$ given by

$$M(t) = \int_{-\infty}^{\infty} e^{tx} dF(x). \tag{4.2.10}$$

The kernel (v) is used when $F(x)$ is a d.f. of a discrete r.v. and this integral transform is called the probability generating function given by

$$Q(t) = \int_{-\infty}^{\infty} t^x dF(x) = \sum_{j=0}^{\infty} t^j p_j, \tag{4.2.11}$$

where $p_j > 0$ are the probabilities that x takes the value j and $\sum_{j=0}^{\infty} p_j = 1$. On substituting the kernel (vi) in (4.2.9) we get the c.f. For details see Lukacs (1960).

As in the case of the c.f., there is a one to one correspondence between a d.f. and all other integral transforms of that d.f. But unlike the c.f., the other integral transforms need not exist always. If an integral transform exists, then we have the corresponding representations for that integral transform of partial maxima and partial minima of a sequence of r.v.s closed under extrema, as that of the c.f.s in Theorem 4.2.1. The next section gives the representations for the k^{th} moment, that is, the integral transform (i) of partial maxima and partial minima and its application in reliability theory.

4.3 Moments of Extrema and their Applications

In this section we obtain the moments of partial extrema (whenever they exist) of a sequence of independent r.v.s closed under extrema. An application of the result is also discussed. The s^{th} moments of partial extrema of a sequence of r.v.s closed under extrema exist, if the s^{th} moment of the underlying distribution exists. The following results hold true under the assumption that the s^{th} moments of the r.v.s corresponding to the family of distributions exist.

Theorem 4.3.1. *Let $\{X_n\}$, be a sequence of independent r.v.s whose d.f.s are from $\mathscr{F}_{X;\boldsymbol{\alpha}}$ and the s^{th} moment of the family exists. If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima with respect to $\boldsymbol{\alpha}$, then*

$$E_{m_n;\underset{\sim}{\alpha}_n}(m_n^s) = E_{X;g(\underset{\sim}{\alpha}_n)}(X^s) \tag{4.3.1}$$

and

$$E_{M_n;\underset{\sim}{\alpha}_n}(M_n^s) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} E_{X;g(\underset{\sim}{\alpha}_{d_k})}(X^s). \tag{4.3.2}$$

If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to $\boldsymbol{\alpha}$, then

$$E_{M_n;\underset{\sim}{\alpha}_n}(M_n^s) \;=\; E_{X;h(\underset{\sim}{\alpha}_n)}(X^s) \tag{4.3.3}$$

and

$$E_{m_n;\underset{\sim}{\alpha}_n}(m_n^s) \;=\; \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} E_{X;h(\underset{\sim}{\alpha}_{d_k})}(X^s). \tag{4.3.4}$$

Proof. Since the s^{th} moment exist, the proof follows on the similar lines of the proof of Theorem 4.2.1, from Theorem 4.1.1. □

Suppose $\{X_n\}$ is a sequence of i.i.d. r.v.s whose d.f.s are from $\mathscr{F}_{X;\boldsymbol{\alpha}}$ and the s^{th} moment of the family exists. Then (4.3.1) reduces to

$$E_{m_n;\alpha}(m_n^s) = E_{X;g_n(\alpha)}(X^s) \tag{4.3.5}$$

and (4.3.2) becomes

$$E_{M_n;\alpha}(M_n^s) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}E_{X;g_k(\alpha)}(X^s), \tag{4.3.6}$$

if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima. Similarly, (4.3.3) becomes

$$E_{M_n;\alpha}(M_n^s) = E_{X;h_n(\alpha)}(X^s) \tag{4.3.7}$$

and (4.3.4) becomes

$$E_{m_n;\alpha}(m_n^s) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}E_{X;h_k(\alpha)}(X^s), \tag{4.3.8}$$

when $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima.

Suppose that the expectations of the r.v.s corresponding to $\mathscr{F}_{X;\boldsymbol{\alpha}}$ exists. Then we have the following corollary.

Corollary 4.3.1. *Let $\{X_n\}$, be a sequence of independent r.v.s whose d.f.s are from $\mathscr{F}_{X;\boldsymbol{\alpha}}$ and the expectation of the r.v.s exist. If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima with respect to $\boldsymbol{\alpha}$, then*

$$E_{m_n;\underset{\sim}{\alpha}_n}(m_n) = E_{X;g(\underset{\sim}{\alpha}_n)}(X) \tag{4.3.9}$$

and

$$E_{M_n;\underset{\sim}{\alpha}_n}(M_n) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} E_{X;g(\underset{\sim}{\alpha}_{d_k})}(X). \tag{4.3.10}$$

If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to $\boldsymbol{\alpha}$, then

$$E_{M_n;\underset{\sim}{\alpha}_n}(M_n) = E_{X;h(\underset{\sim}{\alpha}_n)}(X) \tag{4.3.11}$$

and

$$E_{m_n;\underset{\sim}{\alpha}_n}(m_n) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} E_{X;h(\underset{\sim}{\alpha}_{d_k})}(X). \tag{4.3.12}$$

Proof. Since the expectation of X exist, substituting $s = 1$ in the proof of Theorem 4.3.1, the corollary follows. □

Suppose $\{X_n\}$ is sequence of i.i.d. r.v.s whose expectation exists. If the d.f.s of the r.v.s belong to a family of distributions closed under the minima, from (4.3.9) we get

$$E_{m_n;\alpha}(m_n) = E_{X;g_n(\alpha)}(X) \tag{4.3.13}$$

and from (4.3.10), we have

$$E_{M_n;\alpha}(M_n) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}E_{X;g_k(\alpha)}(X). \tag{4.3.14}$$

Similarly, if the d.f.s of the r.v.s belong to a family of distributions closed under the maxima, from (4.3.11), we have

$$E_{M_n;\alpha}(M_n) = E_{X;h_n(\alpha)}(X) \tag{4.3.15}$$

and from (4.3.12) we get

$$E_{m_n;\alpha}(m_n) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}E_{X;h_k(\alpha)}(X). \tag{4.3.16}$$

Suppose that the second moment of the r.v.s corresponding to the d.f.s of the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ exists. Then we have the following corollary for the variance of partial maxima and the partial minima of a sequence of independent r.v.s closed under extrema.

Corollary 4.3.2. *Let* $\{X_n\}$*, be a sequence of independent r.v.s whose d.f.s are from* $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *and the* 2^{nd} *moment of the family exists. If* $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *is closed under the minima with respect to* $\boldsymbol{\alpha}$*, then*

$$V_{m_n;\underset{\sim}{\alpha}_n}(m_n) = V_{X;g(\underset{\sim}{\alpha}_n)}(X) \tag{4.3.17}$$

and

$$V_{M_n;\underset{\sim}{\alpha}_n}(M_n) \;=\; \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} V_{X;g(\underset{\sim}{\alpha}_{d_k})}(X). \tag{4.3.18}$$

If the family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *is closed under the maxima with respect to* $\boldsymbol{\alpha}$*, then*

$$V_{M_n;\underset{\sim}{\alpha}_n}(M_n) \;=\; V_{X;h(\underset{\sim}{\alpha}_n)}(X) \tag{4.3.19}$$

and

$$V_{m_n;\underset{\sim}{\alpha}_n}(m_n) \;=\; \sum_{k=1}^{n}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} V_{X;h(\underset{\sim}{\alpha}_{d_k})}(X). \tag{4.3.20}$$

Proof. Since the 2^{nd} moment X exist, the 1^{st} moment also exist. Hence, the proof follows from Theorem 4.1.1. □

Suppose $\{X_n\}$ is a sequence of i.i.d. r.v.s whose second moment exists. If the d.f.s of the r.v.s belong to a family of distributions closed under the minima, from (4.3.17) we get

$$V_{m_n;\alpha}(m_n) = V_{X;g_n(\alpha)}(X) \tag{4.3.21}$$

and from (4.3.18), we have

$$V_{M_n;\alpha}(M_n) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} V_{X;g_k(\alpha)}(X). \tag{4.3.22}$$

Similarly, if the d.f.s of the r.v.s belong to a family of distributions closed under the maxima, from (4.3.19), we have

$$V_{M_n;\alpha}(M_n) = V_{X;h_n(\alpha)}(X) \tag{4.3.23}$$

and from (4.3.20) we get

$$V_{m_n;\alpha}(m_n) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}V_{X;h_k(\alpha)}(X). \tag{4.3.24}$$

The next two examples provide an illustration of the results in this section.

Example 4.3.1. *Consider a parallel system consisting of 3 components each having the Weibull distribution with shape parameter α and scale parameters θ_1, θ_2 and θ_3 respectively for components 1, 2 and 3. What will be the expected life time of the system?*
The life time of the j^{th} component, $j = 1, 2, 3$ is distributed as $F_{X;\theta_j}(x) = 1 - e^{-(\frac{x}{\theta_j})^\alpha}$. Since this is a parallel system, it functions as long as at least one of the component functions. So the life of the system is distributed as

$$F_{M_3;\underset{\sim}{\theta}}(x) = \prod_{j=1}^{3}\left(1 - e^{-(\frac{x}{\theta_j})^\alpha}\right).$$

Since the Weibull distribution is closed under the minima when the shape parameter α is fixed, we have

$$F_{m_k;\underset{\sim}{\theta}d_k}(x) = F_{X;g(\underset{\sim}{\theta}_{d_k})}(x),$$

where $d_k \in \mathcal{D}_k$, $k = 1, 2, 3$. Here, $\mathcal{D}_1 = \{1, 2, 3\}$, $\mathcal{D}_2 = \{(1, 2), (1, 3), (2, 3)\}$ and $\mathcal{D}_3 = \{(1, 2, 3)\}$. From Example 3.2.4, we have $g(\underset{\sim}{\theta}_{d_k}) = \left(\sum_{l=1}^{k}\left(\frac{1}{\theta_{j_l}}\right)^\alpha\right)^{-1/\alpha}$.

The expected value of X_j is given by

$$E_{X;\theta_j}(X) = \theta_j \Gamma\left(1 + 1/\alpha\right).$$

Hence,

$$\begin{aligned} E_{m_k;\underset{\sim}{\theta}_{d_k}}(m_k) &= E_{X;g(\underset{\sim}{\theta}_{d_k})}(X) \\ &= \frac{\Gamma\left(1+1/\alpha\right)}{\left(\sum_{l=1}^{k}\left(\frac{1}{\theta_{j_l}}\right)^{\alpha}\right)^{\frac{1}{\alpha}}}. \end{aligned}$$

Therefore,

$$\begin{aligned} E_{M_3;\underset{\sim}{\theta}}(M_3) &= \sum_{k=1}^{3}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k} E_{X;g(\underset{\sim}{\theta}_{d_k})}(X) \\ &= \sum_{k=1}^{3}(-1)^{k-1}\sum_{d_k\in\mathcal{D}_k}\frac{\Gamma(1+1/\alpha)}{\left(\sum_{l=1}^{k}\left(\frac{1}{\theta_{j_l}}\right)^{\alpha}\right)^{1/\alpha}} \\ &= \sum_{d_1\in\mathcal{D}_1}\frac{\Gamma(1+1/\alpha)}{\left(\sum_{l=1}^{1}\left(\frac{1}{\theta_{j_l}}\right)^{\alpha}\right)^{1/\alpha}} - \sum_{d_2\in\mathcal{D}_2}\frac{\Gamma(1+1/\alpha)}{\left(\sum_{l=1}^{2}\left(\frac{1}{\theta_{j_l}}\right)^{\alpha}\right)^{1/\alpha}} + \sum_{d_3\in\mathcal{D}_3}\frac{\Gamma(1+1/\alpha)}{\left(\sum_{l=1}^{3}\left(\frac{1}{\theta_{j_l}}\right)^{\alpha}\right)^{1/\alpha}} \\ &= \Gamma(1+1/\alpha)\Bigg(\theta_1+\theta_2+\theta_3-\frac{\theta_1\theta_2}{(\theta_1^{\alpha}+\theta_2^{\alpha})^{1/\alpha}} \\ &\qquad -\frac{\theta_2\theta_3}{(\theta_2^{\alpha}+\theta_3^{\alpha})^{1/\alpha}}-\frac{\theta_1\theta_3}{(\theta_1^{\alpha}+\theta_3^{\alpha})^{1/\alpha}} \\ &\qquad +\frac{\theta_1\theta_2\theta_3}{(\theta_1\theta_2)^{\alpha}+(\theta_2\theta_3)^{\alpha}+(\theta_1\theta_3)^{\alpha})^{1/\alpha}}\Bigg). \end{aligned}$$

Example 4.3.2. *Consider a parallel system consisting of 5 components each having life distribution $F_{X;\theta}(x) = 1 - e^{-\theta x}$, $x > 0$, $\theta > 0$. What will be the expected life time of the system?*

A parallel system fails if all of its components fail. So the lifetime of the system is given by $F_{M_5;\theta}(x) = P(M_5 \leq x) = (P(X \leq x))^5$. *Since exponential distribution is closed under the minima,* $F_{m_k;\theta}(x) = F_{X;g_k(\theta)}(x) = F_{X;k\theta}(x) = 1 - e^{-k\theta x}$, $k = 1, 2, \ldots, 5$ *and* $E_{m_k;\theta}(m_k) = E_{X;g_k(\theta)}(X) = \frac{1}{k\theta}$. *Therefore,*

$$\begin{aligned} E_{M_5;\theta}(M_5) &= \sum_{k=1}^{5}(-1)^{k-1}\binom{5}{k}E_{X;g_k(\theta)}(X) \\ &= \sum_{k=1}^{5}(-1)^{k-1}\binom{5}{k}\frac{1}{k\theta} \\ &= \frac{137}{60\theta}. \end{aligned}$$

In this chapter, we saw that if the r.v.s are independently distributed and are closed under the minima or maxima we can represent the distributions and integral transforms, whenever they exist, in terms of that of the underlying distribution. In particular, one can express the c.f.s of the partial minima and maxima in terms of the c.f. of the underlying distribution.

CHAPTER

FIVE

Characteristic Functions of Order Statistics

In the last chapter we saw that, the d.f.s of partial maxima can be expressed in terms of the d.f.s of underlying distribution, when the underlying family of distributions is closed under the minima. Similarly, the d.f.s of partial minima can be expressed in terms of the d.f.s of the underlying distribution, when the underlying family of distributions is closed under the maxima. Similar representations hold for the p.d.f.s and p.m.f.s according as the sequence of r.v.s are absolutely continuous with respect to the Lebesgue measure or counting measure respectively. We also saw that, due to the one to one correspondence between the d.f.s and c.f.s, the c.f.s of partial maxima can be expressed in terms of the c.f.s of the underlying distribution, when the underlying family of distributions is closed under the minima. Similarly, the c.f.s of partial minima can be expressed in terms of the c.f.s of the underlying distribution, when the underlying family of distributions is closed under the maxima. We have similar representations for all other integral transforms, whenever they exist. Another measurable function of r.v.s, $(X_1, X_2, \ldots, X_n)$, is the r^{th} order statistic, for

$r = 1, 2, \ldots, n$. When r is equal to 1, we get the minimum, m_n and when r is equal to n, we get the maximum, M_n. Suppose that, the underlying family of distributions is closed under the minima or maxima (i.e., the maxima or minima). Can one express the c.f.s of other order statistics (for $r = 2, 3, \ldots, n-1$) in terms of the c.f.s of the underlying family of distributions? This is the problem investigated in this chapter. The chapter is based on the results in Aparna and Chandran (2018b).

The organization of this chapter is as follows: Section 5.1 defines order statistics of r.v.s $X_1, X_2, \ldots, X_n$ and reviews their d.f.s. This section also describes some applications of order statistics. In Section 5.2 we obtain the representation for the d.f. of $X_{r:n}$ in terms of the d.f.s of partial maxima and the partial minima for independent sequence r.v.s. When these r.v.s are closed under the minima or maxima, the representations for the d.f. of order statistics in terms of the d.f. of the underlying family are also obtained. The results corresponding to the i.i.d. sequence of r.v.s are deduced as special cases. In Section 5.3 the representations for the c.f. of $X_{r:n}$ in terms of the c.f.s of the underlying distributions is obtained. Section 5.4 discusses some applications of the results we proved.

5.1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of r.v.s on a probability space $(\Omega, \mathscr{A}, P)$ with X_j having d.f. $F_{X_j}(x)$ and c.f. $\phi_{X_j}(t)$. Then for a fixed integer n, the order statistics of $(X_1, X_2, \ldots, X_n)$ are defined as follows:

$$\begin{aligned}
X_{n:n} &= \max(X_1, X_2, \ldots, X_n) \\
X_{n-1:n} &= \max(\{X_1, X_2, \ldots, X_n\} - \{X_{n:n}\}) \\
&\vdots \\
X_{r:n} &= \max(\{X_1, X_2, \ldots, X_n\} - \{X_{n:n}, X_{n-1:n}, \ldots, X_{r+1:n}\}) \\
&\vdots \\
X_{1:n} &= \max(\{X_1, X_2, \ldots, X_n\} - \{X_{n:n}, X_{n-1:n}, \ldots, X_{2:n}\}) = \min(X_1, X_2, \ldots, X_n).
\end{aligned}$$

The above order statistics are Borel measurable functions of $X_1, X_2, \ldots, X_n$ and hence are r.v.s on $(\Omega, \mathscr{A}, P)$. That is, if $X_1, X_2, \ldots, X_n$ are r.v.s on $(\Omega, \mathscr{A}, P)$, then $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$, the corresponding order statistics are also r.v.s on the same probability space, $(\Omega, \mathscr{A}, P)$. The r.v. $X_{r:n}$ is known as the r^{th} order statistic of $X_1, X_2, \ldots, X_n$ for $r = 1, 2, \ldots, n$. Even if the r.v.s are independently distributed, order statistics are dependent r.v.s due to the ordering relation among them. The properties and applications of these ordered r.v.s and their measurable functions are studied under the head 'order statistics' in probability theory. The minima and the maxima are particular cases of order statistics. When $r = 1$, we have $X_{1:n} = \min(X_1, X_2, \ldots, X_n) = m_n$ and when $r = n$, we have $X_{n:n} = \max(X_1, X_2, \ldots, X_n) = M_n$.

Some important Borel measurable functions of these order statistics which are quite frequently seen in the statistics literature are; the range, $W = X_{n:n} - X_{1:n}$, which provides a quick estimator of standard deviation, has tremendous applications in quality control and other areas of applied statistics. The extreme deviate from the sample mean, $X_{n:n} - \bar{X}$ is another basic tool which is used for identifying outliers. The linear functions of order statistics can be used systematically for the estimation of location and scale parameters. Order statistics are of key importance for ranking treatment means in the analysis of variance. Another field of application of order statistics is in the study of reliability theory. Suppose that a machine works on n batteries and it works as long as r batteries work or it fails when any of the $(n-r+1)$ batteries fail. Hence, this is an $r-out-of-n : G$ (G-good) system or $(n-r+1)-out-of-n : F$ (F-fail) system. Then we will be interested in knowing the distribution of order statistics $X_{r:n}$ or $X_{n-r+1:n}$.

Let $\{X_n\}$ be a sequence of i.i.d. r.v.s. Then for a fixed n, the d.f. of the r^{th} order statistic $X_{r:n}$ is given by

$$F_{X_{r:n}}(x) = \sum_{i=r}^{n} \binom{n}{i} (F_X(x))^i (1-F_X(x))^{n-i} \quad (5.1.1)$$

and corresponding s.f. is given by

$$\bar{F}_{X_{r:n}}(x) = \sum_{i=n-r+1}^{n} \binom{n}{i} (\bar{F}_X(x))^i (1-\bar{F}_X(x))^{n-i}. \quad (5.1.2)$$

When the r.v.s are independent non-identically distributed, the d.f. of $X_{r:n}$ is given by

$$F_{X_{r:n}}(x) = \sum_{i=r}^{n} \sum_{\mathcal{P}_i} \prod_{l=1}^{i} F_{X_{j_l}}(x) \prod_{l=i+1}^{n} (1-F_{X_{j_l}}(x)) \quad (5.1.3)$$

and the corresponding s.f. is given by

$$\bar{F}_{X_{r:n}}(x) = \sum_{i=n-r+1}^{n} \sum_{\mathcal{P}_i} \prod_{l=1}^{i} \bar{F}_{X_{j_l}}(x) \prod_{l=i+1}^{n} (1-\bar{F}_{X_{j_l}}(x)), \quad (5.1.4)$$

where summation $\mathcal{P}_i$ extends over all permutations $(j_1, j_2, \ldots, j_n)$, of

$(1, 2, \ldots, n)$ for which $j_1 < \ldots < j_i$ and $j_{i+1} < \ldots < j_n$. For a detailed review on order statistics, their measurable functions, d.f.s and other properties see the monograph by David (1970).

5.2 Distributions of Order Statistics

In this section, using (5.1.3) to (5.1.4) of Section 5.1, we obtain the d.f.s of the r^{th} order statistic, $X_{r:n}$, $r = 1, 2, \ldots, n$, in terms of the d.f.s of partial maxima and the partial minima, for sequence of independent r.v.s and a fixed integer n. These representations have great significance when the underlying r.v.s is closed under extrema. From these representations we deduce the representations for the d.f.s of the r^{th} order statistic in terms of the d.f. of the underlying distribution, when the underlying family of distributions is closed under extrema, as defined in Section 3.2. The representations corresponding to a sequence of i.i.d. r.v.s are given as special cases.

Let $\{X_n\}$ be a sequence of independent r.v.s such that $X_j \sim F_{X_j}$. Let $\mathcal{P}_i$ be the set of all permutations, $(j_1, j_2, \ldots, j_n)$, of $\{1, 2, \ldots, n\}$ for which $j_1 < \ldots < j_i$ and $j_{i+1} < \ldots < j_n$. Each such permutation, partition the set $\{1, 2, \ldots, n\}$ into two sets $d_i = \{j_1, j_2, \ldots, j_i\}$ and $d'_i = \{j_{i+1}, j_{i+2}, \ldots, j_n\}$. Let us denote the collection of all d_i by $\mathcal{D}_i$, which is the set of all subsets of $\{1, 2, \ldots, n\}$ having cardinality i, $1 \leq i \leq n$. Let A_{k,d'_i} be the set of all subsets of d'_i having cardinality k, $1 \leq k \leq (n-i)$. We denote an element of A_{k,d'_i} by $a_{k,d'_i} = \{j_1^*, j_2^*, \ldots, j_k^*\}$. Let $m_{d_k} = \min(X_{j_1}, X_{j_2}, \ldots, X_{j_k})$ and $M_{d_k} = \max(X_{j_1}, X_{j_2}, \ldots, X_{j_k})$ with corresponding d.f. $F_{m_{d_k}}$ and $F_{M_{d_k}}$. Let $m_{a_{k,d'_i}} = \min(X_{j_1^*}, X_{j_2^*}, \ldots, X_{j_k^*})$ and $M_{a_{k,d'_i}} = \max(X_{j_1^*}, X_{j_2^*}, \ldots, X_{j_k^*})$ with corresponding d.f. $F_{m_{a_{k,d'_i}}}$ and $F_{M_{a_{k,d'_i}}}$. Note that, for a particular d_i, the set $\{d_i \cup a_{k,d'_i}\} = \mathcal{D}_{i+k}$ and hence the product, $\sum_{d_i \in \mathcal{D}_i} F_{M_{d_i}}(x) \sum_{A_{k,d_i'}} F_{M_{a_{k,d_i'}}}(x)$, is equal to $\sum_{d_{i+k} \in \mathcal{D}_{i+k}} F_{M_{d_{i+k}}}(x)$. We denote the collection of parameters $(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_k})$ of $X_{j_1}, X_{j_2}, \ldots, X_{j_k}$ corresponding to a d_k by $\underset{\sim}{\alpha}_{d_k}$ (i.e., $\underset{\sim}{\alpha}_{d_k} = (\alpha_{j_1}, \ldots, \alpha_{j_k})$). Note that $d_n = \{1, 2, \ldots, n\}$ and $\underset{\sim}{\alpha}_{d_n} = \underset{\sim}{\alpha}_n$. With these notations, the next lemma provides the d.f. of the r^{th} order statistic in terms of the d.f.s of partial maxima.

Lemma 5.2.1. *Let $\{X_n\}$ be a sequence of independent r.v.s such that, $X_j \sim F_{X_j}(x)$. Then for fixed n and $r = 1, 2, \ldots, n$, the d.f. of the r^{th} order statistic, $X_{r:n}$, in terms of the d.f.s of the partial maxima, M_{d_k}, $k = r, (r+1), \ldots, n$, is given by*

$$F_{X_{r:n}}(x) = \sum_{k=r}^{n} (-1)^{k-r} \binom{k-1}{r-1} \sum_{d_k \in \mathcal{D}_k} F_{M_{d_k}}(x). \qquad (5.2.1)$$

Proof. We have,

$$\begin{aligned} F_{r:n}(x) &= \sum_{i=r}^{n} \sum_{\mathcal{P}_i} \prod_{l=1}^{i} F_{j_l}(x) \prod_{l=i+1}^{n} (1 - F_{j_l}(x)) \\ &= \sum_{i=r}^{n} \sum_{\mathcal{P}_i} \prod_{l=1}^{i} F_{j_l}(x) \Bigg(1 - \sum_{j_{i+1} \le j_1^* \le j_n} F_{j_1^*}(x) \\ &\qquad + \sum_{j_{i+1} \le j_1^* < j_2^* \le j_n} F_{j_1^*}(x) F_{j_2^*}(x) - \ldots \\ &\qquad + (-1)^k \sum_{j_{i+1} \le j_1^* < \ldots < j_k^* \le j_n} F_{j_1^*}(x) \ldots F_{j_k^*}(x) + \ldots \\ &\qquad + (-1)^{n-i} \sum_{j_{i+1} \le j_1^* < \ldots < j_{n-i}^* \le j_n} F_{j_1^*}(x) \ldots F_{j_{n-i}^*}(x) \Bigg) \end{aligned}$$

$$
\begin{aligned}
= \quad & \sum_{i=r}^{n} \sum_{\mathcal{P}_i} F_{M_{d_i}}(x) \left(1 - \sum_{A_{1,d'_i}} F_{M_{a_{1,d'_i}}}(x) + \sum_{A_{2,d'_i}} F_{M_{a_{2,d'_i}}}(x) - \dots \right. \\
& \qquad \left. + (-1)^k \sum_{A_{k,d'_i}} F_{M_{a_{k,d'_i}}}(x) + \dots + (-1)^{n-i} \sum_{A_{n-i,d'_i}} F_{M_{a_{n-i,d'_i}}}(x) \right) \\
= \quad & \left(\sum_{\mathcal{P}_r} F_{M_{d_r}}(x) - \sum_{\mathcal{P}_r} F_{M_{d_r}}(x) \sum_{A_{1,d'_r}} F_{M_{a_{1,d'_r}}}(x) + \dots \right. \\
& \qquad + (-1)^k \sum_{\mathcal{P}_r} F_{M_{d_r}}(x) \sum_{A_{k,d'_r}} F_{M_{a_{k,d'_r}}}(x) + \dots \\
& \qquad \left. + (-1)^{n-r} \sum_{\mathcal{P}_r} F_{M_{d_r}}(x) \sum_{A_{n-r,d'_r}} F_{M_{a_{k,d'_i}}}(x) \right) \\
& + \left(\sum_{\mathcal{P}_{r+1}} F_{M_{d_{r+1}}}(x) - \sum_{\mathcal{P}_{r+1}} F_{M_{d_{r+1}}}(x) \sum_{A_{1,d'_{r+1}}} F_{M_{a_{1,d'_{r+1}}}}(x) + \dots \right. \\
& \qquad + (-1)^k \sum_{\mathcal{P}_{r+1}} F_{M_{d_{r+1}}}(x) \sum_{A_{k,d'_{r+1}}} F_{M_{a_{k,d'_{r+1}}}}(x) + \dots \\
& \qquad \left. + (-1)^{n-r-1} \sum_{\mathcal{P}_{r+1}} F_{M_{d_{r+1}}}(x) \sum_{A_{n-r-1,d'_{r+1}}} F_{M_{a_{n-r-1,d'_{r+1}}}}(x) \right) + \dots \\
& + \left(\sum_{\mathcal{P}_{r+k}} F_{M_{d_{r+k}}}(x) - \sum_{\mathcal{P}_{r+k}} F_{M_{d_{r+k}}}(x) \sum_{A_{1,d'_{r+k}}} F_{M_{a_{1,d'_{r+k}}}}(x) + \dots \right. \\
& \qquad + (-1)^k \sum_{\mathcal{P}_{r+k}} F_{M_{d_{r+k}}}(x) \sum_{A_{k,d'_{r+k}}} F_{M_{a_{k,d'_{r+k}}}}(x) + \dots \\
& \qquad \left. + (-1)^{n-r-k} \sum_{\mathcal{P}_{r+k}} F_{M_{d_{r+k}}}(x) \sum_{A_{n-r-k,d'_{r+k}}} F_{M_{a_{n-r-k,d'_{r+k}}}}(x) \right) + \dots \\
& + \sum_{\mathcal{P}_n} F_{M_{d_n}}(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\mathcal{P}_r} F_{M_{d_r}}(x) + \left(\sum_{\mathcal{P}_{r+1}} F_{M_{d_{r+1}}}(x) - \sum_{\mathcal{P}_r} F_{M_{d_r}}(x) \sum_{A_{1,d'_r}} F_{M_{a_{1,d'_r}}}(x) + \ldots \right) \\
&\quad + \left(\sum_{\mathcal{P}_{r+k}} F_{M_{d_{r+k}}}(x) - \sum_{\mathcal{P}_{r+k-1}} F_{M_{d_{r+k-1}}}(x) \sum_{A_{1,d'_{r+k-1}}} F_{M_{a_{1,d'_{r+k-1}}}}(x) + \ldots \right. \\
&\qquad \left. +(-1)^k \sum_{\mathcal{P}_r} F_{M_{d_r}}(x) \sum_{A_{k,d'_r}} F_{M_{a_{k,d'_r}}}(x) \right) + \ldots \\
&\quad + \left(\sum_{\mathcal{P}_n} F_{M_{d_n}}(x) - \sum_{\mathcal{P}_{n-1}} F_{M_{j_{n-1}}}(x) \sum_{A_{1,d'_{n-1}}} F_{M_{a_{1,d'_{n-1}}}}(x) + \ldots \right. \\
&\qquad \left. +(-1)^{n-r} \sum_{\mathcal{P}_r} F_{M_{d_r}}(x) \sum_{A_{n-r,d'_r}} F_{M_{a_{n-r,d'_r}}}(x) \right) \\
&= \sum_{d_r \in \mathcal{D}_r} F_{M_{d_r}}(x) + \sum_{d_{r+1} \in \mathcal{D}_{r+1}} F_{M_{d_{r+1}}}(x) \left(1 - \binom{r+1}{1} \right) + \ldots \\
&\quad + \sum_{d_{r+k} \in \mathcal{D}_{r+k}} F_{M_{d_{r+k}}}(x) \left(1 - \binom{r+k}{1} + \binom{r+k}{2} - \ldots (-1)^k \binom{r+k}{k} \right) + \ldots \\
&\quad + \sum_{\mathcal{P}_n} F_{M_{d_n}}(x) \left(1 - \binom{n}{1} + \binom{n}{2} - \ldots (-1)^{n-r} \binom{n}{n-r} \right) \\
&= \sum_{d_r \in \mathcal{D}_r} F_{M_{d_r}}(x) - \sum_{d_{r+1} \in \mathcal{D}_{r+1}} \binom{r}{1} F_{M_{d_{r+1}}}(x) + \sum_{d_{r+2} \in \mathcal{D}_{r+2}} \binom{r+1}{2} F_{M_{d_{r+1}}}(x) + \ldots \\
&\quad +(-1)^k \sum_{d_{r+k} \in \mathcal{D}_{r+k}} \binom{r+k-1}{k} F_{M_{d_{r+k}}}(x) + \ldots \\
&\quad +(-1)^{n-r} \sum_{d_n \in \mathcal{D}_n} \binom{n-1}{n-r} F_{M_{d_n}}(x) \\
&= \sum_{k=0}^{n-r} (-1)^k \sum_{d_{r+k} \in \mathcal{D}_{r+k}} \binom{r+k-1}{k} F_{M_{d_{r+k}}}(x) \\
&= \sum_{k=0}^{n-r} (-1)^k \binom{r+k-1}{r-1} \sum_{d_{r+k} \in \mathcal{D}_{r+k}} F_{M_{d_{r+k}}}(x)
\end{aligned}
$$

$$= \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\sum_{d_k\in\mathcal{D}_k} F_{M_{d_k}}(x).$$

Hence the proof is complete. □

When $\{X_n\}$ is a sequence of independent r.v.s whose d.f.s are absolutely continuous, the following corollary holds.

Corollary 5.2.1. *Let $\{X_n\}$ be a sequence of independent r.v.s and $X_j \sim F_{X_j}$ with corresponding p.d.f. $f_{X_j}(x)$. Then for fixed n and $r = 1, 2, \ldots, n$, the p.d.f. of the r^{th} order statistic, $X_{r:n}$, in terms of the p.d.f.s partial maxima M_{d_k}, $k = r, (r+1), \ldots, n$, is given by*

$$f_{X_{r:n}}(x) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\sum_{d_k\in\mathcal{D}_k} f_{M_{d_k}}(x). \tag{5.2.2}$$

Proof. Since the p.d.f.s of the r.v.s exist, the proof follows from Lemma 5.2.1. □

When $\{X_n\}$ is a sequence of independent r.v.s whose d.f.s are absolutely continuous with respect to the counting measure, we have the next corollary.

Corollary 5.2.2. *Let $\{X_n\}$ be a sequence of independent r.v.s and $X_j \sim F_{X_j}$ with corresponding p.m.f. $p_{X;\alpha_j}(x)$. Then for fixed n and $r = 1, 2, \ldots, n$, the p.m.f. of he r^{th} order statistic, $X_{r:n}$ in terms of the partial maxima M_{d_k}, $k = r, (r+1), \ldots, n$, is given by*

$$p_{X_{r:n}}(x) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\sum_{d_k\in\mathcal{D}_k} p_{M_{d_k}}(x). \tag{5.2.3}$$

Proof. Since the p.m.f.s of the r.v.s exist, the proof follows from Lemma 5.2.1. □

If $\{X_n\}$ is a sequence of i.i.d. r.v.s, then (5.2.1) reduces to

$$F_{X_{r:n}}(x) = \sum_{k=r}^{n} (-1)^{k-r} \binom{k-1}{r-1} \binom{n}{k} F_{M_k}(x),$$

(5.2.2) reduces to

$$f_{X_{r:n}}(x) = \sum_{k=r}^{n} (-1)^{k-r} \binom{k-1}{r-1} \binom{n}{k} f_{M_k}(x)$$

and (5.2.3) becomes

$$p_{X_{r:n}}(x) = \sum_{k=r}^{n} (-1)^{k-r} \binom{k-1}{r-1} \binom{n}{k} p_{M_k}(x).$$

Now, if $\{X_n\}$ is a sequence of independent r.v.s closed under the maxima, then the d.f. of $X_{r:n}$ have the representations given in the following theorem.

Theorem 5.2.1. *Let $\{X_n\}$ be a sequence of independent r.v.s and $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}_j}$. If $\mathscr{F}_{X;\boldsymbol{\alpha}_j}$ is closed under the maxima with respect to α, then for fixed n and $r = 1, 2, \ldots, n$,*

$$F_{X_{r:n};\underset{\sim}{\alpha}_n}(x) = \sum_{k=r}^{n} (-1)^{k-r} \binom{k-1}{r-1} \sum_{d_k \in \mathcal{D}_k} F_{X;h(\underset{\sim}{\alpha}_{d_k})}(x), \qquad (5.2.4)$$

where $\mathcal{D}_k$ is the class of all subsets of $\{1, 2, \ldots, n\}$ having cardinality k and $h(\underset{\sim}{\alpha}_{d_k})$ is as in Definition 3.2.3.

Proof. Since the d.f.s of the r.v.s belong to a family of distributions closed under the maxima, the theorem follows from Lemma 5.2.1. □

If the family of distributions, $\mathscr{F}_{X;\boldsymbol{\alpha}}$, is absolutely continuous and closed under the maxima, we have the next corollary.

Corollary 5.2.3. *Let* $\{X_n\}$ *be a sequence of independent r.v.s and* $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}_j}$. *Let* $f_{X;\alpha_j}$ *be the p.d.f. corresponding to* $F_{X;\alpha_j}$. *If* $\mathscr{F}_{X;\boldsymbol{\alpha}_j}$ *is closed under the maxima with respect to* α, *then for fixed* n *and* $r = 1, 2, \ldots, n$,

$$f_{X_{r:n};\underset{\sim}{\alpha}_n}(x) \quad = \quad \sum_{k=r}^{n} (-1)^{k-r} \binom{k-1}{r-1} \sum_{d_k \in \mathcal{D}_k} f_{X;h(\underset{\sim}{\alpha}_{d_k})}(x), \qquad (5.2.5)$$

where $\mathcal{D}_k$ *is the class of all subsets of* $\{1, 2, \ldots, n\}$ *having cardinality* k *and* $h(\underset{\sim}{\alpha}_{d_k})$ *is as in Definition 3.2.3.*

Proof. Since the p.d.f.s of the r.v.s exist, the proof follows from Theorem 5.2.1. □

If the family of distributions, $\mathscr{F}_{X;\boldsymbol{\alpha}}$, is a discrete family of distributions and is closed under the maxima, we have the following corollary.

Corollary 5.2.4. *Let* $\{X_n\}$ *be a sequence of independent r.v.s and* $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}_j}$. *Let* $p_{X;\alpha_j}$ *be the p.m.f. corresponding to* $F_{X;\alpha_j}$. *If* $\mathscr{F}_{X;\boldsymbol{\alpha}_j}$ *is closed under the maxima with respect to* α, *then for fixed* n *and* $r = 1, 2, \ldots, n$,

$$p_{X_{r:n};\underset{\sim}{\alpha}_n}(x) \quad = \quad \sum_{k=r}^{n} (-1)^{k-r} \binom{k-1}{r-1} \sum_{d_k \in \mathcal{D}_k} p_{X;h(\underset{\sim}{\alpha}_{d_k})}(x), \qquad (5.2.6)$$

where $\mathcal{D}_k$ *is the class of all subsets of* $\{1, 2, \ldots, n\}$ *having cardinality* k *and* $h(\underset{\sim}{\alpha}_{d_k})$ *is as in Definition 3.2.3.*

Proof. Since the p.m.f.s of the r.v.s exist, the proof follows from Theorem 5.2.1. □

When $\{X_n\}$ is a sequence of i.i.d. r.v.s, from expressions (5.2.4), (5.2.5) and (5.2.5) reduces to

$$F_{X_{r:n};\alpha}(x) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\binom{n}{k}F_{X;h_k(\alpha)}(x), \qquad (5.2.7)$$

$$f_{X_{r:n};\alpha}(x) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\binom{n}{k}f_{X;h_k(\alpha)}(x),$$

and

$$p_{X_{r:n};\alpha}(x) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\binom{n}{k}p_{X;h_k(\alpha)}(x),$$

respectively, where $h_k(\alpha)$ is as in Definition 3.2.4.

The following Lemma is on the representation of the d.f. of the r^{th} order statistic in terms of the d.f. of the partial minima.

Lemma 5.2.2. *Let $\{X_n\}$ be a sequence of independent r.v.s and $X_j \sim F_{X;\alpha_j}$. Then for fixed n and $r = 1, 2, \ldots, n$, the d.f. of the r^{th} order statistic, $X_{r:n}$, in terms of the d.f.s of the partial minima, m_{d_k}, $k = (n-r+1), (n-r+2), \ldots, n$, is given by*

$$F_{X_{r:n}}(x) = \sum_{k=n-r+1}^{n}(-1)^{k-(n-r+1)}\binom{k-1}{n-r}\sum_{d_k\in\mathcal{D}_k}F_{m_{d_k}}(x). \qquad (5.2.8)$$

Proof. Starting from (5.1.4), on expanding and simplifying as in the proof of Lemma 5.2.1, we have

$$\bar{F}_{X_{r:n}}(x) = \sum_{k=n-r+1}^{n}(-1)^{k-(n-r+1)}\binom{k-1}{n-r}\sum_{d_k\in\mathcal{D}_k}\bar{F}_{m_{d_k}}(x). \qquad (5.2.9)$$

Now, by replacing $\bar{F}$ by $1 - F$, we have

$$F_{X_{r:n}}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \sum_{d_k \in \mathcal{D}_k} F_{m_{d_k}}(x).$$

Hence the proof. □

If $\{X_n\}$ is a sequence of independent r.v.s whose d.f.s are absolutely continuous, then the following corollary holds.

Corollary 5.2.5. *Let $\{X_n\}$ be a sequence of independent r.v.s having p.d.f. $f_{X;\alpha_j}$. Then for fixed n and $r = 1, 2, \ldots, n$, the p.d.f. of the r^{th} order statistic, $X_{r:n}$, in terms of the p.d.f.s of the partial minima m_{d_k}, $k = (n-r+1), (n-r+2), \ldots, n$, is given by*

$$f_{X_{r:n}}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \sum_{d_k \in \mathcal{D}_k} f_{m_{d_k}}(x). \quad (5.2.10)$$

Proof. Since the p.d.f.s of the r.v.s exist, the proof follows from Lemma 5.2.2. □

When $\{X_n\}$ is a sequence of independent discrete r.v.s, the next corollary holds.

Corollary 5.2.6. *Let $\{X_n\}$ be a sequence of independent r.v.s r.v.s having p.m.f. $p_{X;\alpha_j}$. Then for fixed n and $r = 1, 2, \ldots, n$, the p.m.f. of he r^{th} order statistic, $X_{r:n}$ in terms of the p.m.f.s of the partial minima, m_{d_k}, $k = (n-r+1), (n-r+2), \ldots, n$, is given by*

$$p_{X_{r:n}}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \sum_{d_k \in \mathcal{D}_k} p_{m_{d_k}}(x). \quad (5.2.11)$$

Proof. Since the p.m.f.s of the r.v.s exist, the proof follows from Lemma 5.2.2. □

When the sequence of r.v.s are i.i.d. the expressions (5.2.8), (5.2.10) and (5.2.11) becomes

$$F_{X_{r:n}}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \binom{n}{k} F_{m_k}(x),$$

$$f_{X_{r:n}}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \binom{n}{k} f_{m_k}(x).$$

and

$$p_{X_{r:n}}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \binom{n}{k} p_{m_k}(x).$$

respectively.

Now, if the sequence, $\{X_n\}$, of independent non-identically distributed r.v.s have d.f.s belonging to a family of distributions closed under the minima, the d.f. of $X_{r:n}$ have the representations given in the following theorem.

Theorem 5.2.2. *Let $\{X_n\}$ be a sequence of independent r.v.s with $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}_j}$. If $\mathscr{F}_{X;\boldsymbol{\alpha}_j}$ is closed under the minima with respect to α, then for fixed n and $r = 1, 2, \ldots, n$,*

$$F_{X_{r:n};\underset{\sim}{\alpha}_n}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \sum_{d_k \in \mathcal{D}_k} F_{X;g(\underset{\sim}{\alpha}_{d_k})}(x), \quad (5.2.12)$$

where $\mathcal{D}_k$ is the class of all subsets of $\{1, 2, \ldots, n\}$ having cardinality k and $g(\underset{\sim}{\alpha}_{d_k})$ is as in Definition 3.2.1.

Proof. Since the r.v.s are from a family of distributions closed under the minima, the theorem follows from Lemma 5.2.2. □

If the family of distributions, $\mathscr{F}_{X;\boldsymbol{\alpha}}$, is a family of absolutely continuous d.f.s, we have the next corollary.

Corollary 5.2.7. *Let* $\{X_n\}$ *be a sequence of independent r.v.s and* $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$. *Let* $f_{X;\alpha_j}$ *be the p.d.f. corresponding to* $F_{X;\alpha_j}$. *If* $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *is closed under the minima with respect to* α, *then for fixed* n *and* $r = 1, 2, \ldots, n$,

$$f_{X_{r:n};\underset{\sim}{\alpha}_n}(x) \quad = \quad \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \sum_{d_k \in \mathcal{D}_k} f_{X;g(\underset{\sim}{\alpha}_{d_k})}(x), \quad (5.2.13)$$

where $\mathcal{D}_k$ *is the class of all subsets of* $\{1, 2, \ldots, n\}$ *having cardinality* k *and* $g(\underset{\sim}{\alpha}_{d_k})$ *is as in Definition 3.2.1.*

Proof. Since the p.d.f.s of the r.v.s exist, the proof follows from Theorem 5.2.2. □

If the family of distributions, $\mathscr{F}_{X;\boldsymbol{\alpha}}$, is a family of discrete distributions, we have the following corollary.

Corollary 5.2.8. *Let* $\{X_n\}$ *be a sequence of independent r.v.s and* $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$. *Let* $p_{X;\alpha_j}$ *be the p.m.f. corresponding to* $F_{X;\alpha_j}$. *If* $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *is closed under the minima with respect to* α, *then for fixed* n *and* $r = 1, 2, \ldots, n$,

$$p_{X_{r:n};\underset{\sim}{\alpha}_n}(x) \quad = \quad \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \sum_{d_k \in \mathcal{D}_k} p_{X;g(\underset{\sim}{\alpha}_{d_k})}(x), \quad (5.2.14)$$

where $\mathcal{D}_k$ *is the class of all subsets of* $\{1, 2, \ldots, n\}$ *having cardinality* k *and* $g(\underset{\sim}{\alpha}_{d_k})$ *is as in Definition 3.2.1.*

Proof. Since the p.m.f.s of the r.v.s exist, the proof follows from Theorem 5.2.2. □

If $\{X_n\}$ is a sequence of i.i.d. r.v.s whose d.f.s are from $\mathscr{F}_{X;\boldsymbol{\alpha}}$, then from (5.2.12), we have

$$F_{X_{r:n};\alpha}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \binom{n}{k} F_{X;g_k(\alpha)}(x), \quad (5.2.15)$$

from (5.2.13), we have

$$f_{X_{r:n};\alpha}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \binom{n}{k} f_{X;g_k(\alpha)}(x),$$

and from (5.2.14), we have

$$p_{X_{r:n};\alpha}(x) = \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \binom{n}{k} p_{X;g_k(\alpha)}(x),$$

respectively, where $g_k(\alpha)$ is as in Definition 3.2.2.

From the above discussions we conclude that, if the sequence of independent r.v.s have d.f.s belonging to a family of distributions closed under the maxima, then for every fixed n, we can represent the d.f.s of the order statistics in terms of the d.f.s of the partial maxima. Similarly, if the sequence of independent r.v.s have d.f.s belonging to a family of distributions closed under the minima, then for every fixed n, we can represent the d.f.s of the order statistics in terms of the d.f.s of the partial minima.

5.3 Characteristic Functions of Order Statistics

In this section, we obtain the c.f.s of order statistics, for fixed n, of a sequence of independent r.v.s closed under extrema.

Theorem 5.3.1. *Let $\{X_n\}$ be a sequence of independent r.v.s with $X_j \sim F_{X;\alpha_j} \in \mathscr{F}_{X;\boldsymbol{\alpha}}$. If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to α, then*

$$\phi_{X_{r:n}\underset{\sim}{\alpha}_n}(t) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\sum_{d_k\in\mathcal{D}_k}\phi_{X;h(\underset{\sim}{\alpha}_k)}(t) \tag{5.3.1}$$

and if the family is closed under the minima with respect to α. Let $\phi_{X;\alpha_j}$ be the c.f. corresponding to $F_{X;\alpha_j}$, then

$$\phi_{X_{r:n};\underset{\sim}{\alpha}_n}(t) = \sum_{k=n-r+1}^{n}(-1)^{k-(n-r+1)}\binom{k-1}{n-r}\sum_{d_k\in\mathcal{D}_k}\phi_{X;g(\underset{\sim}{\alpha}_k)}(t). \tag{5.3.2}$$

where $\mathcal{D}_k$ is the class of all subsets of $\{1,2,\dots,n\}$ having cardinality k, and $g(\underset{\sim}{\alpha}_{d_k})$ and $h(\underset{\sim}{\alpha}_{d_k})$ are as in Definitions 3.2.1 and 3.2.3 respectively.

Proof. By the one to one correspondence between the d.f. and the c.f. of a r.v., (5.3.1) follows from Theorem 5.2.1 and (5.3.2) follows from Theorem 5.2.2. □

Now, when $\{X_n\}$, is a sequence of i.i.d. r.v.s closed under the maxima, (5.3.1) reduces to

$$\phi_{X_{r:n};\alpha}(t) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\binom{n}{k}\phi_{X;h_k(\alpha)}(t) \tag{5.3.3}$$

and if the r.v.s are closed under the minima (5.3.2) reduces to

$$\phi_{X_{r:n};\alpha}(t) = \sum_{k=n-r+1}^{n}(-1)^{k-(n-r+1)}\binom{k-1}{n-r}\binom{n}{k}\phi_{X;g_k(\alpha)}(t), \tag{5.3.4}$$

respectively, where $g_k(\alpha)$ and $h_k(\alpha)$ are as in Definitions 3.2.2 and 3.2.4 respectively.

Hence, if the sequence of independent r.v.s are closed under extrema, for fixed n, we can represent the c.f.s of order statistics in terms of the c.f. of underlying family of distributions. The following example gives an illustration of the above results.

Example 5.3.1. *Let $\{X_n\}$ be a sequence of independent r.v.s and $X_j \sim F_{X;\theta_j} = 1 - e^{-\theta_j x}$. For fixed n, from Example 3.2.1, we know that the corresponding family of distributions is closed under the minima with respect to θ and $g(\underset{\sim}{\theta}) = \sum_{j=1}^{n} \theta_j$. Then $m_n \sim F_{X;\sum_{j=1}^{n}\theta_j}$. From Example 4.2.1, we have*

$$\phi_{m_n;\underset{\sim}{\theta}}(t) = \left(1 - \frac{it}{\sum_{j=1}^{n}\theta_j}\right)^{-1}.$$

Hence,

$$\begin{aligned}
\phi_{X_{r:n};\underset{\sim}{\theta}}(t) &= \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \sum_{d_k \in \mathcal{D}_k} \phi_{X;g(\underset{\sim}{\theta}_{d_k})}(t) \\
&= \sum_{k=n-r+1}^{n} (-1)^{k-(n-r+1)} \binom{k-1}{n-r} \sum_{d_k \in \mathcal{D}_k} \frac{g(\underset{\sim}{\theta}_{d_k})}{g(\underset{\sim}{\theta}_{d_k}) - it},
\end{aligned}$$

where $g(\underset{\sim}{\theta}_{d_k}) = \theta_{j_1} + \ldots + \theta_{j_k}$. Now, let us obtain the expression for fixed values of n and r.

If we fix $n = 4$ and $r = 2$, we get

$$\phi_{X_{2:4};\underset{\sim}{\theta}}(t) = \sum_{k=3}^{4}(-1)^{k-3}\binom{k-1}{2} \sum_{d_k \in \mathcal{D}_k} \phi_{X;g(\underset{\sim}{\theta}_{d_k})}(t)$$

$$
\begin{aligned}
&= \sum_{k=3}^{4}(-1)^{k-3}\binom{k-1}{2}\sum_{d_k\in\mathcal{D}_k}\frac{g(\underset{\sim}{\theta}_{d_k})}{g(\underset{\sim}{\theta}_{d_k})-it}\\
&= \sum_{d_3\in\mathcal{D}_3}\frac{g(\underset{\sim}{\theta}_{d_3})}{g(\underset{\sim}{\theta}_{d_3})-it}-3\sum_{d_4\in\mathcal{D}_4}\frac{g(\underset{\sim}{\theta}_{d_4})}{g(\underset{\sim}{\theta}_{d_4})-it}\\
&= \frac{\theta_1+\theta_2+\theta_3}{\theta_1+\theta_2+\theta_3-it}+\frac{\theta_1+\theta_2+\theta_4}{\theta_1+\theta_2+\theta_4-it}\\
&\quad+\frac{\theta_1+\theta_3+\theta_4}{\theta_1+\theta_3+\theta_4-it}+\frac{\theta_2+\theta_3+\theta_4}{\theta_2+\theta_3+\theta_4-it}\\
&\quad-3\,\frac{\theta_1+\theta_2+\theta_3+\theta_4}{\theta_1+\theta_2+\theta_3+\theta_4-it}.
\end{aligned}
$$

If we denote $\sum_{k=1}^{4}\theta_k=\theta$ *and* $\sum_{\substack{k=1\\k\neq j}}^{4}\theta_k=\theta^j$, *we have*

$$
\phi_{X_{2:4};\underset{\sim}{\theta}}(t) = \frac{\theta^4}{\theta^4-it}+\frac{\theta^3}{\theta^3-it}+\frac{\theta^2}{\theta^2-it}+\frac{\theta^1}{\theta^1-it}-3\frac{\theta}{\theta-it}. \qquad (5.3.5)
$$

In the above example if we consider X_j, $j=1,2,\ldots,n$ to be i.i.d. $\exp(\theta)$, the expression (5.3.5) reduces to

$$
4\times\frac{3\theta}{3\theta-it}-3\times\frac{4\theta}{4\theta-it}=\frac{12\theta^2}{(3\theta-it)(4\theta-it)} \qquad (5.3.6)
$$

and if they are i.i.d. exp(1), then (5.3.6) reduces to

$$
\begin{aligned}
4\times\frac{3}{3-it}-3\times\frac{4}{4-it} &= 12\times\frac{4-it-3+it}{(4-it)(3-it)}\\
&= \frac{12}{(4-it)(3-it)}.
\end{aligned}
$$

When the r.v.s are i.i.d. $\exp(\theta)$, from David (1970, page 18), we have

$$
X_{r:n}\stackrel{d}{=}\frac{1}{\theta}\left(\sum_{j=1}^{r}\frac{Z}{n-j+1}\right), \quad r=1,2,\ldots,n, \qquad (5.3.7)
$$

where $Z \sim$ exponential(1). Hence, for $r = 2$ and $n = 4$, from (5.3.7), we have

$$\begin{aligned} \phi_{X_{2:4};\theta}(t) &= \phi_{X,4\theta}(t)\phi_{X,3\theta}(t) \\ &= \frac{12\theta^2}{(3\theta - it)(4\theta - it)}. \end{aligned}$$

which is same as the expression obtained in (5.3.6).

5.4 Moments of Order Statistics and their Applications

In this section we discuss some applications of the results proved in the previous sections. In Section 4.2, we have already discussed the one to one correspondence between d.f.s and their integral transforms. The s^{th} moment of order statistics of r.v.s closed under extrema exist whenever the s^{th} moment of the underlying r.v.s exists. The following results hold under the assumption that the s^{th} moment of the underlying r.v.s exists.

Theorem 5.4.1. *Let $\{X_n\}$ be a sequence of independent r.v.s and $X_j \sim F_{X;\alpha_j}$ from a family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ and the s^{th} moment of the family exists. If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to α, then the s^{th} moment of $X_{r:n}$ is given by*

$$E_{X_{r:n};\underset{\sim}{\alpha}_n}(X^s_{r:n}) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\sum_{d_k \in \mathcal{D}_k} E_{X;h(\underset{\sim}{\alpha}_{d_k})}(X^s) \tag{5.4.1}$$

and if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima with respect to α, then

$$E_{X_{r:n};\underset{\sim}{\alpha}_n}(X^s_{r:n}) = \sum_{k=n-r+1}^{n}(-1)^{k-(n-r+1)}\binom{k-1}{n-r}\sum_{d_k \in \mathcal{D}_k} E_{X;g(\underset{\sim}{\alpha}_{d_k})}(X^s), \tag{5.4.2}$$

where $\mathcal{D}_k$ is the class of all subsets of $\{1, 2, \ldots, n\}$ having cardinality k and $g(\underset{\sim}{\alpha}_{d_k})$ and $h(\underset{\sim}{\alpha}_{d_k})$ are as in Definitions 3.2.1 and 3.2.3 respectively.

Proof. By the one to one correspondence between the d.f. and the c.f. of a r.v., (5.4.1) follows from Theorem 5.2.1 and (5.4.2) follows from Theorem 5.2.2. □

If the sequence of r.v.s are i.i.d., then from (5.4.1), we have

$$E_{X_{r:n};\alpha}(X^s_{r:n}) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\binom{n}{k}E_{X;h_k(\alpha)}(X^s) \tag{5.4.3}$$

and from (5.4.2), we have

$$E_{X_{r:n};\alpha}(X^s_{r:n}) = \sum_{k=n-r+1}^{n}(-1)^{k-(n-r+1)}\binom{k-1}{n-r}\binom{n}{k}E_{X;g_k(\alpha)}(X^s). \tag{5.4.4}$$

The following is the corollary for the expected value of the r^{th} order statistic of a sequence of independent r.v.s closed under extrema.

Corollary 5.4.1. *Let $\{X_n\}$ be a sequence of independent r.v.s and $X_j \sim F_{X;\alpha_j}$ from a family $\mathscr{F}_{X;\boldsymbol{\alpha}}$ whose expectation exist. If $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to α, then the expectation of $X_{r:n}$ is given by*

$$E_{X_{r:n};\underset{\sim}{\alpha}_n}(X_{r:n}) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\sum_{d_k\in\mathcal{D}_k}E_{X;h(\underset{\sim}{\alpha}_{d_k})}(X), \tag{5.4.5}$$

and if $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima with respect to α, then

$$E_{X_{r:n};\underset{\sim}{\alpha}_n}(X_{r:n}) = \sum_{k=n-r+1}^{n}(-1)^{k-(n-r+1)}\binom{k-1}{n-r}\sum_{d_k\in\mathcal{D}_k}E_{X;g(\underset{\sim}{\alpha}_{d_k})}(X), \tag{5.4.6}$$

where $\mathcal{D}_k$ is the class of all subsets of $\{1, 2, \ldots, n\}$ having cardinality k, and $g(\underset{\sim}{\alpha}_{d_k})$ and $h(\underset{\sim}{\alpha}_{d_k})$ are as in Definitions 3.2.1 and 3.2.3 respectively.

Proof. Since the expectation of the family of distributions exist, the proof follows from Theorem 5.4.1 by substituting $s = 1$. □

When the sequence of r.v.s are i.i.d., (5.4.5) reduces to

$$E_{X_{r:n};\alpha}(X_{r:n}) = \sum_{k=r}^{n}(-1)^{k-r}\binom{k-1}{r-1}\binom{n}{k}E_{X;h_k(\alpha)}(X) \tag{5.4.7}$$

and (5.4.6) reduces to

$$E_{X_{r:n};\alpha}(X_{r:n}) = \sum_{k=n-r+1}^{n}(-1)^{k-(n-r+1)}\binom{k-1}{n-r}\binom{n}{k}E_{X;g_k(\alpha)}(X). \tag{5.4.8}$$

The following is an example for the application of results in this section.

Example 5.4.1. *Consider a machine that works on 4 batteries whose lives are exponentially distributed with parameter θ. Suppose that the machine works as long as 2 of its components work. What will be the expected life time of the machine?*

Here the system fails when any of the 3 *batteries fails. So we want to calculate the expected value of the* 3^{rd} *order statistic. We know that for* $\exp(\theta)$

$$E_{X;\theta}(X) = \frac{1}{\theta} \quad \textit{and} \quad E_{X;g_k(\theta)}(X) = \frac{1}{k\theta}.$$

Hence, from (5.4.8), we have

$$\begin{aligned} E_{X_{3:4};\theta}(X_{(3)}) &= \sum_{k=2}^{4}(-1)^{k-2}\binom{k-1}{1}\binom{4}{k}E_{X;g_k(\alpha)}(X) \\ &= \sum_{k=2}^{4}(-1)^{k-2}(k-1)\binom{4}{k}\frac{1}{k\theta} \\ &= \frac{6}{2\theta} - \frac{8}{3\theta} + \frac{3}{4\theta} \\ &= \frac{13}{12\theta}. \end{aligned}$$

In this chapter, we derived the d.f.s of order statistics of a sequence of independent r.v.s, for fixed n, closed under extrema in terms of the d.f.s of the underlying distribution. By the one to one correspondence between d.f.s and their integral transforms like c.f., similar results hold for the corresponding c.f.s and all other integral transforms whenever they exist.

CHAPTER

SIX

Families of Bivariate Distributions Closed under Extrema

Till now all our discussions were on univariate r.v.s. In many situations, one may be interested in multivariate r.v.s or random vectors (R.V.s). This chapter extents the concept of closure property under extrema of univariate distributions to multidimensional distributions. Even though our discussions are on two-dimensional R.V.s, the results hold true for all n-dimensional R.V.s. Some properties of such classes of distributions are also discussed. The chapter is based on Aparna and Chandran (2018c).

The chapter is organized as follows: Section 6.1 introduces basics of two-dimensional R.V.s. Section 6.2 defines the concepts of componentwise maxima and componentwise minima. Section 6.3 discusses some basic concepts in copula theory. In Section 6.4 we define bivariate distributions closed under extrema. The necessary and sufficient conditions for a family of bivariate distributions to be closed under extrema are obtained and the copulas closed under extrema are defined in Section 6.4. In Section 6.5, the discussion is on how the bivariate closure property changes under strictly monotone transformations of marginal r.v.s. The changes in bivariate closure property under extrema on truncation of the marginal r.v.s are discussed in Section 6.6. Section 6.7 is on c.f.s of componentwise extrema.

6.1 Introduction

Random vectors (R.V.s) are multivariate generalisation of r.v.s. such that, each component of a R.V. is a univariate r.v. on the same probability space say, $(\Omega, \mathscr{A}, P)$. We can define a R.V. $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$ as a mapping from Ω to $\mathbb{R}_n$ such that, if $B_n \in \mathscr{B}_n$, the Borel *sigma*-field over $\mathbb{R}_n$,

$$\begin{aligned} \underset{\sim}{X}^{-1}(B_n) &= \{\omega \in \Omega : \underset{\sim}{X}(\omega) \in B_n\} \in \mathscr{A}. \\ &= \{\omega \in \Omega : (X_1(\omega), X_2(\omega), \ldots, X_n(\omega)) \in B_n\} \in \mathscr{A}. \end{aligned}$$

As in the univariate case, the R.V. $\underset{\sim}{X}$ induces a probability measure $P_{\underset{\sim}{X}}$ on the measurable space $(\mathbb{R}_n, \mathscr{B}_n)$ given by

$$\begin{aligned} P_{\underset{\sim}{X}}(B_n) &= P(\underset{\sim}{X}^{-1}(B_n)), \quad \forall \quad B_n \in \mathscr{B}_n \\ &= P(\omega \in \Omega : \underset{\sim}{X}(\omega) \in B_n). \end{aligned} \tag{6.1.1}$$

Then $(\mathbb{R}_n, \mathscr{B}_n, P_{\underset{\sim}{X}})$ is the probability space induced by the R.V. $\underset{\sim}{X}$. Note that, if $\underset{\sim}{X}$ is an n-dimensional R.V. and $g : \mathbb{R}_n \to \mathbb{R}_k$ is measurable, then $g(\underset{\sim}{X})$ is a k-dimensional R.V. (see, Laha and Rohatgi (1979), page 20). Now, let us concentrate on 2-dimensional R.V.s.

Let (X,Y) be a two-dimensional R.V. Hence, it is a mapping from Ω to $\mathbb{R}_2$ such that, if $B_2 \in \mathscr{B}_2$, the Borel *sigma*-field over $\mathbb{R}_2$,

$$(X,Y)^{-1}(B_2) \quad = \quad \{\omega \in \Omega : (X(\omega), Y(\omega)) \in B_2\} \in \mathscr{A}. \tag{6.1.2}$$

Then (6.1.1) can be written as:

$$P_{X,Y}(B_2) = P(\omega \in \Omega : (X(\omega), Y(\omega)) \in B_2), \quad B_2 \in \mathscr{B}_2, \tag{6.1.3}$$

known as the probability space induced by the R.V. (X,Y). Then the function H on $\mathbb{R}_2$ defined by

$$\begin{aligned} H_{X,Y}(x,y) &= P_{X,Y}\{(-\infty,x] \times (-\infty,y]\}, \quad \forall \quad (x,y) \in \mathbb{R}_2 \\ &= P(X^{-1}(-\infty,x], Y^{-1}(-\infty,y]), \quad \forall \quad (x,y) \in \mathbb{R}_2 \\ &= P\{\omega \in \Omega : X(\omega) \le x, Y(\omega) \le y\}, \quad \forall \quad (x,y) \in \mathbb{R}_2 \\ &= P(X \le x, Y \le y), \quad \forall \quad (x,y) \in \mathbb{R}_2 \end{aligned} \tag{6.1.4}$$

is called the joint d.f. of the R.V. (X,Y). The joint d.f. H satisfies the following conditions:

i) $\lim_{x \to -\infty} H(x,y) = H(-\infty, y) = 0 = \lim_{y \to -\infty} H(x,y) = H(x, -\infty)$.

ii) $\underset{x,y \to +\infty}{H}(x,y) = H(+\infty, +\infty) = 1$.

iii) H is right continuous in each of its arguments.

iv) For every $x_1, x_2, y_1, y_2 \in I$ such that $x_1 \le x_2$ and $y_1 \le y_2$,

$$H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \ge 0. \tag{6.1.5}$$

Conversely, any function satisfying the above conditions uniquely determines a probability measure P_H on $\mathscr{B}_2$. The conditions (i) and (iv) above implies that H is non-decreasing in each of its arguments. Unlike the s.f. of a r.v., the joint s.f. of a R.V., denoted by $\bar{H}$, is not the compliment of H. For all $(x, y) \in \mathbb{R}_2$, the joint s.f. of a R.V. (X, Y) is given by

$$\begin{aligned}\bar{H}_{X,Y}(x,y) &= P(X > x, Y > y)\\ &= 1 - P(X \leq x) - P(Y \leq y) + P(X \leq x, Y \leq y)\\ &= 1 - F_X(x) - G_Y(y) + H_{X,Y}(x,y). \qquad (6.1.6)\end{aligned}$$

For a R.V. (X, Y) on some probability space, the transformation given by

$$\phi_{(X,Y)}(t_1, t_2) = E\left(e^{i(t_1 X + t_2 Y)}\right), \quad t_1, t_2 \in \mathbb{R} \qquad (6.1.7)$$

$$= \int_{\mathbb{R}_2} e^{i(t_1 x + t_2 y)} dH_{X,Y}(x,y) \qquad (6.1.8)$$

is called the c.f. of the R.V. (X, Y).

6.2 Componentwise Minima and Componentwise Maxima

Let $\{(X_n, Y_n)\}_{n=1}^{\infty}$ be a sequence of independent pairs of r.v.s with $(X_j, Y_j) \sim H_{X_j,Y_j}(x, y)$ and joint s.f. $\bar{H}_{X_j,Y_j}(x, y)$. Let $F_{X_j}(x)$ be the marginal d.f. of X_j and $G_{Y_j}(y)$ that of Y_j. Let us denote $X_{M_n} = \max\{X_1, X_2, \ldots, X_n\}$ and $Y_{M_n} = \max\{Y_1, Y_2, \ldots, Y_n\}$. Then the sequence $\{(X_{M_n}, Y_{M_n})\}$ is called the sequence of componentwise maxima. Similarly, let $X_{m_n} = \min\{X_1, X_2, \ldots, X_n\}$ and $Y_{m_n} = \min\{Y_1, Y_2, \ldots, Y_n\}$. Then the sequence $\{(X_{m_n}, Y_{m_n})\}$ is known as the sequence of componentwise minima. From the discussions on the d.f.s and the s.f.s of partial maxima and partial minima in Section 2.3, we have the d.f. of X_{M_n} and Y_{M_n} given by

$$\begin{aligned} F_{X_{M_n}}(x) &= (F_X(x))^n \\ G_{Y_{M_n}}(y) &= (G_Y(y))^n \end{aligned}$$

and the s.f. of X_{m_n} and Y_{m_n} given by

$$\begin{aligned} \bar{F}_{X_{m_n}}(x) &= (\bar{F}_X(x))^n \\ \bar{G}_{Y_{m_n}}(y) &= (\bar{G}_Y(y))^n. \end{aligned}$$

The joint d.f. of componentwise maxima, (X_{M_n}, Y_{M_n}), $n \geq 1$ is given by

$$\begin{aligned} H_{X_{M_n}, Y_{M_n}}(x, y) &= P(X_{M_n} \leq x, Y_{M_n} \leq y) \\ &= \prod_{j=1}^{n} H_{X_j, Y_j}(x, y) \end{aligned} \quad (6.2.1)$$

and when the R.V.s are i.i.d., we have

$$H_{X_{M_n}, Y_{M_n}}(x, y) = (H_{X,Y}(x, y))^n \quad (6.2.2)$$

The joint s.f. of componentwise minima, (X_{m_n}, Y_{m_n}), $n \geq 1$ is given by

$$\begin{aligned} \bar{H}_{X_{m_n}, Y_{m_n}}(x, y) &= P(X_{m_n} > x, Y_{m_n} > y) \\ &= \prod_{j=1}^{n} \bar{H}_{X_j, Y_j}(x, y) \end{aligned} \quad (6.2.3)$$

and when the R.V.s are i.i.d., we have

$$\bar{H}_{X_{m_n},Y_{m_n}}(x,y) = (\bar{H}_{X,Y}(x,y))^n. \tag{6.2.4}$$

For a given bivariate distribution we can uniquely identify the marginal d.f.s. But the converse is not unique. That is, for a pair of univariate distributions the joint d.f. is not unique.

Example 6.2.1. *Consider the following two bivariate d.f.s with support* $x \geq 0$, $y \geq 0$ *and parameters* $\alpha > 0$, $\beta > 0$ *and* $0 \leq \theta \leq 1$.

$$H^1_{X,Y}(x,y) = (1 - e^{-\alpha x})(1 - e^{-\beta x}) \tag{6.2.5}$$

$$H^2_{X,Y}(x,y) = 1 - e^{-\alpha x} - e^{-\beta y} + e^{-(\alpha x + \beta y + \theta \alpha \beta x y)} \tag{6.2.6}$$

Both (6.2.5) and (6.2.6) have marginals as $X \sim \exp(\alpha)$ and $Y \sim \exp(\beta)$ but the joint distributions are different. Copulas are functions which connects a joint d.f. and the corresponding marginal d.f.s. i.e., copula evaluated at the margins gives the corresponding joint d.f. This is the content of Sklar's theorem (Sklar 1959). Hence, the marginal distributions along with a copula uniquely determines a joint distribution. We can also view this as a joint d.f. whose one dimensional margins are uniform (0,1). It is also referred in the literature as uniform representation and dependence function. It is a scale free measure of dependence and a starting point of construction of bivariate or multivariate distributions.

6.3 Copulas

In this section, with the help of Nelson (1999), we review the basic concepts in copula theory that we require in the subsequent sections. The statements given in the previous section about copulas are not actual definitions. The definition of copulas is as follows.

Definition 6.3.1. *A two-dimensional copula is a function C from $[0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:*

i) For every $u, v \in I$,

$$C(u,0) = 0 = C(0,v) \tag{6.3.1}$$

and

$$C(u,1) = u \quad \text{and} \quad C(1,v) = v. \tag{6.3.2}$$

ii) For every $u_1, u_2, v_1, v_2 \in I$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0. \tag{6.3.3}$$

The conditions (6.3.1), (6.3.2) and (6.3.3) above means that; C is grounded, C has marginals and C is 2-increasing respectively. The Sklar's theorem, a celebrated theorem in the literature of copula theory, sheds light on the relationship between multivariate d.f.s and their univariate marginal d.f.s. It is stated below, for the bivariate case.

Theorem 6.3.1 (Sklar's Theorem). *Let $H_{X,Y}$ be a joint d.f. with marginal d.f.s F_X and G_Y. Then there exists a copula C such that for all x, y in $\overline{\mathbb{R}}$ (the* *extended real line),*

$$H_{X,Y}(x,y) = C(F_X(x), G_Y(y)). \tag{6.3.4}$$

If F_X and G_Y are continuous d.f.s, then the copula C is unique; otherwise, C is uniquely determined on $\mathsf{Range}F \times \mathsf{Range}G$. Conversely, if C is a copula and F_X and F_Y are d.f.s, then the function $H_{X,Y}$ defined by (6.3.4) is a joint d.f. with marginal d.f.s F_X and G_Y.

Proof. See Nelson(1999). □

Survival copulas are copulas that joins the joint s.f. to their one dimensional marginal s.f.s. The joint s.f. of a bivariate R.V. (X, Y) with joint d.f. $H_{X,Y}(x,y) = C(F_X(x), G_Y(y))$ is given by

$$\bar{H}_{X,Y}(x,y) = \bar{F}_X(x) + \bar{G}_Y(y) - 1 + C(1 - \bar{F}_X(x), 1 - \bar{G}_Y(y)). \tag{6.3.5}$$

The function $\hat{C}$ is defined from $I \times I \to I$ by $\hat{C}(u,v) = u+v-1+C(1-u, 1-v)$ is a copula. Then we have

$$\bar{H}_{X,Y}(x,y) = \hat{C}(\bar{F}_X(x), \bar{G}_Y(y)) \tag{6.3.6}$$

and $\hat{C}$ is called the survival copula of (X, Y). Survival copula couples the joint s.f.s to its univariate marginal s.f.s. The following theorem gives the copula of componentwise maxima of i.i.d. bivariate R.V.s having copula C.

Theorem 6.3.2. *If C is a copula and n a positive integer, then the function C_{M_n} given by*

$$C_{M_n}(u,v) = C^n(u^{1/n}, v^{1/n}), \quad \textit{for } u, v \in I \tag{6.3.7}$$

is a copula. Furthermore, if (X_j, Y_j), $j = 1, 2, \ldots, n$ are i.i.d. pairs of r.v.s *with copula C, then C_{M_n} is the copula of X_{M_n} and Y_{M_n}.*

Proof. See Nelson(1999). □

Now, if (X_j, Y_j), $j = 1, 2, \ldots, n$ are i.i.d. pairs of r.v.s with survival copula $\hat{C}$, then

$$\hat{C}_{m_n}(u, v) = \hat{C}^n(u^{1/n}, v^{1/n}), \quad \text{for } u, v \in I \tag{6.3.8}$$

is the survival copula of X_{m_n} and Y_{m_n}.

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be i.i.d. pairs of r.v.s having a common copula C. Let C_{M_n} be the copula of componentwise maxima. From (6.3.7), we have $C_{M_n}(u, v) = C^n(u^{1/n}, v^{1/n})$. limit of the sequence, $\{C_{M_n}(u, v)\}$ leads to the notion of extreme value copula. If there exists a copula C such that

$$\underset{n\to\infty}{C^n}(u^{1/n}, v^{1/n}) = C^*(u, v), \tag{6.3.9}$$

then C^* is known as the extreme value copula or max-stable copula. Furthermore, C is said to be in the MDA of C^*. Some examples of extreme valu copulas are:

i) $C(u, v) = uv$

ii) $C(u = v) = \min(u, v)$

iii) $C(u, v) = e^{(-[(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta})}, \quad \theta \in [1, \infty)$

iv) $C(u, v) = \min(u^{1-\alpha} v, uv^{1-\beta}), \quad 0 < \alpha, \beta < 1.$

In this section, we reviewed the concept 'copula'. How this function connects a joint d.f. to its marginal d.f.s is discussed. The copulas (survival copulas) corresponding to the componentwise maxima (minima) and extreme value copulas are also discussed.

6.4 Families of Bivariate Distributions Closed under Extrema

In this section we extend the concept of 'closure under extrema' of a family of univariate distributions to a family of bivariate distributions. The necessary and sufficient conditions for R.V.s to be closed under extrema are also obtained.

Let $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ be a family of independent bivariate distributions. Let $\mathscr{F}_{X;\boldsymbol{\alpha}}$ and $\mathscr{G}_{Y;\boldsymbol{\beta}}$ be the corresponding families of marginal distributions and $\boldsymbol{\theta}$ be the dependence parameter. The following is the definition of family of bivariate distributions closed under the minima.

Definition 6.4.1. *A family of bivariate distributions, $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$, is said to be closed under the minima with respect to $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$, if for every $H_{X,Y;\alpha_j,\beta_j,\theta_j} \in \mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$, $j = 1,2,\ldots,n$, $n \geq 1$, the joint d.f.s of component-wise minima, $H_{X_{m_n},Y_{m_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n} = H_{X,Y;g(\underset{\sim}{\alpha}_n),g^*(\underset{\sim}{\beta}_n),\eta(\underset{\sim}{\theta}_n)}$ belong to $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ with parameter $(g(\underset{\sim}{\alpha}_n), g^*(\underset{\sim}{\beta}_n), \eta(\underset{\sim}{\theta}_n))$ depending on $(\underset{\sim}{\alpha}_n, \underset{\sim}{\beta}_n, \underset{\sim}{\theta}_n)$.*

Remark 6.4.1. *A set of R.V.s is said to be closed under the minima, if the corresponding family of bivariate distributions is closed under the minima.*

Remark 6.4.2. *When the bivariate r.v.s are i.i.d., we denote $g(\underset{\sim}{\alpha}_n) = g_n(\alpha)$, $g^*(\underset{\sim}{\beta}_n) = g^*_n(\beta)$ and $\eta(\underset{\sim}{\theta}_n) = \eta_n(\theta)$ respectively, which are functions of n and the corresponding parameters.*

Remark 6.4.3. *If X_j and Y_j, $j = 1,2,\ldots,n$ are identically distributed then α=β and $g_n(\alpha) = g^*_n(\beta)$.*

Example 6.4.1. *Let (X_j, Y_j) $j = 1,2,\ldots,n$ follows bivariate exponential dis-*

tribution (Gumbel (1960)) with joint d.f.

$$H_{X,Y;\alpha_j,\beta_j,\theta_j}(x,y) = 1 - e^{-\alpha_j x} - e^{-\beta_j y} + e^{-\alpha_j x+\beta_j y+\theta_j\alpha_j\beta_j xy},$$
$$\alpha_j > 0, \beta_j > 0, 0 \leq \theta_j \leq 1.$$

Then

$$\begin{aligned} H_{X_{m_n},Y_{m_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n}(x,y) &= 1 - e^{-\sum_{j=1}^{n}\alpha_j x} - e^{-\sum_{j=1}^{n}\beta_j y} + e^{-\sum_{j=1}^{n}(\alpha_j x+\beta_j y+\theta_j\alpha_j\beta_j xy)} \\ &= H_{X,Y;g(\underset{\sim}{\alpha}_n),g^*(\underset{\sim}{\beta}_n),\eta(\underset{\sim}{\theta}_n)}(x,y), \qquad (6.4.1) \end{aligned}$$

where $g(\underset{\sim}{\alpha}_n) = \sum_{j=1}^{n}\alpha_j$, $g^*(\underset{\sim}{\beta}_n) = \sum_{j=1}^{n}\beta_j$ *and* $\eta(\underset{\sim}{\theta}_n) = \left(\frac{\sum_{j=1}^{n}\alpha_j\beta_j\theta_j}{\sum_{j=1}^{n}\alpha_j\sum_{j=1}^{n}\beta_j}\right)$.

In the above example if the R.V.s are i.i.d., then $g_n(\alpha) = n\alpha$, $g_n^*(\beta) = n\beta$ and $\eta_n(\theta) = \frac{\theta}{n}$.

Next, we have the definition of family of bivariate distributions closed under the maxima.

Definition 6.4.2. *A family of bivariate distributions,* $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$*, is said to be closed under the maxima with respect to* $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$*, if for every* $H_{X,Y;\alpha_j,\beta_j,\theta_j} \in \mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$, $j = 1, 2, \ldots, n$, $n \geq 1$, *the joint d.f.s of componentwise maxima,* $H_{X_{M_n},Y_{M_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n} = H_{X,Y;h(\underset{\sim}{\alpha}_n),h^*(\underset{\sim}{\beta}_n),\nu(\underset{\sim}{\theta}_n)}$ *belong to* $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ *with parameters* $(h(\underset{\sim}{\alpha}_n), h^*(\underset{\sim}{\beta}_n), \nu(\underset{\sim}{\theta}_n))$ *depending on* $(\underset{\sim}{\alpha}_n, \underset{\sim}{\beta}_n, \underset{\sim}{\theta}_n)$.

Remark 6.4.4. *A set of R.V.s is said to be closed under the maxima, if the corresponding family of bivariate distributions is closed under the maxima.*

Remark 6.4.5. *When the bivariate r.v.s are i.i.d.,* $h(\underset{\sim}{\alpha}_n) = h_n(\alpha)$, $h^*(\underset{\sim}{\beta}_n) = h_n^*(\beta)$ *and* $\nu(\underset{\sim}{\theta}_n) = \nu_n(\theta)$*, which are functions of* n *and the corresponding parameters.*

Remark 6.4.6. *If* X_j *and* Y_j, $j = 1, 2, \ldots, n$ *are identically distributed then* $\alpha{=}\beta$ *and* $h_n(\alpha) = h_n^*(\beta)$.

Remark 6.4.7. *Definition 6.4.1 and Definition 6.4.2 hold true even if we replace d.f.s by corresponding s.f.s.*

So for the sake of convenience, we will be using joint and marginal d.f.s while discussing closure under maxima and joint and marginal s.f.s will be used while discussing closure under minima.

The following theorem gives the necessary condition for a family of bivariate distributions to be closed under the minima.

Theorem 6.4.1. *If a family of bivariate distributions is closed under the minima, then so are its marginal families of distributions.*

Proof. Suppose $\bar{H}_{X_{m_n},Y_{m_n};\alpha,\beta,\theta}(x,y) = \bar{H}_{X,Y;g_n(\alpha),g_n^*(\beta),\eta_n(\theta)}(x,y)$. Then

$$\begin{aligned}
\bar{F}_{X_{m_n};\alpha}(x) &= \bar{H}_{X_{m_n},Y_{m_n};\alpha,\beta,\theta}(x,\infty) \\
&= \bar{H}_{X,Y;g_n(\alpha),g_n^*(\beta),\eta_n(\theta)}(x,\infty) \\
&= \bar{F}_{X;g_n(\alpha)}(x).
\end{aligned}$$

Hence, X is closed under the minima with respect to $\boldsymbol{\alpha}$. Similarly, Y is closed under the minima with $\boldsymbol{\beta}$. Hence the proof. □

A necessary condition for a family of bivariate distributions to be closed under the maxima is given in the next theorem.

Theorem 6.4.2. *If a family of bivariate distribution is closed under the maxima, then so are its marginal families of distributions.*

Proof. The proof is similar to that of Theorem 6.4.1. □

The conditions given in Theorem 6.4.1 and Theorem 6.4.2 are not sufficient for a family of bivariate distributions to be closed under the minima or maxima respectively.

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be i.i.d. R.V.s. Then consider the following examples.

Example 6.4.2. *For the s.f.s* $\bar{F}_{X;\alpha}(x) = e^{-\alpha x}, x > 0, \alpha > 0$ *and* $\bar{G}_{Y;\beta}(x) = e^{-\beta y}, y > 0, \beta > 0$*, the bivariate exponential distribution (Gumbel (1960)) have joint s.f. given by*

$$\bar{H}_{X,Y;\alpha,\beta,\theta}(x, y) = e^{-(\alpha x+\beta y+\theta\alpha\beta xy)}, \quad 0 \le \theta \le 1. \tag{6.4.2}$$

Hence,

$$\begin{aligned} \bar{H}_{X_{m_n},Y_{m_n};\alpha,\beta,\theta}(x, y) &= e^{-n(\alpha x+\beta y+\theta\alpha\beta xy)} \\ &= \bar{H}_{X,Y;n\alpha,n\beta,\frac{\theta}{n}}(x, y). \end{aligned}$$

The survival copula corresponding to the joint s.f. in Example 6.4.1 is given by

$$\hat{C}_\theta(u, v) = uve^{-\theta \ln u \ln v}, 0 \le \theta \le 1. \tag{6.4.3}$$

For this copula,

$$C_\theta^n(u^{1/n}, v^{1/n}) = uve^{-\frac{1}{n}\theta \ln u \ln v} = C_{\frac{\theta}{n}}(u, v).$$

Now, for the same marginal s.f.s as in Example 6.4.1, consider the joint s.f. given in the following example.

Example 6.4.3. *Consider the bivariate distribution with joint s.f. given by*

$$\bar{H}_{X,Y;\alpha,\beta,\theta}(x,y) = \frac{e^{-(\alpha x+\beta y)}}{1-\theta(1-e^{-\alpha x})(1-e^{\beta y})}. \tag{6.4.5}$$

Then

$$\bar{H}_{X_{m_n},Y_{m_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n}(x,y) = \frac{e^{-(\sum_{j=1}^{n}\alpha_j x+\sum_{j=1}^{n}\beta_j y)}}{\prod_{j=1}^{n}(1-\theta_j(1-e^{-\alpha_j x})(1-e^{\beta_j y}))}$$

is not of the form $\bar{H}_{X,Y;g(\underset{\sim}{\alpha}_n),g^*(\underset{\sim}{\beta}_n),\eta(\underset{\sim}{\theta}_n)}(x,y)$.

Here marginal distributions are exponential which have closure property under the minima by Example 3.2.1, but the joint d.f. is not so. So the converse of Theorem 6.4.1 and Theorem 6.4.2 need not be true. i.e, marginal distributions are closed under the minima (maxima) need not imply that the joint distribution is closed under the minima (maxima). Hence, we need some additional conditions on how they are dependent. Here comes the importance of 'Copula'. Recall that, the bivariate exponential distribution given in Example 6.4.1 is closed under the minima.

The copula corresponding to Example 6.4.3 is

$$\hat{C}_{\theta}(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}, \quad -1 \le \theta \le 1, \tag{6.4.6}$$

known as the Ali-Mikhail-Haq family of copulas (Hutchinson and Lai (1990)). For this copula,

$$C_{\theta}^{n}(u^{1/n},v^{1/n}) = \frac{uv}{(1-\theta(1-u^{1/n})(1-v^{1/n}))^{n}} \neq C_{\eta(\theta)}(u,v).$$

Let us consider another example.

Example 6.4.4. *Consider the copula in (6.4.3) and the marginal s.f.s given by* $\bar{F}_{X;\alpha}(x) = 1 - (\frac{x}{\lambda})^{\alpha}$, $0 < x < \lambda$, $\lambda > 0$, $\alpha > 0$ *and* $\bar{G}_{X;\alpha}(y) = 1 - (\frac{y}{\gamma})^{\beta}$, $0 < x < \gamma$, $\gamma > 0$, $\beta > 0$. *Then the corresponding joint s.f. is*

$$\bar{H}_{X,Y;\alpha,\beta,\theta}(x,y) = (1-(x/\lambda)^{\alpha})(1-(y/\gamma)^{\beta})e^{-\theta\ln(1-\left(\frac{x}{\lambda}\right)^{\alpha})\ln(1-\left(\frac{y}{\gamma}\right)^{\beta})}. \quad (6.4.8)$$

Hence,

$$\begin{aligned}\bar{H}_{X_{m_n},Y_{m_n};\alpha,\beta,\theta}(x,y) &= (1-(x/\lambda)^{\alpha})^n(1-(y/\gamma)^{\beta})^n e^{-n\theta\ln(1-\left(\frac{x}{\lambda}\right)^{\alpha})\ln(1-\left(\frac{y}{\gamma}\right)^{\beta})} \\ &\neq \bar{H}_{X_{m_n},Y_{m_n};g_n(\alpha),g_n^*(\beta),\eta_n(\theta)}(x,y).\end{aligned}$$

The above example shows that even if the copula satisfies the condition $C_{\theta}^n(u^{1/n}, v^{1/n}) = C_{\eta(\theta)}(u,v)$ (here $\eta_n(\theta) = \theta/n$), the joint s.f. is not closed under the minima when the marginal s.f.s are not so (see Example 3.2.5). The following theorem gives a necessary and sufficient conditions for a family of bivariate distributions to be closed under the minima.

Theorem 6.4.3. *Let* $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ *be a family of bivariate distributions with corresponding families of marginal distributions* $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *and* $\mathscr{G}_{Y;\boldsymbol{\beta}}$, *and copula* $C_{\theta}(u,v)$. *Then* $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ *is closed under the minima with respect to* (α,β,θ) *if, and only if,* $\mathscr{F}_{X;\boldsymbol{\alpha}}$ *is closed under the minima with respect to* $\boldsymbol{\alpha}$, $\mathscr{G}_{Y;\boldsymbol{\beta}}$ *is closed under the minima with respect to* $\boldsymbol{\beta}$ *and* $\hat{C}_{\theta}^n(u^{1/n}, v^{1/n}) = \hat{C}_{\eta_n(\theta)}(u,v)$.

Proof. Suppose that $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the minima with respect to (α,β,θ). Then by Theorem 6.4.1, $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the minima with respect to α, $\mathscr{G}_{Y;\boldsymbol{\beta}}$ is closed under the minima with respect to β. i.e., $\bar{F}_{X_{m_n};\alpha}(x) =$

$\bar{F}_{X;g_n(\alpha)}(x)$ and $\bar{G}_{Y_{m_n};\beta}(y) = \bar{G}_{Y;g_n^*(\beta)}(y)$.

$$\begin{aligned}
&\bar{H}_{X_{m_n},Y_{m_n};\alpha,\beta,\theta}(x,y) = \bar{H}_{X,Y;g_n(\alpha),g_n^*(\beta),\eta_n(\theta)}(x,y) \\
&\Rightarrow \hat{C}_\theta^n\left(\bar{F}^{1/n}_{X_{m_n};\alpha}(x), \bar{G}^{1/n}_{Y_{m_n};\beta}(y)\right) = \hat{C}_{\eta_n(\theta)}\left(\bar{F}_{X,g_n(\alpha)}(x), \bar{G}_{Y;g_n^*(\beta)}(y)\right) \\
&\Rightarrow \hat{C}_\theta^n(u^{1/n}, v^{1/n}) = \hat{C}_{\eta_n(\theta)}(u,v).
\end{aligned}$$

To prove the converse, suppose that $\bar{F}_{X_{m_n};\alpha}(x) = \bar{F}_{X;g_n(\alpha)}(x)$, $\bar{G}_{Y_{m_n};\beta}(y) = \bar{G}_{Y;g_n^*(\beta)}(y)$ and $\hat{C}_\theta^n(u^{1/n}, v^{1/n}) = \hat{C}_{\eta_n(\theta)}(u,v)$. Now, substituting $u = \bar{F}_{X_{m_n}}$ and $v = \bar{G}_{Y_{m_n}}$, we have

$$\begin{aligned}
\hat{C}_\theta^n\left(\bar{F}^{1/n}_{X_{m_n};\alpha}(x), \bar{G}^{1/n}_{Y_{m_n};\beta}(y)\right) &= \hat{C}_{\eta_n(\theta)}\left(\bar{F}_{X_{m_n};\alpha}(x), \bar{G}_{Y_{m_n};\beta}(y)\right) \\
&= \hat{C}_{\eta_n(\theta)}\left(\bar{F}_{X,g_n(\alpha)}(x), \bar{G}_{Y;g_n^*(\beta)}(y)\right)
\end{aligned}$$

which implies

$$\bar{H}_{X_{m_n},Y_{m_n};\alpha,\beta,\theta}(x,y) = \bar{H}_{X,Y;g_n(\alpha),g_n^*(\beta),\eta_n(\theta)}(x,y).$$

Hence the proof. □

The necessary and sufficient condition for a family of bivariate distributions to be closed under the maxima is given in the next theorem.

Theorem 6.4.4. *Let $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ be a family of bivariate distributions with corresponding families of marginal distributions $\mathscr{F}_{X;\boldsymbol{\alpha}}$ and $\mathscr{G}_{Y;\boldsymbol{\beta}}$, and copula $C_\theta(u,v)$. Then $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the maxima with respect to (α,β,θ) if, and only if, $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to $\boldsymbol{\alpha}$, $\mathscr{G}_{Y;\boldsymbol{\beta}}$ is closed under the maxima with respect to $\boldsymbol{\beta}$ and $C_\theta^n(u^{1/n}, v^{1/n}) = C_{\eta_n(\theta)}(u,v)$.*

Proof. Suppose that $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the maxima with respect to

(α, β, θ). Then by Theorem 6.4.2, $\mathscr{F}_{X;\boldsymbol{\alpha}}$ is closed under the maxima with respect to α, $\mathscr{G}_{Y;\boldsymbol{\beta}}$ is closed under the maxima with respect to β. i.e., $F_{X_{m_n};\alpha}(x) = F_{X;g_n(\alpha)}(x)$ and $G_{Y_{m_n};\beta}(y) = G_{Y;g_n^*(\beta)}(y)$.

$$\begin{aligned}
& H_{m_n;\alpha,\beta,\theta}(x,y) = H_{X,Y;g_n(\alpha),g_n^*(\beta),\eta_n(\theta)}(x,y) \\
& \Rightarrow C_\theta^n\left(F^{1/n}_{X_{m_n};\alpha}(x), G^{1/n}_{Y_{m_n};\beta}(y)\right) = C_{\eta_n(\theta)}\left(F_{X,g_n(\alpha)}(x), G_{Y;g_n^*(\beta)}(y)\right) \\
& \Rightarrow C_\theta^n(u^{1/n}, v^{1/n}) = C_{\eta_n(\theta)}(u,v)
\end{aligned}$$

To prove the converse, suppose that $\bar{F}_{X_{m_n};\alpha}(x) = \bar{F}_{X;g_n(\alpha)}(x)$, $\bar{G}_{Y_{m_n};\beta}(y) = \bar{G}_{Y;g_n^*(\beta)}(y)$ and $C_\theta^n(u^{1/n}, v^{1/n}) = C_{\eta_n(\theta)}(u,v)$. Now, on substituting $u = \bar{F}_{X_{m_n}}$ and $v = \bar{G}_{Y_{m_n}}$, we have

$$\begin{aligned}
C_\theta^n\left(F^{1/n}_{X_{m_n};\alpha}(x), G^{1/n}_{Y_{m_n};\beta}(y)\right) &= C_{\eta_n(\theta)}\left(F_{X_{m_n};\alpha}(x), G_{Y_{m_n};\beta}(y)\right) \\
&= C_{\eta_n(\theta)}\left(F_{X,g_n(\alpha)}(x), G_{Y;g_n^*(\beta)}(y)\right)
\end{aligned}$$

which implies

$$H_{X_{m_n},Y_{m_n};\alpha,\beta,\theta}(x,y) = H_{X,Y;g_n(\alpha),g_n^*(\beta),\eta_n(\theta)}(x,y).$$

Hence the proof. □

Now, we define the closure under extrema for a copula C.

Definition 6.4.3. *A copula satisfying the condition* $C_\theta^n(u^{1/n}, v^{1/n}) = C_{\eta_n(\theta)}(u,v)$ *for all* $n \geq 1$*, integer, is called 'copula closed under extrema'.*

Remark 6.4.8. *The set of extreme value copulas discussed in Section 6.3 is a subset of copulas closed under extrema.*

The following table gives some examples of copulas closed under extrema.

$C_\theta(u,v)$	$C_\theta^n(u^{1/n},v^{1/n})$	$\eta(\theta)$	Range of θ
uv	uv		
$\min(u,v)$	$\min(u,v)$		
$e^{(-[(-\ln u)^\theta+(-\ln v)^\theta]^{1/\theta})}$	$e^{(-\{(-\ln u)^\theta+(-\ln v)^\theta\}^{1/\theta})}$	θ	$[1,\infty)$
$[\max(u^{-\theta}+v^{-\theta}-1,0)]^{-1/\theta}$	$[\max(u^{-\theta/n}+v^{-\theta/n}-1,0)]^{-n/\theta}$	θ/n	$[1,\infty)\setminus\{0\}$
$uv/[1+(1-u^\theta)(1-v^\theta)]^1/\theta$	$uv/[1+(1-u^{\theta/n})(1-v^{\theta/n})]^n/\theta$	θ/n	$(0,1]$
$(u^{-1/\theta}+v^{-1/\theta}-1)^{-\theta}$	$(u^{-1/n\theta}+v^{-1/n\theta}-1)^{-n\theta}$	$n\theta$	$(0,\infty)$

Table 6.1: Examples of Copulas Closed under Extrema

6.5 Monotone Transformations and Bivariate Closure Property under Extrema

In Section 3.3, we discussed how the closure property under extrema of univariate r.v.s changes under strictly increasing or strictly decreasing measurable transformations. In this section, we extend this to bivariate case and discuss how closure property under extrema of R.V.s changes under strictly increasing or strictly decreasing measurable transformations of marginal r.v.s.

Theorem 6.5.1. *Let the family $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ be closed under extrema with respect to $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$ and, ξ and ψ be Borel measurable and strictly increasing on $\mathsf{Ran}X$ and $\mathsf{Ran}Y$ respectively. Suppose $\mathscr{G}_{\xi(X),\psi(Y);\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is the family of distributions of $(\xi(X),\psi(Y))$. Then the following results hold.*

i) If $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the maxima, then so is $\mathscr{G}_{\xi(X),\psi(Y);\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$.

ii) If $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the minima, then so is $\mathscr{G}_{\xi(X),\psi(Y);\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$.

Furthermore, the corresponding $(g(\underset{\sim}{\alpha}_n),g^*(\underset{\sim}{\beta}_n),\eta(\underset{\sim}{\theta}_n))$ or $(h(\underset{\sim}{\alpha}_n),h^*(\underset{\sim}{\beta}_n),\nu(\underset{\sim}{\theta}_n))$ for both the families will be the same.

Proof. Let $G_{\xi(X),\psi(Y);\alpha_j,\beta_j,\theta_j}$ be the joint d.f. of $(\xi(X),\psi(Y))$. Since ξ and ψ are strictly increasing, we have

$$\begin{aligned}
G_{\xi(X),\psi(Y);\alpha_j,\beta_j,\theta_j}(x,y) &= P_{\alpha_j,\beta_j,\theta_j}(\xi(X)\le x,\psi(Y)\le y)\\
&= P_{\alpha_j,\beta_j,\theta_j}(X\le \xi^{-1}(x),Y\le \psi^{-1}(y))\\
&= H_{X,Y;\alpha_j,\beta_j,\theta_j}(\xi^{-1}(x),\psi^{-1}(y)).
\end{aligned}$$

Let $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ be closed under maxima. Then for all $H_{X,Y;\alpha_j,\beta_j,\theta_j}$, $j=1,2,\ldots,n$ belonging to $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$; $H_{X_{M_n},Y_{M_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n}=H_{X,Y;h(\underset{\sim}{\alpha}_n),h^*(\underset{\sim}{\beta}_n),\nu(\underset{\sim}{\theta}_n)}$. Hence,

$$\begin{aligned}
G_{\xi(X)_{M_n},\psi(Y)_{M_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n}(x,y) &= \prod_{j=1}^{n} G_{\xi(X),\psi(Y);\alpha_j,\beta_j,\theta_j}(x,y)\\
&= \prod_{j=1}^{n} H_{X,Y;\alpha_j,\beta_j,\theta_j}(\xi^{-1}(x),\psi^{-1}(y))\\
&= H_{X,Y;h(\underset{\sim}{\alpha}_n),h^*(\underset{\sim}{\beta}_n),\nu(\underset{\sim}{\theta}_n)}(\xi^{-1}(x))\\
&= G_{\xi(X),\psi(Y);h(\underset{\sim}{\alpha}_n),h^*(\underset{\sim}{\beta}_n),\nu(\underset{\sim}{\theta}_n)}(x,y).
\end{aligned}$$

i.e., $\mathscr{G}_{\xi(X),\psi(Y);\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the maxima with $(h(\underset{\sim}{\alpha}_n),h^*(\underset{\sim}{\beta}_n),\nu(\underset{\sim}{\theta}_n))$ of $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$. Let $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ be closed under minima. Then for all $H_{X,Y;\alpha_j,\beta_j,\theta_j}$, $j=1,2,\ldots,n$ belonging to $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$; $H_{X_{m_n},Y_{m_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n}=H_{X,Y;g(\underset{\sim}{\alpha}_n),g^*(\underset{\sim}{\beta}_n),\eta(\underset{\sim}{\theta}_n)}$. Hence,

$$\begin{aligned}
\bar{G}_{\xi(X)_{M_n},\psi(Y)_{M_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n}(x,y) &= \prod_{j=1}^{n} \bar{G}_{\xi(X),\psi(Y);\alpha_j,\beta_j,\theta_j}(x,y)\\
&= \prod_{j=1}^{n} \bar{H}_{X,Y;\alpha_j,\beta_j,\theta_j}(\xi^{-1}(x),\psi^{-1}(y))\\
&= \bar{H}_{X,Y;g(\underset{\sim}{\alpha}_n),g^*(\underset{\sim}{\beta}_n),\eta(\underset{\sim}{\theta}_n)}(\xi^{-1}(x))\\
&= \bar{G}_{\xi(X),\psi(Y);g(\underset{\sim}{\alpha}_n),g^*(\underset{\sim}{\beta}_n),\eta(\underset{\sim}{\theta}_n)}(x,y).
\end{aligned}$$

i.e., $\mathscr{G}_{\xi(X),\psi(Y);\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the minima with $(g(\underset{\sim}{\alpha}_n), g^*(\underset{\sim}{\beta}_n), \eta(\underset{\sim}{\theta}_n))$ of $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$. Hence the proof. □

Theorem 6.5.2. *Let the family $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ be closed under extrema with respect to $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$ and, ξ and ψ be Borel measurable and strictly decreasing on $\mathsf{Ran}X$ and $\mathsf{Ran}Y$ respectively. Suppose $\mathscr{G}_{\xi(X),\psi(Y);\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is the family of distributions of $(\xi(X),\psi(Y))$. Then the following results hold.*

i) If $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the maxima, then $\mathscr{G}_{\xi(X),\psi(Y);\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the minima with corresponding $(g(\underset{\sim}{\alpha}_n), g^*(\underset{\sim}{\beta}_n), \eta(\underset{\sim}{\theta}_n)) = (h(\underset{\sim}{\alpha}_n), h^*(\underset{\sim}{\beta}_n), \nu(\underset{\sim}{\theta}_n))$ of $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$.

ii) If $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the minima, then $\mathscr{G}_{\xi(X),\psi(Y);\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the maxima with corresponding $(h(\underset{\sim}{\alpha}_n), h^*(\underset{\sim}{\beta}_n), \nu(\underset{\sim}{\theta}_n)) = (g(\underset{\sim}{\alpha}_n), g^*(\underset{\sim}{\beta}_n), \eta(\underset{\sim}{\theta}_n))$ of $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$.

Proof. Since ξ and ψ are strictly decreasing, $-\xi$ and $-\psi$ is strictly increasing. Hence, by Theorem 6.5.1 and the relation

$$(X_{m_n}, Y_{m_n}) = (-\max\{-X_1, -X_2, \ldots, -X_n\}, -\max\{-Y_1, -Y_2, \ldots, -Y_n\})$$

The proof follows on the similar lines of Theorem 3.3.2. □

Result 6.5.1. *Let the family $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$, $P(X \neq 0) = 1$ and $P(Y \neq 0) = 1$, be closed under componentwise minima (maxima) with respect to $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$. Then the family $\mathscr{H}_{\frac{1}{X},\frac{1}{Y};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the maxima (minima) with respect to $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$.*

Result 6.5.2. *Let the family $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ be closed under componentwise minima (maxima)with respect to $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$. Then the family $\mathscr{H}_{-X,-Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is closed under the maxima (minima) with respect to $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$.*

Result 6.5.3. *Closure under componentwise maxima (minima) is invariant under change of origin and scale.*

Result 6.5.4. *Let the family* $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ *be closed under componentwise minima (maxima)with respect to* $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$. *Then*

1. *the family* $\mathscr{H}_{X^{\gamma},Y^{\delta};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$, $X \geq 0$, $Y \geq 0$ *a.s., is closed under the minima (maxima) with respect to* $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}$, *for* $\gamma > 0$, $\delta > 0$ *constants.*

2. *the family* $\mathscr{H}_{X^{\gamma},Y^{\delta};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$, $X > 0$, $Y > 0$ *a.s., is closed under the maxima (minima) with respect to* $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}$, *for* $\gamma > 0$, $\delta > 0$ *constants.*

6.6 Truncation and Bivariate Closure Property under Extrema

Recall that, in Section 3.4, we have seen that truncated distributions are obtained by restricting the domain of a probability distribution. We also discussed how closure property under extrema of r.v.s changes on truncation. In this section, we are going to discuss how closure property under extrema changes on the truncation of marginal r.v.s. The probability distribution of (X,Y) conditioned on $X > a$, $Y > c$ is called left-truncated distribution of (X,Y) truncated at (a,c) and the probability distribution of (X,Y) conditioned on $X < b$, $Y < d$ is called right-truncated distribution of (X,Y) truncated at (b,d). The joint d.f. of (X,Y) truncated at both ends, or doubly truncated, is denoted by $H_{X_{T(a,b)},Y_{T(c,d)}}$ and is given by

$$H_{X_{T(a,b)},Y_{T(c,d)}}(x,y) = \frac{H_{X,Y}(x,y) - H_{X,Y}(x,c) - H_{X,Y}(a,y) + H_{X,Y}(a,c)}{H_{X,Y}(b,d) - H_{X,Y}(b,c) - H_{X,Y}(a,d) + H_{X,Y}(a,c)} \tag{6.6.1}$$

with corresponding joint s.f.

$$\bar{H}_{X_{T(a,b)},Y_{T(c,d)}}(x,y) = \frac{\bar{H}_{X,Y}(x,y) - \bar{H}_{X,Y}(x,d) - \bar{H}_{X,Y}(b,y) + \bar{H}_{X,Y}(b,d)}{\bar{H}_{X,Y}(a,c) - \bar{H}_{X,Y}(a,d) - \bar{H}_{X,Y}(b,c) + \bar{H}_{X,Y}(b,d)}. \tag{6.6.2}$$

for $x_a \leq a < x < b \leq x_b$, where x_a and x_b are the end points of the support of X and $y_c \leq c < y < d \leq y_d$. If $a = x_a$, $b < x_b$, $c = y_c$ and $d < y_d$ we get a right truncated distribution. Similarly, if $a > x_a$, $b = x_b$, $c > y_c$ and $d = y_d$ we get a left truncated distribution. Hence, the joint d.f. of a right truncated distribution truncated at (b, d) is given by

$$H_{X_{T(b)},Y_{T(d)}}(x,y) = \frac{H_{X,Y}(x,y)}{H_{X,Y}(b,d)} \tag{6.6.3}$$

and the joint s.f. of a left truncated distribution truncated at (a, c) is given by

$$\bar{H}_{X_{T(a)},Y_{T(c)}}(x,y) = \frac{\bar{H}_{X,Y}(x,y)}{\bar{H}_{X,Y}(a,c)}. \tag{6.6.4}$$

In this section we see how the closure property changes under right and left truncation of bivariate distributions. From (6.6.1) and (6.6.2) we can easily say that closure under extrema is not preserved under truncation at both the ends. But, from (6.6.3) and (6.6.4) we observe that closure under the maxima is invariant under right truncation and closure under the minima is invariant under left truncation respectively. This is the content of the following two theorems.

Theorem 6.6.1. *Let the family $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ be closed under the maxima. Then the right truncated family $\mathscr{H}_{X_{T(b)},Y_{T(d)};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ is also closed under the maxima.*

Proof. We have $H_{X_{M_n},Y_{M_n};\alpha,\beta,\theta}(x,y) = H_{X,Y;h_n(\alpha),h_n^*(\beta),\nu(\theta)}$. Then

$$\begin{aligned} H_{X_{T(b)_{M_n}},Y_{T(d)_{M_n}};\alpha,\beta,\theta}(x,y) &= \prod_{j=1}^{n} H_{X_{T(b)},Y_{T(d)};\alpha,\beta,\theta}(x,y) \\ &= \prod_{j=1}^{n}\left(\frac{H_{X,Y;\alpha,\beta,\theta}(x,y)}{H_{X,Y;\alpha,\beta,\theta}(b,d)}\right) \\ &= \frac{H_{X,Y;h_n(\alpha),h_n^*(\beta),\nu(\theta)}(x,y)}{H_{X,Y;h_n(\alpha),h_n^*(\beta),\nu(\theta)}(b,d)} \\ &= H_{X_{T(b)},Y_{T(d)};h_n(\alpha),h_n^*(\beta),\nu(\theta)}(x,y). \end{aligned}$$

Hence the proof. □

Theorem 6.6.2. *Let the family* $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ *be closed under the minima. Then the left truncated family* $\mathscr{H}_{X_{T(b)},Y_{T(d)};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ *is also closed under the minima.*

Proof. We have $\bar{H}_{X_{m_n},Y_{m_n};\alpha,\beta,\theta}(x,y) = H_{X,Y;g_n(\alpha),g_n^*(\beta),\eta(\theta)}$. Then from $\bar{H}_{X_{T(b)_{M_n}},Y_{T(d)_{M_n}};\alpha,\beta,\theta}(x,y)$ on simplification we get $\bar{H}_{X_{T(b)},Y_{T(d)};g_n(\alpha),g_n^*(\beta),\eta(\theta)}(x,y)$. Hence the proof. □

6.7 Characteristic Functions of Componentwise Extrema

Let $\{(X_n, Y_n)\}$ be a sequence of independent bivariate r.v.s. Due to the one to one correspondence between the joint d.f. and the joint c.f, if the joint d.f. of R.V.s belong to a family of bivariate distributions closed under the minima, then one can represent the joint c.f.s of componentwise minima in terms of the joint c.f. of the underlying distribution.

Theorem 6.7.1. *Let* $\{X_n, Y_n\}$ *be a sequence of independent bivariate r.v.s with* $(X_j, Y_j) \sim H_{X_j,Y_j}(x,y) \in \mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ *and let the c.f.* (X_j, Y_j) *be* $\phi_{X,Y;\alpha_j,\beta_j,\theta_j}$.

i) If $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ *is closed under the componentwise minima with respect to* $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$*, then* $\phi_{X_{m_n},Y_{m_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n} = \phi_{X,Y;g(\underset{\sim}{\alpha}_n),g^*(\underset{\sim}{\beta}_n),\eta(\underset{\sim}{\theta}_n)} \in \mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$.

ii) f $\mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$ *is closed under the componentwise maxima with respect to* $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$*, then* $\phi_{X_{M_n},Y_{M_n};\underset{\sim}{\alpha}_n,\underset{\sim}{\beta}_n,\underset{\sim}{\theta}_n} = \phi_{X,Y;h(\underset{\sim}{\alpha}_n),h^*(\underset{\sim}{\beta}_n),\nu(\underset{\sim}{\theta}_n)} \in \mathscr{H}_{X,Y;\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}$.

Can one represent the joint c.f.s of componentwise maxima in terms of the joint c.f. of the underlying family of distributions? This will be possible if one can represent the joint d.f. of componentwise maxima in terms of the joint d.f. of the underlying family of distributions. Similar question exists, when a family bivariate of distributions is closed under the maxima. This is the problem addressed in this section.

Suppose $(X_1, Y_1) \sim H_1(x, y)$ and $(X_2, Y_2) \sim H_2(x, y)$ are independently distributed. Then the joint d.f. of their maxima is given by

$$\begin{aligned}
H_{X_{M_2},Y_{M_2}}(x,y) &= \prod_{j=1}^{2} P(X_j \le x, Y_j \le y) \\
&= \prod_{j=1}^{2}(1 - \bar{F}_{X_j}(x) - \bar{G}_{Y_j}(y) + \bar{H}_{X_j,Y_j}(x,y)) \\
&= 1 - \bar{F}_{X_1}(x) - \bar{F}_{X_2}(x) - \bar{G}_{Y_1}(y) - \bar{G}_{Y_2}(y) \\
&\quad + \bar{H}_{X_1,Y_1}(x,y) + \bar{H}_{X_2,Y_2}(x,y) \\
&\quad + \bar{F}_{X_1}(x)\bar{F}_{X_2}(x) + \bar{G}_{Y_1}(y)\bar{G}_{Y_2}(y) + \bar{H}_{X_1,Y_1}(x,y)\bar{H}_{X_2,Y_2}(x,y) \\
&\quad + \bar{F}_{X_1}(x)\bar{G}_{Y_2}(y) + \bar{F}_{X_2}(x)\bar{G}_{Y_1}(x) \\
&\quad - \bar{F}_{X_1}(x)\bar{H}_{X_2,Y_2}(x,y) - \bar{F}_{X_2}(x)\bar{H}_{X_1,Y_1}(x,y) \\
&\quad - \bar{G}_{Y_1}(y)\bar{H}_{X_2,Y_2}(x,y) - \bar{G}_{Y_2}(y)\bar{H}_{X_1,Y_1}(x,y). \qquad (6.7.1)
\end{aligned}$$

Now, if (X_1, Y_1) and (X_2, Y_2) are i.i.d. having joint d.f. $H_{X,Y}(x, y)$, then (6.7.1) becomes

$$\begin{aligned}
H_{X_{M_2},Y_{M_2}}(x,y) &= 1 - 2\bar{F}(x) - 2\bar{G}(y) + 2\bar{H}(x,y) \\
&\quad + \bar{F}^2(x) + \bar{G}^2(y) + \bar{H}^2(x,y) + 2\bar{F}(x)\bar{G}(y) \\
&\quad - 2\bar{F}(x)\bar{H}(x,y) - 2\bar{G}(y)\bar{H}(x,y). \\
&= 1 - 2\bar{F}_{X_{m_1}}(x) - 2\bar{G}_{Y_{m_1}}(y) + 2\bar{H}_{X_{m_1},Y_{m_1}}(x,y) \\
&\quad + \bar{F}_{X_{m_2}}(x) + \bar{G}_{Y_{m_2}}(y) + \bar{H}_{X_{m_2},Y_{m_2}}(x,y) + 2\bar{F}_{X_{m_1}}(x)\bar{G}_{Y_{m_1}}(y) \\
&\quad - 2\bar{F}_{X_{m_1}}(x)\bar{H}_{X_{m_1},Y_{m_1}}(x,y) - 2\bar{G}_{Y_{m_1}}(y)\bar{H}_{X_{m_1},Y_{m_1}}(x,y). \\
&= 2H_{X_{m_1},Y_{m_1}}(x,y) - H_{X_{m_2},Y_{m_2}}(x,y) \\
&\quad + 2\bar{H}_{X_{m_2},Y_{m_2}}(x,y) + 2\bar{F}_{X_{m_1}}(x)\bar{G}_{Y_{m_1}}(y) \\
&\quad - 2\bar{F}_{X_{m_1}}(x)\bar{H}_{X_{m_1},Y_{m_1}}(x,y) - 2\bar{G}_{Y_{m_1}}(y)\bar{H}_{X_{m_1},Y_{m_1}}(x,y)
\end{aligned} \tag{6.7.2}$$

From (6.7.1) and (6.7.2), we conclude that in the bivariate case, the joint d.f.s of componentwise maxima does not have a representation in terms of the partial joint d.f.s of componentwise minima. Similarly, the joint d.f.s of componentwise minima does not have a representation in terms of the partial joint d.f.s of componentwise maxima. Hence, the corresponding joint c.f.s does not have similar representations.

CHAPTER

SEVEN

Conclusion

The d.f.s of partial maxima and partial minima of a sequence of independent r.v.s can be represented in terms of the d.f. of underlying distribution. But in many situations they are hard to handle. However, when the sequence of r.v.s have closure property under extrema, the study of partial minima, partial maxima, and all other order statistics for every fixed n, reduces to the study of underlying distribution. In this case it is possible to express the distributions, c.f.s and other integral transforms in terms of that of underlying distribution. That is, if the sequence of r.v.s is closed under the minima or maxima, then the study of these statistics reduces to the study of underlying distribution. The thesis identifies the class of distributions closed under extrema.

In the case of bivariate or multivariate r.v.s, the concept can not be extended completely. That is, even if the sequence of R.V.s is closed under the minima it is not possible to express the joint d.f. of componentwise maxima in terms of the partial joint d.f. of componentwise minima. Hence, similar representations does not exist for joint c.f.s and other integral transforms, even if they exist.

Similar is the case when the sequence of R.V.s is closed under the maxima. 'Copulas closed under extrema' can be used as a starting point for construction of families of bivariate distributions closed under the minima or maxima. By joining the d.f.s of univariate distributions having closure property under the maxima using a copula closed under extrema we get a bivariate distribution belonging to a family closed under the maxima. Similarly, by joining s.f.s of univariate distributions having closure property under the minima using a copula closed under extrema, we get a bivariate distribution belonging to a family closed under the minima.

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LIST OF PAPER PUBLICATIONS & PRESENTATIONS

PUBLICATIONS

i) Aparna Aravindakshan, M. and Chandran, C. (2017). On the Characteristic Function of Extrema with Some Applications, *ProbStat Forum*, Vol 10, 85-93.

ii) Aparna Aravindakshan, M. and Chandran, C. (2018a). On the Extrema of Observations from Certain Families of Distributions, *ProbStat Forum*, communicated.

iii) Aparna Aravindakshan, M. and Chandran, C. (2018b). Order Statistics of Observations from Certain Families of Distributions, to be communicated.

iv) Aparna Aravindakshan, M. and Chandran, C. (2018c). Families of Bivariate Distributions Closed under Extrema, to be communicated.

PRESENTATIONS

i) **On the Characteristic Function of Maxima and Minima with Some Applications**, at Second International Conference on Statistics for Twenty-first Century (ICSTC-2016), organized by the Department of Statistics, University of Kerala, held during 21-23 December 2016.

ii) **On the Copulas of Componentwise Maxima and Minima**, at National Conference on Advances in Statistical Sciences (NCASS-2017), organized by the Department of Statistical Sciences, Kannur University, in conjunction with the Annual Conference of Kerala Statistical Association (KSA), held during 17-18 February 2017.

iii) **On the Characteristic Function of Extrema: Independent Non-Identical Case**, at the International Conference on Statistics for Twenty-first Century (ICSTC-2017), organized by the Department of Statistics, University of Kerala, held during 14-16 December 2017.

iv) **On the Characteristic Function of r^{th} Order Statistic** at the International Conference on Theory and Applications of Statistics and Information Sciences (TASIS-2018), organized by the Department of Statistics, Bharathiar University, Coimbatore, in conjunction with XXXVII annual conference of Indian Society for Probability and Statistics (ISPS) and Indian Bayesian Society (IBS), held during 05-07 January 2018.

v) **Copula of Bivariate Distributions Closed under Extrema** at National Seminar on Recent Trends in Statistics (NSRTS), organized by the Department of Statistics, University of Calicut, held during 13-15 March 2018.