STUDY ON GEOMETRIC STABLE LAWS

Thesis submitted to

University of Calicut

for the award of degree of

Doctor of Philosophy

in

Statistics

under Faculty of Science

by

SAJAYAN. T

under the guidance of

Prof. (Dr.) K. JAYAKUMAR

Department of Statistics

University of Calicut

Kerala - 673 635

INDIA

October 2022

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALICUT

Prof. (Dr.) K. JAYAKUMAR
Senior Professor and Head

Calicut University (P.O.)
Kerala, INDIA 673 635.
Phone : 0494 2407341 / 340
Mob : 09847533374
Email : jkumar19@rediffmail.com

Date : 02.11.2022

CERTIFICATE

I hereby certify that the work reported in this thesis entitled **STUDY ON GEOMETRIC STABLE LAWS** that is being submitted by Sri. **Sajayan. T** for the award of Doctor of Philosophy, to the University of Calicut, is based on the bonafide research work carried out by him under my supervision and guidance in the Department of Statistics, University of Calicut. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any Degree or Diploma of any other University or Institution. Also certify that the contents of the thesis have been checked using anti plagiarism data base and no unacceptable similarity was found through the software check.

Prof. (Dr.) K. Jayakumar
Research Supervisor

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALICUT

Dr. K. Jayakumar
Senior Professor

Calicut University (P.O.)
Kerala, INDIA 673 635.
Mob : 9847533374
Email : jkumar19@rediffmail.com

Date : 25.09.2023

CERTIFICATE

Certified that the corrections/suggestions from the adjudicators, of the PhD thesis entitled STUDY ON GEOMETRIC STABLE LAWS submitted by Sri. SAJAYAN T, Research scholar of the Department under my supervision and guidance, have been incorporated in this copy of the thesis.

Prof. (Dr.) K. Jayakumar
Research Supervisor

DECLARATION

I hereby declare that this thesis entitled **STUDY ON GEOMETRIC STABLE LAWS** submitted to the University of Calicut for the award of the degree of **Doctor of Philosophy** under the Faculty of Science is an independent work done by me under the guidance and supervision of **Dr. K. Jayakumar**, Senior Professor and Head, Department of Statistics, University of Calicut.

I also declare that this thesis contains no material which has been accepted for the award of any other degree or diploma of any university or institution and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference made in the text of the thesis.

Calicut University Campus
02.11.2022

Sajayan. T.

ACKNOWLEDGEMENT

I am deeply grateful to my supervising teacher ***Prof.(Dr.) K. Jayakumar****, Senior Professor and Head, Department of Statistics, University of Calicut, for his scholarly guidance, continual supervision, constructive criticism and inspiration that enabled me to complete this research work. I am particularly thankful to him for his cooperation and willingness for thoughtful discussions that made this difficult task easy and feasible.*

I am extremely thankful to ***Prof. (Dr.) N. Raju (Rtd.), Prof. (Dr.) M. Manoharan*** *and* ***Prof. (Dr). C. Chandran*** *and all the faculty members for the constant encouragement and valuable support and help bestowed upon me during my research period. I express my sincere gratitude to all of them. My thanks are also due to the librarian and office staffs of the Department of Statistics, University of Calicut for providing the necessary facilities during the span of my study.*

I am very lucky to have Sri. ***Dr Sankaran K.K*** *and* ***Dr. Girish Babu M*** *as my research colleagues. I would like to thank all of them for the support I have received. Also, I wish to thank* ***Dr. V.M Chacko*** *and* ***Dr Deepthi KS*** *for their support. I am thankful to the College Development Council, University of Calicut, for their timely help during the period of study.*

Memories of love, affection and care of my father ***Sri. Damodaran T (Late)*** *have been a great inspiration and force which lead me to achieve higher levels in my life. I owe thanks to a very special person, my wife,* ***Reethu K.*** *for all her love, continuing support and encouragement.*

Lastly, but not least, I remember with sincere thanks my extended family members and friends for bearing with me throughout this research work with their unstinting patience and cooperation.

SAJAYAN. T

STUDY ON GEOMETRIC STABLE LAWS

CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Introduction

Geometric stable(GS) laws is well suited for modeling heavy-tailed phenomena. Modeling and predicting the behavior of financial asset returns has attracted attention of numerous researchers over the years. Bachelier(1900) proposed normal distribution to model stock returns. His main idea came from the Central Limit Theorem: normal distribution provides good approximation for sum of independent, identically distributed random variables with finite variance. Since the price change Y over a given period of time can be regarded as the sum of changes X_i over shorter periods (that is, monthly change = sum of daily changes), the distribution of Y can be approximated by normal law under the assumption of independence, identical distribution, and finite variance of X_i's.

Further studies, however, revealed that empirical distributions of financial data had more kurtosis ("fatter tails") than that predicted by the normal approximation. It is not unusual for a stock price to have a relatively large jump, which is not consistent with the normal hypothesis. In response to these findings, Mandelbrot(1963b) and Fama(1965) proposed symmetric stable distributions for modeling asset returns.

Stable distributions provide approximations for sums of independent and identically distributed(i.i.d.) random variables that have heavy tails, and thus seemed appropriate for modeling leptokurtic data. However, a number of recent studies, showed inconsistencies with the Paretian stable model, and alternatives to the stable laws have been proposed for modeling asset returns. Extending the stability concept of Mandelbrot, Mittnik and Rachev (1993) considered other distributions, stable with respect to various operations (e.g., minimum, maximum, random summation). Fitting these alternative stable distributions to the stock-index data, they found that the Weibull distribution, which arises in geometric summation(summation variable follows geometric distribution)scheme, dominated all other alternative stable laws.

Geometric stable distributions approximate geometric random sums of i.i.d. random variables, which naturally arise in a variety of applied problems and are particularly appropriate in modeling heavy tailed phenomena. In finance, it was observed that the number of "individual effects" that produce a price change during a period of time is random. Namely, if T is the (random) number of transactions in one day, and X_i's represent price changes between successive transactions, then

$$Y = \sum_{i=1}^{T} X_i \tag{1.1}$$

represents the daily price change of a particular stock or commodity. If T has a geometric distribution and if T is large, then (appropriately normalized) sum

(1.1) can be approximated by a geometric stable law.

The objective of this research work is to study on geometric stable distributions and their extensions. The study mainly focus on inference, circular modeling, univariate and multivariate extensions and autoregressive models of additive structure.

1.2 Review of Literature

Now we consider some basic concepts along with a review of distributions used in the forthcoming chapters.

1.2.1 Self decomposability

Definition 1.2.1. *A charactristic function ϕ is self-decomposable if for every $a \in (0,1)$, there exists a characteristic function ϕ_a such that $\phi(t) = \phi(at)\phi_a(t)$, $\forall t \in \mathcal{R}$*

The corresponding distribution is said to belong to class $\mathcal{L}$

Definition 1.2.2. *A charactristic function ϕ is semi self-decomposable if for some $a \in (0,1)$, there exists a characteristic function ϕ_a such that $\phi(t) = \phi(at)\phi_a(t)$, $\forall t \in \mathcal{R}$*

1.2.2 Autoregressive Processes

A time series is a set of observations x_t, each one being recorded at a specific time t. A *discrete time series* is one in which the set T_0 of times at which observations are made is a discrete set, for example, observations are made at fixed time intervals. *Continuous time series* are obtained when observations are recorded continuously over some time interval.

When we analyse a time series using formal statistical methods, we view the collection of observations $\{X_n, n = 1, 2, \cdots\}$ as a particular realization of the stochastic process $\{X_k\}$. Hence a complete description of a time series, observed as a collection of n random variables at arbitrary time points $t_1, t_2, \cdots, t_n$ is provided by the joint distribution function $F(x_1, x_2, \cdots, x_n) = P(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \cdots, X_{t_n} \leq x_n)$. A special class of time series, is stationary time series. If the joint probability distribution of $\{X_t\}$ at any set of times $t_1, t_2, \cdots, t_n$ is same as the joint probability distribution at times $t_1 + k, t_2 + k, \cdots, t_n + k$, where k is an any arbitrary shift in time, then $\{X_n\}$ is called a strictly stationary time series.

An autoregressive time series model of order $p \geq 1$, abbreviated as $AR(p)$, is defined as

$$X_n = \rho_1 X_{n-1} + \rho_2 X_{n-2} + \cdots + \rho_p X_{n-p} + \epsilon_n$$

where $\{\epsilon_n\}$ is a sequence of independent and identically distributed random variables, and $\rho_1, \rho_2, \cdots, \rho_p$ are constants.
$AR(1)$, autoregressive process of order 1, is obtained as

$$X_n = \rho X_{n-1} + \epsilon_n$$

and ρ must satisfy the condition $|\rho| < 1$ to ensure the stationarity of the process.

1.2.3 Non-Gaussian autoregressive models

Classical time series analysis is based on the normality assumption of the error variable. But there are many occasions in which the time series are non-normal. We have later witnessed the emergence of many non Gaussian autoregressive processes in discrete time. The fact is that many naturally occurring time series are non Guassian. Consider the linear additive autoregreessive equation

$$X_n = \rho X_{n-1} + \epsilon_n, n = 0, \pm 1, \pm 2, \ldots, |\rho| \leq 1 \tag{1.2}$$

where $\{\epsilon_n\}$ is i.i.d. and ϵ_n is independent of $\{X_{n-1}, X_{n-2}, \ldots\}$. Lawrance(1978) derived the gamma and the Laplace solution of equation (1.2). Gaver and Lewis (1980) obtained the exponential solution. Jayakumar(1997) developed autoregressive model using semi $\alpha-$Laplace as marginal distribution. For more details see, Andel(1983), Dewald and Lewis(1985), Sim(1993), Seetha Lekshmi and Jose(2006).

1.2.4 Angular observations and related measures

Circular data arise in various ways. The two main ways correspond to the two principal circular measuring instruments, the *compass* and the *clock*. Typical observations measured by the compass include wind directions and directions of birds. Data of similar type arise from measurements by spirit level or protractor. Typical observations measured by the clock include the arrival times(on a 24-hour clock) of patients at a casualty unit in a hospital. Data of a similar type arise as times of year(or times of month) of appropriate events. A circular observation can be regarded as a point on a circle of unit radius, or a unit vector (that is, a *direction*)in the plane. Once an initial direction and an orientation of the circle has been chosen, each circular observation can be specified by the angle from the initial direction to the point on the circle corresponding to the observation.

Circular data arises in different fields such as earth sciences, meteorology, biology and image analysis. In meteorology, wind directions provide a natural source of circular data. A distribution of wind directions may arise either as marginal distribution of the wind speed and direction or as a conditional distribution for a given speed. Other circular data arising in meteorology include the times of day at which thunderstorms occur and the times of year at which heavy rain occurs. In earth science, spherical data arise readily as the surface of the earth is approximately a sphere. For example, in estimation of relative rotations of tectonic plates, the points on the earth's surface considered to be the observations. In biology, studies of animal navigation lead to circular data. The incidents of onsets of a particular disease(or of deaths due to the disease) at various times of year provides circular data in the medicine fields. For more details see, Mardia and Jupp(2000).

A circular distribution is a probability distribution whose total probability concentrated on the circumference of a unit circle. Since such a distribution is a way of assigning probabilities to different directions or defining a directional distribution, the range of a circular random variable Θ, measured in radians, may be taken to be $[0, 2\pi]$ or $[-\pi, \pi]$. A continuous circular probability density function $f(\theta)$ exists and has the following basic properties: (i) $f(\theta) \geq 0; \forall\theta$(ii)$\int_0^{2\pi} f(\theta)d\theta = 1$ and (iii)$f(\theta) = f(\theta + 2k\pi)$ for any integer k. The distribution function $F(\theta)$ can be defined over any interval (θ_1, θ_2) by $F(\theta_2) - F(\theta_1) = \int_{\theta_1}^{\theta_2} f(\theta)d\theta$. If an initial direction and orientation of the unit circle have been chosen(generally 0^0 and anticlockwise orientation), then $F(\theta)$ is defined as $F(\theta) = \int_0^{\theta} f(\theta)d\theta$.

The characteristic function of a circular random variable Θ having distribution function $F(\theta)$ is defined by

$$\Phi(t) = E[e^{it\Theta}] = \int_0^{2\pi} e^{it\theta} dF(\theta).$$

Since Θ is a periodic random variable having the same distribution as $\Theta + 2\pi$, the characteristic function of such a random variable has the property,

$$\Phi(t) = E[e^{it\Theta}] = E[e^{it(\Theta+2\pi)}] = e^{it2\pi}\Phi(t)$$

Hence $e^{it2\pi} = 1$, whenever there is a $\Phi(t)$ with $|\Phi(t)| \neq 0$. This suggests that for circular random variables the characteristic function needs to be defined only for integer values. Therefore, the characteristic function of a random angle Θ is the doubly-infinite sequence of complex numbers $\Phi(p) : p = 0, \pm 1, \pm 2, \cdots$ given by

$$\Phi(p) = E[e^{ip\Theta}] = \int_0^{2\pi} e^{ip\theta} dF(\theta), p = 0, \pm 1, \pm 2, \cdots \tag{1.3}$$

$$= \rho_p e^{i\mu_p^0} \tag{1.4}$$

Let us write $\Phi(p) = \alpha_p + i\beta_p$; Then

$$\alpha_p = E(\cos(p\Theta)) \qquad \text{and}$$
$$\beta_p = E(\sin(p\Theta))$$

The complex numbers $\Phi(p) : p = 0, \pm 1, \pm 2, \cdots$ are the Fourier coefficients of F (see, Feller (1971)). It is possible to write

$$dF(\theta) \sim \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \Phi(p) e^{-ip\theta}$$

If

$$\sum_{p=1}^{\infty} {\alpha_p}^2 + {\beta_p}^2 < \infty, \tag{1.5}$$

then the random variable Θ has a density f which is defined almost every where by

$$f(\theta) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \Phi(p) e^{-ip\theta}. \tag{1.6}$$

Equation (1.6) can be written as

$$f(\theta) = \frac{1}{2\pi} \left\{ 1 + 2 \sum_{p=1}^{\infty} [\alpha_p \cos(p\theta) + \beta_p \sin(p\theta)] \right\} \tag{1.7}$$

and the distribution function is given by

$$F(\theta) = \frac{1}{2\pi} \left\{ \theta + 2 \sum_{p=1}^{\infty} [\alpha_p \sin(p\theta) + \beta_p (1 - \cos(p\theta))]/p \right\} \tag{1.8}$$

The p^{th} trigonometric moment of Θ is the same as $\Phi(p)$. When $p = 1$,

$$\begin{aligned} \Phi(1) =& \alpha_1 + i\beta_1 \\ =& \rho_1 e^{i\mu_1^0} \end{aligned}$$

where

$$\begin{aligned} \mu_1^0 =& \mu = \arctan(\frac{\beta_1}{\alpha_1}) \text{ is the mean direction and} \\ \rho_1 =& \rho = \sqrt{\alpha_1{}^2 + \beta_1{}^2} \text{ is the mean resultant length.} \end{aligned}$$

The circular variance is given by, $V_0 = 1 - \rho$

The circular standard deviation is given by, $\sigma_0 = \sqrt{-2log(1 - V_0)}$

The coefficient of skewness is given by, $\zeta_1^0 = \frac{\bar{\beta_2}}{(1-\rho)^{3/2}}$.

The coefficient of kurtosis is given by, $\zeta_2^0 = \frac{\bar{\alpha_2} - \rho^4}{(1-\rho)^2}$.

1.2.5 Generalized von Mises distribution

An angular random variable Θ is said to follow the Generalized von Mises distribution over $(0, 2\pi]$ if its probability density function is

$$f(\theta) = \frac{1}{2\pi G_0(\delta, \kappa_1, \kappa_2)} \exp\{\kappa_1 \cos(\theta - \mu_1) + \kappa_2 \cos 2(\theta - \mu_2)\},$$

for $0 \in [0, 2\pi), \mu_1 \in [0, 2\pi), \mu_2 = [0, \pi), \delta = \mu_1 - \mu_2 \mod (2\pi), \kappa_1, \kappa_2 > 0$, where $G_0(\delta, \kappa_1, \kappa_2) = \int_0^{2\pi} \exp\{\kappa_1 \cos\theta + \kappa_2 \cos 2(\theta + \delta)\}\mathrm{d}\theta$
We write $\Theta \sim \mathrm{GvM}(\mu_1, \mu_2, \kappa_1, \kappa_2, \delta)$

1.2.6 Wrapped distributions

Circular distributions can be obtained by wrapping distributions on the real line around unit circle. In general, if X is any random variable on the real line, with probability density function $g(x)$, and distribution function $G(x)$, we can obtain circular random variable Θ by defining

$$\Theta \equiv X \mod (2\pi).$$

The probability density function of Θ, $f(\theta)$, is obtained by wrapping $g(x)$ around the circumference of a unit circle and summing up the overlapping points:

$$f(\theta) = \sum_{k=-\infty}^{\infty} g(\theta + 2\pi k), 0 \leq \theta \leq 2\pi.$$

The cumulative distribution function is

$$F(\theta) = \sum_{k=-\infty}^{\infty} [G(\theta + 2\pi k) - G(2\pi k)].$$

The probability density function $f(\theta)$ of the random variable θ, which has a period 2π, can be written as an infinite sum of *sine* and *cosine* functions on the interval $[0, 2\pi)$. That is,

$$f(\theta) = \frac{1}{2\pi}\left\{1 + 2\sum_{p=1}^{\infty}[\alpha_p \cos(p\theta) + \beta_p \sin(p\theta)]\right\}$$

where α_p and β_p are defined by

$$\alpha_p = \int_0^{2\pi} \cos(p\theta) dF(\theta), \text{and}$$
$$\beta_p = \int_0^{2\pi} \sin(p\theta) dF(\theta)$$

1.2.7 Wrapped variance Gamma distribution

An angular random variable Θ is said to follow the wrapped variance gamma distribution over $(0, 2\pi]$ if its probability density function is

$$f(\theta) = \frac{\gamma^{2\lambda}\exp\{\beta(\theta-\mu)\}}{\sqrt{\pi}\Gamma(\lambda)(2\alpha)^{\lambda-\frac{1}{2}}}\sum_{m=-\infty}^{\infty}\frac{\exp\{\beta m 2\pi\}K_{\lambda-\frac{1}{2}}(\alpha|\theta+2m\pi-\mu|)}{|\theta+2m\pi-\mu|^{\lambda-\frac{1}{2}}}$$

for $\theta \in [0, 2\pi), \alpha > 0, \beta > 0, 0 \leq |\beta| < \alpha, \lambda > 0, 0 \leq |\mu| < \alpha, \gamma = \sqrt{\alpha^2 - \beta^2} > 0$ where $K_\lambda(.)$ is the modified Bessel function of the third kind. We write $\Theta \sim \text{WvG}(\mu, \lambda, \alpha, \beta, \gamma)$

1.2.8 Infinite divisibility

A random variable X is said to be infinitely divisible if for every positive integer 'n', X can be written as

$$X \stackrel{d}{=} X_{n,1} + X_{n,2} + \cdots + X_{n,n}$$

where $X_{n,1}, X_{n,2}, \cdots, X_{n,n}$ are independently and identically distributed random variables. Thus the distribution function $F(x)$ of X, is said to be infinitely divisible if for every positive integer 'n', there exists a distribution function $F_n(x)$ such that

$$F(x) = \underbrace{F_n(x) * \cdots * F_n(x)}_{\text{n times}}$$

which implies that $F(x)$ is the n-fold convolution of $F_n(x)$. Equivalently, a characteristic function $\Phi(t)$ of a random variable X is said to be infinitely divisible if for every positive integer 'n', there exists a characteristic function $\phi_n(t)$ such that

$$\phi(t) = (\phi_n(t))^n.$$

For more details, see, Laha and Rohatgi (1979). Analogous to this, an angular random variable Θ (and its probability distribution) is said to be infinitely divisible if for every positive integer 'n', there exist identically and independently distributed angular random variables $\Theta_1, \Theta_2, \cdots, \Theta_n$ such that

$$\Theta \stackrel{d}{=} \Theta_1 + \Theta_2 + \cdots + \Theta_n \mod (2\pi).$$

Equivalently, if the characteristic function of Θ, ϕ_p, can be factored as

$$\phi_p = (\tilde{\phi}_p)^n, \text{ for every } n \geq 1$$

where $\tilde{\phi}_p$ is a characteristic function of Θ_1, then Θ is said to be infinitely divisible (see, Mardia (1972)).

1.2.9 Stable distributions

A random variable X is said to have a stable distribution if it has a domain of attraction, that is, if there is a sequence of independent and identically distributed random variables $Y_1, Y_2, \ldots$ and sequences of positive numbers $\{d_n\}$ and real numbers $\{a_n\}$ such that $\frac{\sum_{i=1}^n Y_i}{d_n} + a_n \xrightarrow{d} X$. Stable distribution best described by its characteristic function(see, Samorodnitski and Taqqu(1994)).

The random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2, \sigma > 0, -1 \leq \beta \leq 1$, and μ real such that its characteristic function, $\phi(t)$ has the following form:

$$\phi(t) = \exp\{-\sigma^{\alpha}|t|^{\alpha}\omega_{\alpha,\beta}(t) + i\mu t\} \tag{1.9}$$

with

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \text{sign}(x)\tan(\pi\alpha/2), & \text{if} \quad \alpha \neq 1, \\ 1 + i\beta(2/\pi)\text{sign}(x)\log|x|, & \text{if} \quad \alpha = 1. \end{cases}$$

The parameter α is the index of stability and

$$\text{sign(x)} = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

1.2.10 Geometric infinite divisibility

The concept of geometric infinite divisibility was introduced by Klebanov *et al.*(1984). A random variable X is said to be geometrically infinitely divisible if

$$X \stackrel{d}{=} \sum_{i=1}^{N_p} X_p^{(i)} \tag{1.10}$$

where N_p is a geometric random variable with probability mass function

$$P(N_p = k) = (1-p)^{k-1}p, k = 1, 2, \cdots, p \in (0, 1); \tag{1.11}$$

$X_p^{(i)}, i = 1, 2, \cdots$ are independent and identically distributed random variables and N_p and $X_p^{(i)}(i = 1, 2, \cdots)$ are independent. The relation (1.10) is equivalent to

$$\psi(t) = \sum_{k=1}^{\infty} \phi(t)^k (1-p)^{k-1} p$$
$$= \frac{p\phi(t)}{1-(1-p)\phi(t)}$$

where $\psi(t)$ and $\phi(t)$ are the characteristic functions of X and $X_p^{(i)}$ respectively. The class of geometric infinite divisible distributions is a proper subclass of infinitely divisible distributions. Klebanov *et al.*(1984) established that a distribution function F with characteristic function $\psi(t)$ is geometric infinite divisible if and only if $\exp\{1 - \frac{1}{\psi(t)}\}$ is infinitely divisible. Distributions such as exponential and Laplace are examples of geometric infinite divisible distributions. For more details (see, Klebanov *et al.*(1984), Mohan *et al.*(1993), Fujita(1993) and Pillai(1990))

Jammalamadaka and Kozubowski (2003) defined the concept of geometric infinite divisibility of an angular random variable. An angular random variable Θ is said to be geometrically infinitely divisible if there exist independent and identically distributed angular random variables $\Theta_1, \Theta_2, \cdots, \Theta_n$ such that

$$\Theta \stackrel{d}{=} \Theta_1 + \Theta_2 + \cdots + \Theta_{N_p} \mod (2\pi)$$

where N_p has the geometric distribution (1.11) and N_p and Θ_i are independent.

1.2.11 Geometric stable distributions

Geometric stable distributions arise as limiting class in the random summation scheme, when the number of terms is geometrically distributed. Let N_p be a geometric random variable with mean $1/p$:

$$P(N_p = k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

Let $Y, X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables independent of N_p. If there exist deterministic $a = a(p) > 0$ and $b = b(p) \in \Re$ such that

$$a(p) \sum_{i=1}^{N_p} (X_i + b(p)) \xrightarrow{d} Y, \quad \text{as} \quad p \to 0, \tag{1.12}$$

(see, Kozubowski(1994)), we say that the limiting random variable Y (and its distribution)is geometric stable(GS). Mittnik and Rachev(1991) obtained the one-to-one correspondence between characteristic functions of geometric stable and stable distributions: Y is geometric stable if, and only if, its characteristic function ψ has the form

$$\psi(t) = E(\exp itY) = (1 - \log \phi(t))^{-1},$$

where $\phi(t)$, the characteristic function of the stable distribution, has the expression as defined in (1.9). Therefore, the characteristic function, $\psi(t)$ of the geometric stable distribution has the following representation:

$$\psi(t) = [1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t]^{-1} \tag{1.13}$$

with

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \text{sign}(x) \tan(\pi\alpha/2), & \text{if} \quad \alpha \neq 1, \\ 1 + i\beta(2/\pi)\text{sign}(x) \log|x|, & \text{if} \quad \alpha = 1, \end{cases}$$

where $\alpha (0 < \alpha \leq 2)$ is the index of stability, $\beta(-1 \leq \beta \leq 1)$ is the skewness parameter, and $\mu \in \Re$ and $\sigma \geq 0$ control location and scale, respectively.

The most important parameter is the index α, determining the tails of a geometric stable law. In the special case $\alpha = 2$, all moments of Y exist, and the distribution is not heavy tailed. For $\alpha < 2$ the variance is infinite, and the mean is finite only if $1 < \alpha < 2$.

Strictly geometric stable distributions have the characteristic function

$$\psi(t) = [1 + \lambda|t|^{\alpha} \exp(-i\pi\alpha\tau \text{ sign(t)}/2)]^{-1}$$

where $0 < \alpha \leq 2, \lambda > 0$, and $|\tau| \leq \min(1, 2/\alpha - 1)$

1.2.12 Normal-Laplace Distribution

The normal-Laplace distribution was introduced by Reed and Jorgensen (2004), as the convolution of independent normal and Laplace random variables. Normal-Laplace distribution is a distribution which (in its symmetric form) behaves somewhat like the normal distribution in the middle of its range, and like the Laplace distribution in its tails.

A Normal-Laplace (NL) random variable X with parameters μ,σ^2,α and β can be represented as

$$X \stackrel{d}{=} Y + W \tag{1.14}$$

where Y and W are independent random variables with Y following normal distribution with mean μ and variance σ^2 and W following an asymmetric Laplace distribution with probability density function,

$$f(w) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} e^{\beta w}, & for\ w \leq 0, \\ \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha w}, & for\ w > 0, \end{cases}$$

where $-\infty < \mu < \infty$,$\sigma^2 > 0$,$\alpha > 0$ and $\beta > 0$.

The probability density function of X can shown to be

$$g(x) = \frac{\alpha\beta}{\alpha+\beta}\phi(\frac{x-\mu}{\sigma})\left[R\left(\alpha\sigma - \frac{x-\mu}{\sigma}\right) + R\left(\beta\sigma + \frac{x-\mu}{\sigma}\right)\right]$$

where $R(z) = \frac{1-\Phi(z)}{\phi(z)}$ is the Mill's ratio where $\Phi(z)$ and $\phi(z)$ are the cumulative distribution function (c.d.f.) and the probability density function of standard normal distribution.

We shall refer to this as the Normal-Laplace distribution and write $X \sim NL(\alpha, \beta, \mu, \sigma^2)$ to indicate that X follows this distribution.
A closed-form expression for the c.d.f of $NL(\alpha, \beta, \mu, \sigma^2)$ can be obtained as

$$G(x) = \Phi(\frac{x-\mu}{\sigma}) - \phi(\frac{x-\mu}{\sigma})\frac{\beta R\left(\alpha\sigma - \frac{x-\mu}{\sigma}\right) - \alpha R\left(\beta\sigma + \frac{x-\mu}{\sigma}\right)}{\alpha+\beta}$$

Since a Laplace random variable can be expressed as the difference between two exponentially distributed variates, a $NL(\alpha, \beta, \mu, \sigma^2)$ random variable, X can be expressed as

$$X \stackrel{d}{=} \mu + \sigma Z + E_1/\alpha + E_2/\beta \tag{1.15}$$

where E_1, E_2 are independent standard exponential random variables and Z is a standard normal random variable, independent of E_1 and E_2.

From the representation (1.14), it follows that the characteristic function of $NL(\alpha, \beta, \mu, \sigma^2)$ is the product of the characteristic functions of its normal and Laplace components. Precisely it is

$$\phi_X(t) = \frac{\alpha\beta \exp(i\mu t - \frac{t^2\sigma^2}{2})}{(\alpha - it)(\beta - it)} \tag{1.16}$$

It is clear that as $\sigma \to 0$, the distribution tends to an asymmetric Laplace distribution, and as $\alpha, \beta \to \infty$, it tends to a normal distribution. If only $\beta \to \infty$, the distribution is that of the sum of independent normal and Exponential components and has a fatter tail than the normal distribution in its upper tail. In this case the probability density function is

$$g_1(x) = \alpha\phi(\frac{x-\mu}{\sigma}) \left[R\left(\alpha\sigma - \frac{x-\mu}{\sigma}\right)\right].$$

If only $\alpha \to \infty$, the distribution exhibits extra-normal variation only in the lower tail and the probability density function is

$$g_2(x) = \beta\phi(\frac{x-\mu}{\sigma}) \left[R\left(\beta\sigma + \frac{x-\mu}{\sigma}\right)\right]$$

$NL(\alpha, \beta, \mu, \sigma^2)$ probability density function can be represented as a mixture of the above probability density functions as

$$g(x) = \frac{\beta}{\alpha+\beta} g_1(x) + \frac{\alpha}{\alpha+\beta} g_2(x)$$

The symmetric NL distribution arises when $\alpha = \beta$, with probability density function

$$g(x) = \frac{\alpha}{2}\phi(\frac{x-\mu}{\sigma}) \left[R\left(\alpha\sigma - \frac{x-\mu}{\sigma}\right) + R\left(\alpha\sigma + \frac{x-\mu}{\sigma}\right)\right]$$

Reed and Jorgensen(2004) also introduced a generalized normal-Laplace distribution, which is useful in financial applications for obtaining an alternative stochastic process model to Brownian motion for logarithmic prices, in which the increments exhibit fatter tails than the normal distribution. Reed (2007) developed Brownian-Laplace motion for modelling financial asset price returns.

1.2.13 Mittag-Leffler distribution

The function $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}$ was first introduced by Mittag-Leffler in 1903. Many properties of the function follow from Mittag-Leffler integral representation

$$E_\alpha(z) = \frac{1}{2\pi i}\int_C \frac{t^{\alpha-1}e^t}{t^\alpha - z}$$

where the path of integration C is a loop which starts and ends at $-\infty$ and encircles the circular disc $|t| \le z^{\frac{1}{\alpha}}$. Pillai(1990a) proved that

$$F_\alpha(x) = 1 - E_\alpha(-x^\alpha) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{k\alpha}}{\Gamma(1+k\alpha)}, x \ge 0, 0 < \alpha \le 1$$

are distribution functions, having Laplace transforms $\psi(t) = (1+t^\alpha)^{-1}, t > 0$. He called $F_\alpha(x)$, for $0 < \alpha < 1$, a Mittag-Leffler distribution. The Mittag-Leffler distribution is a generalization of the exponential distribution, since for $\alpha = 1$, we get exponential distribution. Mittag-Leffler distributions can also be used as waiting-time distributions as well as first-passage time distributions for certain renewal processes.

Pillai(1985) developed $\alpha-$Laplace distribution with characteristic function given by $(1+|t|^\alpha)^{-1}, 0 < \alpha \le 2$. This distribution is also known as Linnik distribution. Jose *et al.*(2010) introduced generalized Mittag-Leffler distribution and developed first order autoregressive processes with generalized Mittag-Leffler marginals. A random variable with support over $(0,\infty)$ is said to follow the generalized Mittag-Leffler distribution with parameters α and β if its Laplace transform is given by

$$\psi(t) = (1+t^\alpha)^{-\beta}, 0 < \alpha \le 1, \beta > 0, t > 0. \qquad (1.17)$$

The cumulative distribution function corresponding to (1.17) is given by

$$F_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\beta+k) x^{\alpha(\beta+k)}}{k! \Gamma(\beta) \Gamma(1+\alpha(\beta+k))}$$

It easily follows that when $\beta = 1$, we get Pillai's Mittag-Leffler distribution. When $\alpha = 1$, we get the gamma distribution. When $\alpha = 1, \beta = 1$ we get the exponential distribution. This family may be regarded as the positive counterpart of Pakes generalized Linnik distribution characterized by the characteristic function

$$(1 + |t|^{\alpha})^{-\beta}, 0 < \alpha \leq 2, \beta > 0.$$

(see, Pakes(1998)).

1.2.14 Geometric Mittag-Leffler distribution

The geometric exponential distribution(GED (μ))introduced by Pillai(1990b) has Laplace transform given by

$$\left[1 + \log(1 + \mu t)\right]^{-1}$$

Pillai (1990b) developed renewal processes with geometric exponential as waiting time distribution. Geometric exponential distribution can be extended to obtain the geometric gamma distribution denoted by GGD(μ, λ)whose Laplace transform is

$$\left[1 + \lambda \log(1 + \mu t)\right]^{-1}.$$

We say that a random variable X on $[0,\infty)$ has the geometric Mittag-Leffler distribution and write $X \stackrel{d}{=} GML(\alpha)$ if it has the distribution function

$$F_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^\infty \frac{\Gamma(k+t) x^{\alpha(k+t)} e^{-t}}{\Gamma(t)\Gamma(1+\alpha(k+t))}.$$

The Laplace transform of $F_\alpha(x)$ is

$$\phi_\alpha(t) = E(e^{-tX}) = \frac{1}{1+\log(1+t^\alpha)}, 0 < \alpha \leq 1, t > 0.$$

Note that $\alpha = 1$, we get the geometric exponential distribution having density function

$$g(x) = e^{-x} \int_0^\infty \frac{e^{-t} x^{t-1}}{\Gamma(t)} dt$$

(see, Jayakumar and Ajitha(2003)). Jose *et al.*(2010) introduced geometric generalized Mittag-Leffler distributions having the Laplace transform

$$(1+t^\alpha)^{-\beta}, 0 < \alpha \leq 1, \beta > 0.$$

and discussed the applications in various areas like astrophysics, space sciences, meteorology, financial modeling and reliability modeling. Seetha Lekshmi and Jose(2006) introduced geometric Pakes generalized Linnik distribution and studied its properties. A random variable X on $(-\infty,\infty)$ is said to follow geometric Pakes generalized Linnik distribution and write $X \stackrel{d}{=} GPGLD(\alpha,\lambda)$ if it has the characteristic function

$$\frac{1}{1+\lambda\log(1+|t|^\alpha)}, 0 < \alpha \leq 2, \lambda > 0.$$

If $\lambda = 1$, geometric Pakes generalized Linnik distribution reduces to geometric α-Laplace distribution.

1.2.15 Multivariate Laplace distributions

Multivariate Laplace distribution is an important stochastic model that accounts for asymmetry and heavier than Gaussian tails, while still ensuring the existence of the second moments. A d-dimensional random vector $\mathbf{X} \in \Re^d$ is said to follow multivariate symmetric Laplace laws, with parameter $\mathbf{\Sigma}$ if it has the characteristic function

$$\phi(\mathbf{t}) = \frac{1}{1 + \frac{1}{2}\mathbf{t}'\mathbf{\Sigma}\mathbf{t}},$$

where $\mathbf{t} \in \Re^d, \mathbf{\Sigma}$ is a dxd nonnegative definite matrix(see, Kotz *et al.*(2001)).

Asymmetric Laplace laws can be defined in various equivalent ways, which we express in the form of their characterizations and representations. Their significance comes from the fact that they are the only distributional limits for (appropriately normalized) random sums,

$$\mathbf{X}^{(\mathbf{1})} + \mathbf{X}^{(\mathbf{2})} + \ldots + \mathbf{X}^{(N_p)} \tag{1.18}$$

of independent and identically distributed random vectors (r.v.'s) with finite second moments, where N_p has geometric distribution with the mean 1/ p (independent of $X^{(i)}$'s):

$$P(N_p = k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

and p converges to zero (see, Mittnik and Rachev (1991)). Since the sums such as (1.18) frequently appear in many applied problems in biology, economics, insurance mathematics, reliability, and other fields, asymmetric Laplace distributions should have a wide variety of applications. In particular this class seems to be suitable for modeling heavy-tailed asymmetric multivariate data for which one is reluctant to sacrifice the property of finiteness of moments.

A random vector $\mathbf{Y} \in \mathfrak{R}^d$ is said to have a multivariate asymmetric Laplace distribution, if its characteristic function is given by

$$\psi(\mathbf{t}) = \frac{1}{1 + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t} - i\boldsymbol{\mu}'\mathbf{t}},$$

where $\boldsymbol{\mu} \in \mathfrak{R}^d$ and $\boldsymbol{\Sigma}$ is a dxd non negative definite symmetric matrix(see, Kozubowski and Podgórski(2000))

Multivariate extension of the normal-Laplace distribution of Reed and Jorgenson(2004), namely multivariate normal-Laplace distribution, introduce in Jose and Manu(2014)as the convolution of multivariate normal(with parameters $\boldsymbol{\eta}$ and $\boldsymbol{\mathcal{T}}$) and multivariate asymmetric Laplace (with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$). The ch.f of multivariate normal-Laplace distribution is given by

$$\exp\{i\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\} \left[1 + \frac{1}{2}\left(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\right) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-1}, \mathbf{t}, \boldsymbol{\mu}, \boldsymbol{\eta} \in \Re^{\mathbf{p}}, \boldsymbol{\mathcal{T}} > \mathbf{0}, \boldsymbol{\Sigma} > \mathbf{0}.$$

1.3 Slash Distributions

Kafadar(1988) proposed slash normal distribution, which is a heavy tailed alternative to the normal distribution. Wang and Genton(2006) generalized the univariate slash normal distribution to multivariate slash normal and introduced multivariate skew slash distribution. The standard slash normal distribution is obtained as the distribution of the ratio $Y = \frac{X}{U^{1/q}}$, where X is a standard normal random variable, U is an independent uniform random variable over the interval (0,1) and $q > 0$. As $q \to \infty$, we get the standard normal distribution. For $q = 1$, it has the probability density function,

$$f(y,1) = \begin{cases} \frac{\phi(0)-\phi(y)}{y^2}, & x \neq 0, \\ \frac{\phi(0)}{2}, & x = 0, \end{cases}$$

where $\phi(.)$ is the probability density function of standard normal distribution.

Tan and Peng(2005) have introduced slash Student's t and skew slash Student's t distributions and studied their properties. Its probability density function is,

$$f(y;m,q) = \int_0^1 u^{1/q} f(yu^{1/q};m)du$$

where $f(.)$ denotes the probability density function of the Student's t distribution with m degrees of freedom.

The slash distributions are widely used in simulation studies and robust procedures for statistical analysis.

1.4 Objectives of the Study

The present study has been undertaken with the following objectives:

1. To introduce generalization of univariate geometric stable distributions and study its properties

2. To develop estimation procedures for parameters of geometric stable and generalized geometric stable distributions.

3. To introduce distributions related to geometric stable distributions and develop autoregressive time series models using these distributions.

4. To introduce normal-geometric stable models and its g.i.d versions and to derive autoregressive time series models.

5. To introduce circular versions of new models and study the properties.

6. To extend the models to multivariate case viz multivariate generalized geometric stable and multivariate generalized normal geometric stable distributions, study their properties and to develop related processes.

7. Apply the distributions to real data sets and compare the performances of new models.

1.5 Summary of the Present Work

The thesis is organized into seven chapters. Chapter 1 is introductory which gives preliminary concepts to the topic of research such as self-decomposability, geometric infinite divisibility, circular data, wrapped distributions, stable distributions etc. The concepts of time series and non-Gaussian autoregressive models are discussed. We also consider Gaussian non-Gaussian distributions. Recents works on geometric stable laws are presented.

In Chapter 2, we study the geometric stable distributions and its properties. Representation of geometric stable variate is developed for simulation. We derived the moments of the log-transformed GS random variable. Distribution of weighted sum of independent geometric stable variables is obtained. Estimation of geometric stable parameters based on log-moments is done. Asymptotic normality of the estimates are discussed. We introduced a generalization of geometric stable(GGS) distributions and its distributional properties are studied. Histograms for different parametric values are presented. The special cases of GGS distributions mentioned. The absolute and signed fractional order moments of GGS random variables are derived. Generalized strictly geometric sable studied and first order autoregressive process of its g.i.d versions developed. Moments of the log-transformed

GGS variables and weighed sums of GGS independent random variables are derived. We also extended the parameter estimation of geometric stable disributions based on log-moments to the the parameters of GGS distributions and a simulation study conducted to check the performance of the estimation techniques. We here further extended the GGS distributions to generalized normal geometric stable distributions(GNGS) and studied its properties.

Geometric GGS distributions(GeoGGS) introduced in Chapter 3 and discussed its different properties. First order autoregressive process with GeoGGS marginals developed and extended it to k^{th} order. We also introduced Geometric GNGS distributions(GeoGNGS) and its properties are discussed. Autoregressive time series models with GeoGNGS marginals developed.

Circular distributions studied in Chapter 4. Wrapped versions of GGS (WGGS))distributions introduced and different measures including trigonometric moments derived. The problem of estimation of parameters is considered. WGGS distributions further generalized to wrapped generalized normal geometric stable(WGNGS) distributions. A representation of WGNGS derived and its infinite divisibility property is also established. Different measures including trigonometric moments and other parametrs are derived. The maximum likelihood procedure developed for the estimation of parameters.

Chapter 5 is devoted to multivariate extensions. A multivariate generalization of GGS distributions introduced and its properties discussed. A representation of multivariate GGS random vector is presented. Multivariate slash generalized geometric stable distributions is introduced. The multivariate geometric generalized geometric stable distributions introduced and its properties are studied. First order autoregressive process with multivariate

GeoGGS marginals derived and extended it to k^{th} order. Multivariate GNGS introduced and its properties studied. Multivariate slash GNGS introduced. Multivariate geometric GNGS distributions introduced and its properties are studied. First order autoregressive process with multivariate GeoGNGS marginals is derived

Applications of new models in various context discusses in Chapter 6.

Recommendations are presented in Chapter 7.

The results of this thesis have been presented in various National and International conferences and have been published/ submitted for publication of research papers in National/ International journals which are listed below.

Papers Presented in National/International Conferences

1. 'q-Geometric Stable Distributions and Processes' in the International Conference on Statistics for Twenty-first Centuary-2015 (ICSTC-2015) organized by the Department of Statistics, University of Kerala, Trivandrum, Kerala, India during 17-19, December 2015.

2. 'A Generalization of Mittag-Leffler Distributions and Related Processes' in the National Conference on Advances in Statisical Sciences and Annual Conference of the Kerala Statistical Association held in the Department of Statistical Science, Kannur University, Kerala, India during 17-18, February 2017.

3. 'Density Parameter Estimation of Skewed Geometric stable Distributions' in the National Seminar on Innovative Appproches in

Statistics organized by the Department of Statistics, St. Thomas'College (Autonomous), Thrissur, Kerala, India during 15-17 February 2018.

4. 'Wrapped Generalized Geometric Stable Distributions' in the National Seminar on Recent Trends in Statistical Science(RTSS-2019) and 40th Annual Conference of Kerala Statistical Association organized by the Department of Statistics, University of Kerala, Trivandrum, Kerala, India during 07-09 March, 2019.

Publications

1. K. Jayakumar and T. Sajayan(2020). On Estimation of Geometric Stable Distributions, *Journal of the Indian Society for Probability and Statistics* **21**, 329–347.

2. T. Sajayan and K. Jayakumar(2022). A pathway model of Mittag-Leffler distributions and related processes, *Far East Journal of Theoretical Statistics* **65**, 55-70.

3. K. Jayakumar and T. Sajayan(2022). Wrapped Generalized Geometric Stable Distributions with an Application to Wind Direction, *Far East Journal of Theoretical Statistics*, accepted.

CHAPTER 2

PARAMETRIC ESTIMATION AND UNIVARIATE GENERALIZATION OF GEOMETRIC STABLE DISTRIBUTIONS

2.1 Introduction

The Geometric stable distributions best described by its characteristic function. The characteristic function $\psi(t)$ of a $GS(\alpha, \beta, \sigma, \mu)$ random variable U is

$$\psi(t) = (1 + \sigma^{\alpha}|t|^{\alpha}\omega_{\alpha,\beta}(t) - i\mu t)^{-1} \tag{2.1}$$

with

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \operatorname{sign}(x) \tan(\pi\alpha/2), & \text{if} \quad \alpha \neq 1, \\ 1 + i\beta \ (2/\pi) \operatorname{sign}(x) \log|x|, & \text{if} \quad \alpha = 1, \end{cases} \tag{2.2}$$

and

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x = 0, \\ -1, & \text{if} \quad x < 0. \end{cases} \tag{2.3}$$

where $\alpha(0 < \alpha \leq 2)$ is the index of stability, $\beta(-1 \leq \beta \leq 1)$ is the skewness parameter, and $\mu \in \Re$ and $\lambda \geq 0$ control location and scale, respectively. The most important parameter is the index α, determining the tails of a geometric stable law. In the special case $\alpha = 2$, all moments of Y exist, and the distribution becomes asymmetric Laplace.

The geometric stable random variable U has the representation

$$U = \begin{cases} \mu Z + Z^{\frac{1}{\alpha}} X, & \text{if} \quad \alpha \neq 1, \\ \mu Z + ZX + \sigma Z\beta(2/\pi)\log(\sigma Z), & \text{if} \quad \alpha = 1, \end{cases} \tag{2.4}$$

where $X \ \sim \text{S}(\alpha, \beta, \sigma, 0)$(see, Samorodnitski and Taqqu(1994)), Z is unit exponential with distribution $F_Z(x) = 1 - \exp(-x), x > 0$, and X and Z are independent. We write $U \ \sim \text{GS}(\alpha, \beta, \sigma, \mu)$. Note that the characteristic function of the random variable X having stable distribution $\text{S}(\alpha, \beta, \sigma, \mu)$ is

$$\phi(t) = \exp\{-\sigma^{\alpha}|t|^{\alpha}\omega_{\alpha,\beta}(t) + i\mu t\} \tag{2.5}$$

with $\omega_{\alpha,\beta}(x)$ and sign(x) as in (2.2)and (2.3) respectively, where $0 < \alpha \leq 2, \sigma > 0, -1 \leq \beta \leq 1$, and $\mu \in \Re$. Mittnik and Rachev(1991) showed the one-to-one correspondence between characteristic functions of geometric stable and stable distributions: Y is geometric stable if, and only if, its characteristic function ψ has the form

$$\psi(t) = E(\exp itY) = (1 - \log \phi(t))^{-1} \tag{2.6}$$

where $\phi(t)$ is the characteristic function of the stable variable X as defined in (2.5).

Mittag-Leffler and Linnik distributions are the two special cases of geometric stable laws, studied extensively in recent years(see, Jayakumar and Pillai(1993), Kozubowski(2001) and Jayakumar *et al.*(2010)). Its generalizations and applications to financial data are studied by different authors.

In the section below we present a representation for simulation geometric stable random variable

2.2 Simulation of GS distributions

The most widely used technique of simulation of random variables is the inversion method. It is based on the following fact: if a random variable Y has distribution function F, then

$$Y \stackrel{d}{=} F^{-1}(U),$$

where F^{-1} is a function inverse to the distribution function F and U is a uniform random variable on (0,1). In the above representation of geometric stable random variable, we should simulate the stable random variable X to simulate the geometric stable random varaible U. The inversion method is not suitable in the case of stable laws since there is no analytic expressions for F of stable laws, except for few special cases.

However, we can represent a stable random variable as a function of two independent random variables (uniform and exponential) (see, Chambers et al.(1976)). Then, via the representation of geometric stable laws given by (2.4) we should be able to express a geometric stable random variable U as follows(see, Kozubowski and Rachev(1994)):

$$U = \begin{cases} \mu Z + (Z/L)^{\frac{1}{\alpha}} L\sigma H_{\alpha\beta}(\pi(S-\frac{1}{2})), & \alpha \neq 1, \\ \mu Z + \sigma Z K_{\alpha\beta}(\pi(S-\frac{1}{2}), L) + \sigma Z \beta (2/\pi)\log(\sigma Z), & \alpha = 1, \end{cases} \tag{2.7}$$

where Z, L and S are independent with Z, $L \sim \exp(1)$ and $S \sim U(0,1)$. $H_{\alpha\beta}(x)$ and $K_{\alpha\beta}(x,y)$ are defined as

$$H_{\alpha\beta}(x) = \frac{\sin[\alpha(x-c)]}{(\cos x)^{1/\alpha}}(\cos[x-\alpha(x-c)])^{\frac{1-\alpha}{\alpha}}, \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}), \tag{2.8}$$

with

$$c = c(\alpha,\beta) = \begin{cases} -\beta.(\frac{1}{2}\pi)K(\alpha)/\alpha, & \text{if} \quad \alpha < 1, \\ \beta.(\frac{1}{2}\pi)K(\alpha)/\alpha, & \text{if} \quad \alpha > 1, \end{cases}$$

where $K(\alpha) = \min(\alpha, 2-\alpha)$, and

$$K_{\alpha\beta}(x,y) = \frac{2}{\pi}\left((\frac{1}{2}\pi + \beta x)\tan x - \beta \log(\frac{\frac{1}{2}\pi y \cos x}{\frac{1}{2}\pi + \beta x}) \right), \; y > 0, \; x \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{2.9}$$

In the present chapter, we focus on properties of the geometric stable distributions and the problem of estimation of its parameters. A representation of geometric stable variate for the purpose of simulation developed. We derive the moments of the log-transformed geometric stable random variable. Distribution of weighted sum of independent geometric stable variables developed. Based on the moments of log-transformed variable and weighted sum property, we developed an estimation technique for the parameters of geometric stable distributions. Asymptotic normality of parameters is discussed.

2.3 Moments of the log-transformed geometric stable random variable U'

We derive the log-moments of the random variable U in (2.4). Applying the log-transformation to the mixture representation (2.4) for the case $\mu = 0$ and $\alpha \neq 1$, we obtain

$$U' = \frac{1}{\alpha} Z' + X' \tag{2.10}$$

where $U' = \log|U|, Z' = \log(Z),$ and $X' = \log|X|$

It is straight forward to show the following non-central moments of the random variable Z':(see Cahoy(2013))

$$\mathbf{E}(Z') = -\mathbb{C}, \qquad \mathbf{E}(Z'^2) = \mathbb{C}^2 + \frac{\pi^2}{6} = \mathbb{C}^2 + \psi_1$$
$$\mathbf{E}(Z'^3) = -\mathbb{C}^3 - \frac{\mathbb{C}\pi^2}{2} - 2\zeta(3), \qquad \mathbf{E}(Z'^4) = \mathbb{C}^2(\mathbb{C}^2 + \pi^2) + \frac{3\pi^4}{20} + 8\mathbb{C}\zeta(3)$$

where $\mathbb{C} = 0.57721566\cdots = -\psi_0,\ \psi_1 = \frac{\pi^2}{6}, \zeta(3) = 1.2020569\ldots$

Here $\mathbb{C}$ is the Euler's constant and ψ_0 is the digamma function evaluated at 1. The functions ψ_1, ψ_2 are the first and second differentials of $\psi(\tau)$ (or polygamma) evaluated at 1, and $\zeta(\tau)$ is the Riemann zeta function. The following are the log-moments of the random variable $X \sim S(\alpha, \beta, \sigma, 0)$ (see Kuruoğlu(2001)):

$$L_1 = \mathbf{E}(X') = \psi_0\left(1 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\log\left|\frac{\gamma}{\cos\theta}\right| \tag{2.11}$$

$$L_2 = \mathbf{E}(X' - L_1)^2 = \psi_1\left(\frac{1}{2} + \frac{1}{\alpha^2}\right) - \frac{\theta^2}{\alpha^2} \tag{2.12}$$

$$L_3 = \mathbf{E}(X' - L_1)^3 = \psi_2\left(1 - \frac{1}{\alpha^3}\right) \tag{2.13}$$

where $\gamma = \sigma^\alpha$ and $\theta = \arctan(\beta\tan(\frac{\alpha\pi}{2}))$.

Taking expectation of (2.10) and using the above moments, we get the mean and variance

$$L_1' = \mathbf{E}(U') = -\mathbb{C} + \frac{1}{\alpha}\log\left|\frac{\kappa}{\cos\theta}\right| \quad \text{and} \quad L_2' = \mathbf{V}(U') = \frac{\pi^2}{\alpha^2 3} + \frac{\pi^2}{12} - \frac{\theta^2}{\alpha^2}. \tag{2.14}$$

A similar calculation yields the third central moment as

$$L_3' = \mathbf{E}(U' - L_1')^3 = \psi_2\left(1 - \frac{3}{\alpha^3}\right). \tag{2.15}$$

Note that higher order moments $L_4{}', L_5{}', \ldots$ can be calculated in a similar manner.

2.4 Weighted Sums of Independent GS Variates

Let $Y_k \sim GS(\alpha, \beta, \sigma, \mu)$ be independent geometric stable variates that are identically distributed. Then the distribution of a weighted sum of these variables with the weights a_k can be derived using a set of $S(\alpha, \beta, \sigma, 0)$ random variables X_k. Define $T = \sum a_k X_k$ and $Y_k = \mu Z + Z^{\frac{1}{\alpha}} X_k$. Then

$$T = \sum_{k=1}^{n} a_k X_k \sim S\left(\alpha, \frac{\sum_{k=1}^{n} a_k{}^{\langle\alpha\rangle}}{\sum_{k=1}^{n} |a_k|^{\alpha}}\beta, \sum_{k=1}^{n} |a_k|^{\alpha}\sigma, 0\right),$$

where $x^{\langle p\rangle} = \text{sign}(x)|x|^p$ (see, Kuruoğlu(2001)) and $Z \sim \exp(1)$ and is independent of X_k. Then,

$$\begin{aligned}
\sum a_k Y_k &= \sum a_k \left(\mu Z + Z^{\frac{1}{\alpha}} X_k\right) \\
&= \left(\sum_{k=1}^{n} a_k \mu\right) Z + Z^{\frac{1}{\alpha}} \left(\sum_{k=1}^{n} a_k X_k\right) \\
&\sim GS\left(\alpha, \frac{\sum_{k=1}^{n} a_k{}^{\langle\alpha\rangle}}{\sum_{k=1}^{n} |a_k|^{\alpha}}\beta, \sum_{k=1}^{n} |a_k|^{\alpha}\sigma, \sum_{k=1}^{n} a_k \mu\right).
\end{aligned}$$

This provides a convenient way to generate sequences of independent geometric stable random variables with $\mu = 0$, $\beta = 0$, or with zero values for both μ and β(except when $\alpha = 1$). We call these the centered, deskewed and symmetrized sequences, respectively:

$$\begin{aligned}
Y_k{}^{C} &= Y_{3k} + Y_{3k-1} - 2Y_{3k-2} \\
&\sim GS\left(\alpha, \left[\frac{2 - 2^{\alpha}}{2 + 2^{\alpha}}\right]\beta, [2 + 2^{\alpha}]\sigma, 0\right),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
Y_k{}^{D} &= Y_{3k} + Y_{3k-1} - 2^{1/\alpha} Y_{3k-2} \\
&\sim GS\left(\alpha, 0, 4\sigma, [2 - 2^{1/\alpha}]\mu\right),
\end{aligned} \tag{2.17}$$

$$Y_k{}^{S} = Y_{2k} - Y_{2k-1} \sim GS\left(\alpha, 0, 2\sigma, 0\right). \tag{2.18}$$

Using such sequences, we may apply methods for symmetric variates to skewed variates and we may apply skew-estimation methods for centered variates to noncentered variates, with the effective loss of some sample.

2.5 Estimation for $GS(\alpha, \beta, \sigma, \mu)$ distributions

We discuss in this section the issue of parameter estimation of geometric stable laws. Kozubowski(1999) proposed an estimation procedure for the parameters of geometric stable distributions based on empirical characteristic function. The draw back of the method include the lack of optimality properties for estimators, and possible difficulties with choosing the required constants. We utilizes here the concept of Cahoy(2013)for estimation, where the method of moments, based on moments of log-transformed random variable for the estimation of parameters Mittag-Leffler distributions. Here we use the moments of the random variable U' defined in Section 2.3 for the estimation of parameters of geometric stable laws. Equating the sample moments and actual moments, we may readily solve for the characteristic exponent α using the L_3' and θ from L_2' by substituting estimate of α. The estimate of γ obtained from L_1' by substituting both the estimate of α and θ. However estimation based on higher order moments is not a good practice. We therefore adopt the centro-symmetrization procedure; therefore we solve for α using L_2'. This α estimate may then be used to solve the L_2' of the skewed process for the skew parameter β. Similarly, L_1' is solved for σ.

The resulting estimators may be summarised as follows:

Logarithmic estimator for α: Apply centro-symmetrization as given by equation (2.18) to the observed data to obtain transformed data.

Estimate ${L_2}'$ for the transformed data and hence, estimate

$$\hat{\alpha} = \left(\frac{3}{\pi^2} \left({L_2}' - \frac{\pi^2}{12} \right) \right)^{-1/2}. \tag{2.19}$$

Logarithmic estimator for β: Assume an estimate of α is available and that data with $\mu = 0$ has been obtained, by centering as given by equation (2.16). Estimate ${L_2}'$ for the data, and hence

$$|\theta| = \left(\alpha^2 \left(\frac{\pi^2}{12} - {L_2}' \right) + \frac{\pi^2}{3} \right)^{1/2}. \tag{2.20}$$

Estimate β using $\beta = \frac{\tan\theta}{\tan(\alpha\pi/2)}$. Since we have applied centering, it is necessary to transform the resulting β by dividing by $\frac{2-2^{\hat{\alpha}}}{2+2^{\hat{\alpha}}}$

Logarithmic estimator for σ: As for β estimate, we assume data with $\mu = 0$. Estimate ${L_1}'$ for this data, and hence

$$\hat{\gamma} = \cos(\theta) \exp \left\{ \alpha \left({L_1}' + \mathbb{C} \right) \right\} \tag{2.21}$$

and hence

$$\hat{\sigma} = (\hat{\gamma})^{1/\hat{\alpha}}.$$

Since, we apply centering, transform the resulting σ by dividing $2 + 2^{\hat{\alpha}}$ to obtain the actual estimate of σ.

2.6 Interval estimation for $GS(\alpha, \beta, \sigma, \mu)$ distribution

We study the limiting distribution of our estimator $\hat{\alpha}$ and $\hat{\sigma}$ from the geometric stable distribution $GS(\alpha, 0, \sigma, 0)$ for $\alpha \neq 1$. If we let

$$\hat{L_1}' = \hat{\mu_{U'}} = \frac{1}{n}\sum_{j=1}^{n} U_j{}' \text{ and } \hat{L_2}' = \hat{\sigma_{U'}{}^2} = \sum_{j=1}^{n}\left(U_j' - \overline{U}\right)^2/n,$$

then, the standard two dimensional central limit theorem implies, as $n \to \infty$, the following convergence,

$$\sqrt{n}\begin{pmatrix} \hat{L_1}' - L_1{}' \\ \hat{L_2}' - L_2{}' \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} L_2{}' & L_3{}' \\ L_3{}' & L_4{}' - L_2{}'^2 \end{pmatrix}\right),$$

where $L_1{}', L_2{}', L_3{}', L_4{}'$ are the moments defined in section 2.3.

Now to show the asymptotic normality of the estimators, we use Cramer's theorem (see, Ferguson(1996)). Let

$$g(L_1{}', L_2{}') = g(\mu_{U'}, \sigma_{U'}{}^2) = \exp(L_1{}' + \mathbb{C}).$$

Then the gradient becomes $\dot{g}(L_1', L_2') = \begin{pmatrix} \exp(L_1' + \mathbb{C}) \\ 0 \end{pmatrix}$. This implies that $\sqrt{n}(\hat{\sigma} - \sigma) \xrightarrow{d} N(0, {\sigma_\sigma}^2)$ where

$$\begin{aligned} {\sigma_\sigma}^2 =& \dot{g}(L_1', L_2')' \begin{pmatrix} L_2' & L_3' \\ L_3' & L_4' - {L_2'}^2 \end{pmatrix} \dot{g}(L_1', L_2') \\ =& \left(\frac{\pi^2}{3\alpha^2} + \frac{\pi^2}{12}\right) \exp(2(L_1' + \mathbb{C})) \\ =& \left(\frac{\pi^2}{3\alpha^2} + \frac{\pi^2}{12}\right) \sigma^2. \end{aligned}$$

Similarly, $\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, {\sigma_\alpha}^2)$, where

$$\begin{aligned} {\sigma_\alpha}^2 =& \left(\frac{-12\pi}{(12L_2' - \pi^2)^{3/2}}\right)^2 \left(L_4' - {L_2'}^2\right) \\ =& \frac{144\pi^2}{(12L_2' - \pi^2)^3} \left(L_4' - {L_2'}^2\right). \end{aligned}$$

The above expression for ${\sigma_\alpha}^2$ obtained by substituting

$$g(L_1', L_2') = \frac{2\pi}{\sqrt{12L_2' - \pi^2}}$$

and

$$\dot{g}(L_1', L_2') = \begin{pmatrix} 0 \\ \frac{-12\pi}{(12L_2' - \pi^2)^{3/2}} \end{pmatrix}.$$

Therefore, we have shown that our estimates are normally distributed (asymptotically unbiased) as the sample size n goes large. Consequently, we can approximate the $(1 - \epsilon)\%$ confidence interval for α and σ as $\hat{\alpha} \pm z_{\epsilon/2}\sqrt{\frac{\hat{\sigma_\alpha}^2}{n}}$ and $\hat{\sigma} \pm z_{\epsilon/2}\sqrt{\frac{\hat{\sigma_\sigma}^2}{n}}$ respectively, where

$$\hat{\sigma_\sigma}^2 = \left(\frac{\pi^2}{3\hat{\alpha}^2} + \frac{\pi^2}{12}\right) \sigma^2$$

and

$$\hat{\sigma_\alpha}^2 = \frac{144\pi^2}{(12L_2{}' - \pi^2)^3}\left(\hat{L_4}' - \hat{L_2}'^2\right),$$

$z_{\epsilon/2}$ is the $(1-\epsilon/2)^{th}$ quantile of the standard normal distribution, and $0 < \epsilon < 1$.

2.7 Generalized geometric stable distributions

Now we introduce and study, a new class of distributions called generalized geometric stable(GGS) distributions.

Definition 2.7.1. *A random variable V is said to have generalized geometric stable distribution* $GGS(\lambda, \alpha, \beta, \sigma, \mu)$ *if there are parameters* $0 < \alpha \leq 2, \lambda > 0, -1 \leq \beta \leq 1, \sigma > 0$, *and* μ *real such that its characteristic function,* $\phi(t)$ *has the following form:*

$$\phi(t) = [1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t]^{-\lambda} \tag{2.22}$$

where

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta sign(x)\tan(\pi\alpha/2), & if \quad \alpha \neq 1, \\ 1 + i\beta(2/\pi) sign(x)\log|x|, & if \quad \alpha = 1. \end{cases}$$

and

$$sign(x) = \begin{cases} 1, & if \quad x > 0, \\ 0, & if \quad x = 0, \\ -1, & if \quad x < 0. \end{cases}$$

Special cases: GGS laws becomes geometric stable laws when $\lambda = 1$. For $\beta = 0$ and $\mu = 0$ it becomes generalized Linnik (see Pakes(1998))and Linnik if $\lambda = 1$, $\beta = 0$ and $\mu = 0$. Generalized Mittag-Leffler (see Jose *et al.*(2010))distributions, which are GGS with $\lambda \neq 1$, $0 < \alpha < 1$, $\sigma =$

$\sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha}$, $\beta = 1$ and $\mu = 0$.

Detailed list of special cases of GGS laws is presented in the Table 2.1.

Distribution	Charcteristic function	Parametric values
Geometric Stable	$[1+\sigma^\alpha\lvert t\rvert^\alpha\omega_{\alpha,\beta}(t)-i\mu t]^{-1}$	$0<\alpha\le 2, \lambda=1, -1\le\beta\le 1,$ $\sigma>0, \mu\in \mathrm{R}$
Generalized Linnik	$[1+\sigma^\alpha\lvert t\rvert^\alpha]^{-\lambda}$	$0<\alpha\le 2, \lambda>0, \beta=0,$ $\sigma>0, \mu=0$
Linnik	$[1+\sigma^\alpha\lvert t\rvert^\alpha]^{-1}$	$0<\alpha\le 2, \lambda=1, \beta=0,$ $\sigma>0, \mu=0$
Generalized Asymmmetric Laplace	$[1+\sigma^2\lvert t\rvert^2-i\mu t]^{-\lambda}$	$\alpha=2, \lambda>0, \beta=0,$ $\sigma>0, \mu\in \mathrm{R}$
Generalized Symmmetric Laplace	$[1+\sigma^2\lvert t\rvert^2]^{-\lambda}$	$\alpha=2, \lambda>0, \beta=0,$ $\sigma>0, \mu=0$
Asymmmetric Laplace	$[1+\sigma^2\lvert t\rvert^2-i\mu t]^{-1}$	$\alpha=2, \lambda=1, \beta=0,$ $\sigma>0, \mu\in \mathrm{R}$
Symmetric Laplace	$[1+\sigma^2\lvert t\rvert^2]^{-1}$	$\alpha=2, \lambda=1, \beta=0,$ $\sigma>0, \mu=0$
Generalized Mittag-Leffler	$[1+\sigma^\alpha(-it)^\alpha]^{-\lambda}$	$0<\alpha<1,\ \sigma=\sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha},$ $\beta=1$ and $\mu=0,\ \lambda>0$
Mittag-Leffler	$[1+\sigma^\alpha(-it)^\alpha]^{-1}$	$0<\alpha<1,\ \sigma=\sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha},$ $\beta=1$ and $\mu=0,\ \lambda=1$
Gamma	$[1-\mu it]^{-\lambda}$	$\alpha=1,\ \sigma=0,$ $\beta=1$ and $\mu>0,\ \lambda>0$
Exponential	$[1-\mu it]^{-1}$	$\alpha=1,\ \sigma=0,$ $\beta=1$ and $\mu>0,\ \lambda=1$

Table 2.1: Special cases of GGS laws.

Theorem 2.7.1. *Let X be a $GGS(\frac{1}{\delta},\alpha,\beta,\delta^{\frac{1}{\alpha}}\sigma,\delta\mu)$ random variable, we write $X \sim DeS(\delta,\alpha,\beta,\sigma,\mu)$. Then X becomes $S(\alpha,\beta,\sigma,\mu)$ with characteristic function given in (1.9), as $\delta\to 0$.*

Proof. Since $X \sim GGS(\frac{1}{\delta},\alpha,\beta,\delta^{\frac{1}{\alpha}}\sigma,\delta\mu))$,

$$\phi_X(t) = \left[1+\delta\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\delta\mu t\right]^{-\frac{1}{\delta}}$$

Therefore,

$$\begin{aligned}\lim_{\delta\to 0}\phi_X(t) &= \lim_{\delta\to 0}\left[1+\delta\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\delta\mu t\right]^{-\frac{1}{\delta}}\\ &= \exp\{-\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)+i\mu t\}.\end{aligned}$$

□

Proposition 2.7.2. *Let V∼ GGS($\lambda, \alpha, \beta, \sigma, \mu$) and X ∼S($\alpha, \beta, \sigma, 0$). Then*

$$V = \begin{cases} \mu W + W^{\frac{1}{\alpha}} X, & \text{if } \alpha \neq 1, \\ \mu W + WX + \sigma W \beta (2/\pi) \log(W), & \text{if } \alpha = 1, \end{cases} \tag{2.23}$$

where W is gamma distributed with scale parameter 1 and shape parameter λ and is independent of X.

Proof. Case 1: $\alpha \neq 1$

$$\begin{aligned} \phi_V(t) &= E[e^{itV}] \\ &= E_{\mathrm{w}} E[e^{itV} | W = w] \\ &= E_{\mathrm{w}} E_X[e^{it(\mu \mathrm{w} + \mathrm{w}^{\frac{1}{\alpha}} X)} | W = w] \\ &= E_{\mathrm{w}}[e^{it\mu \mathrm{w}} E_X[e^{i(t\mathrm{w}^{\frac{1}{\alpha}})X}]] \\ &= E_W[e^{it\mu \mathrm{w}} e^{-\sigma^\alpha |t\mathrm{w}^{\frac{1}{\alpha}}|^\alpha \omega_{\alpha,\beta}(t\mathrm{w}^{\frac{1}{\alpha}})}] \\ &= E_W[e^{-\mathrm{w}\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) + it\mu \mathrm{w}}] \\ &= E_W[e^{-(\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) + it\mu)\mathrm{w}}] \\ &= [1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - it\mu]^{-\lambda}. \end{aligned}$$

Case2: $\alpha = 1$

$$\begin{aligned}
\phi_V(t) &= E[e^{itV}] \\
&= E_W[e^{it(\mu W + WX + \sigma W\beta(2/\pi)\log(W))}] \\
&= E_W[e^{it(\mu W + \sigma W\beta(2/\pi)\log(W))}E_X[e^{i(tW)X}]] \\
&= E_W[e^{it(\mu W + \sigma W\beta(2/\pi)\log(W))}e^{-|\sigma tW|\omega_{\alpha,\beta}(tW)}] \\
&= E_W[e^{it(\mu W + \sigma W\beta(2/\pi)\log(W))}e^{-|\sigma tW|[1+i\beta(2/\pi)\text{sign}(tw)\log|tw|]}] \\
&= E_W[e^{it\mu W - \sigma|t|W - i\beta\sigma|t|W(2/\pi)\text{sign}(t)\log|t|}] \\
&= E_W[e^{W\{it\mu - \sigma|t| - i\beta\sigma|t|(2/\pi)\text{sign}(t)\log|t|\}}] \\
&= [1 - it\mu + \sigma|t| + i\beta\sigma|t|(2/\pi)\text{sign}(t)\log|t|]^{-\lambda} \\
&= [1 + \sigma|t|(1 + i\beta(2/\pi)\text{sign}(t)\log|t|) - it\mu]^{-\lambda} \\
&= [1 + \sigma|t|\omega_{\alpha,\beta}(t) - it\mu]^{-\lambda}.
\end{aligned}$$

□

For the purpose of simulation, we derived the representation of GGS random variable using (2.23). Then we have the random variable U having GGS distribution admits the representation:

$$U = \begin{cases} \mu W + (W/L)^{\frac{1}{\alpha}} L\sigma H_{\alpha\beta}(\pi(s - \frac{1}{2})), & \text{if} \quad \alpha \neq 1, \\ \mu W + W\sigma K_{\alpha\beta}(\pi(s - \frac{1}{2}), L) + \sigma W\beta(2/\pi)\log(\sigma W), & \text{if} \quad \alpha = 1, \end{cases} \tag{2.24}$$

where $H_{\alpha\beta}(x)$ and $K_{\alpha\beta}(x, y)$ are as defined in (2.8) and (2.9) respectively and, W, L and S are independent with $W \sim G(1, \lambda)$, $L \sim \exp(1)$ and $S \sim U(0, 1)$.

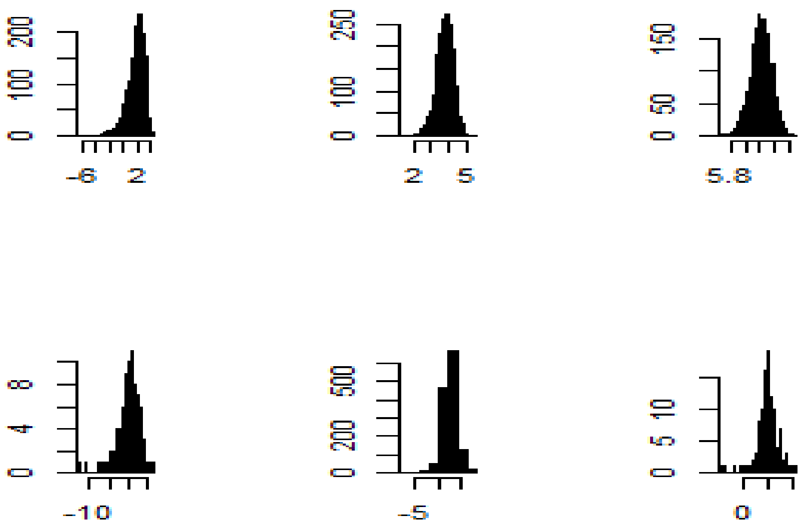

Figure 2.1: Histograms of GGS($\lambda = 1, 5, 50$) for $\alpha = 1.2, \sigma = 2, \beta = 0$ and $\mu = 1$(top) and for $\alpha = 1, \sigma = 2, \beta = 0$ and $\mu = 1$ (bottom).

The Figure 2.1 shows the histograms of simulated data of 2850 observations drawn from GGS distributions for different values of λ viz $\lambda = 1, 5, 50$. The top row panels shows histograms for $\alpha \neq 1$(that is, $\alpha = 1.2$) and bottom panels for $\alpha = 1$, for fixed $\beta = 0, \sigma = 2, \mu = 1$. It captures the variations of the model especially peakedness, for the values of λ.

2.7.1 Limits of random sums:

Recall that geometric stable distributions are the only possible (weak) limiting distributions of (normalized) geometric random sums (1.10) as $p \to 0$. A similar result holds true for the GGS distributions under *Negative Binomial*(NB) random summation. Let $N_{p,\lambda}$ be an NB random variable with parameters $p \in (0, 1), \lambda > 0$, so that

$$P(N_{p,\lambda} = k) = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)\Gamma(k+1)} p^{\lambda}(1-p)^{k-1}, \; k = 1, 2, \ldots . \tag{2.25}$$

and let $Y, X_1, X_2, \ldots$ be a sequence of independent and identically distributed GGS random variables independent of $N_{p,\lambda}$ and there exist deterministic $a = a(p) > 0$ and $b = b(p) \in \Re$ such that

$$a(p) \sum_{i=1}^{N_{p,\lambda}} (X_i + b(p)) \xrightarrow{d} Y, \quad \text{as} \quad p \to 0. \tag{2.26}$$

2.8 Moments of generalized geometric stable distributions

Here we derived the absolute and signed fractional order moments of GGS random variable V

Theorem 2.8.1. *Let $V \sim GGS(\lambda, \alpha, \beta, \sigma, 0)$. Then for $\alpha \neq 1$*

$$E|V|^q = \frac{\Gamma(\lambda + \frac{q}{\alpha})}{\Gamma\lambda} \frac{\Gamma(1 - \frac{q}{\alpha})}{\Gamma(1 - q)} \left| \frac{\gamma}{\cos\theta} \right|^{\frac{q}{\alpha}} \frac{\cos(\frac{q\theta}{\alpha})}{\cos(\frac{q\pi}{2})} \tag{2.27}$$

for $q \in (-1, \alpha) \cap (-\frac{\lambda}{\alpha}, \infty)$ where $\gamma = \sigma^\alpha$ and $\theta = \arctan(\beta \tan(\frac{\alpha\pi}{2}))$

Proof. Since, $V = W^{\frac{1}{\alpha}} X$ where W and X are independent random variables defined in (2.23). Therefore

$$\begin{aligned} E|V|^q =& E[W^{\frac{1}{\alpha}} X]^q \\ =& E[W^{\frac{q}{\alpha}}] E[X]^q \\ =& \frac{\Gamma(\lambda + \frac{q}{\alpha})}{\Gamma\lambda} \frac{\Gamma(1 - \frac{q}{\alpha})}{\Gamma(1 - q)} \left| \frac{\gamma}{\cos\theta} \right|^{\frac{q}{\alpha}} \frac{\cos(\frac{q\theta}{\alpha})}{\cos(\frac{q\pi}{2})} \end{aligned}$$

Here we used the fact that $E[W^{\frac{q}{\alpha}}] = \frac{\Gamma(\lambda + \frac{q}{\alpha})}{\Gamma\lambda}$ for $q \in (-\frac{\lambda}{\alpha}, \infty)$ and $EX^q = \frac{\Gamma(1 - \frac{q}{\alpha})}{\Gamma(1-q)} \left| \frac{\gamma}{\cos\theta} \right|^{\frac{q}{\alpha}} \frac{\cos(\frac{q\theta}{\alpha})}{\cos(\frac{q\pi}{2})}$ for $q \in (-1, \alpha)$ (see, Kuruoğlu(2001)) □

Special case: Let $\lambda = 1, \beta = 1, \sigma = \sigma(\cos(\frac{\pi\alpha}{2}))^{\frac{1}{\alpha}}$, then we get

$$EV^q = \frac{q\sigma^q\pi}{\alpha\Gamma(1-q)\sin(\frac{q\pi}{\alpha})},$$

this is the qth fractional moment of the Mittag-Leffler random varaible $ML(\alpha, \sigma)$

Theorem 2.8.2. *Let $V \sim GGS(\lambda, \alpha, \beta, \sigma, 0)$. Then, for $\alpha \neq 1$*

$$E[V^{<q>}] = \frac{\Gamma(\lambda + \frac{q}{\alpha})}{\Gamma\lambda} \frac{\Gamma(1-\frac{q}{\alpha})}{\Gamma(1-q)} \left|\frac{\gamma}{\cos\theta}\right|^{\frac{q}{\alpha}} \frac{\sin(\frac{q\theta}{\alpha})}{\sin(\frac{q\pi}{2})} \tag{2.28}$$

for $q \in [(-1, \alpha) \cup (-2, -1)] \cap [(-\frac{\lambda}{\alpha}, \infty)]$

Proof.

$$\begin{aligned}
E[V^{<q>}] &= E[\text{sign}(V)|V|^q] \\
&= E[\text{sign}(W^{\frac{1}{\alpha}}X)|W^{\frac{1}{\alpha}}X|^q] \\
&= E[\text{sign}(X)W^{\frac{q}{\alpha}}|X|^q] \\
&= E[W^{\frac{q}{\alpha}}]E[\text{sign}(X)|X|^q] \\
&= E[W^{\frac{q}{\alpha}}]E[X^{<q>}] \\
&= \frac{\Gamma(\lambda + \frac{q}{\alpha})}{\Gamma\lambda} \frac{\Gamma(1-\frac{q}{\alpha})}{\Gamma(1-q)} \left|\frac{\gamma}{\cos\theta}\right|^{\frac{q}{\alpha}} \frac{\sin(\frac{q\theta}{\alpha})}{\sin(\frac{q\pi}{2})},
\end{aligned}$$

here we used the result $E[X^{<q>}] = \frac{\Gamma(1-\frac{q}{\alpha})}{\Gamma(1-q)} \left|\frac{\gamma}{\cos\theta}\right|^{\frac{q}{\alpha}} \frac{\sin(\frac{q\theta}{\alpha})}{\sin(\frac{q\pi}{2})}$ for $q \in (-1, \alpha) \cup (-2, -1)$(see, Kuruoğlu(2001)) □

2.9 Generalized strictly geometric stable(GStGS) distribution

Generalized strictly GS distributions have the characteristic function

$$\psi(t) = [1 + \sigma^\alpha |t|^\alpha \exp(-i\pi\alpha\beta\,\mathrm{sign(t)}/2)]^{-\lambda},$$

where $0 < \alpha \leq 2, \sigma > 0, \lambda > 0$ and $|\beta| \leq \min(1, 2/\alpha - 1)$.

2.9.1 Self decomposabilty

Consider the characteristic function of GStGS distribution

$$\psi_X(t) = [1 + \sigma^\alpha |t|^\alpha \exp(-i\pi\alpha\beta\,\mathrm{sign(t)}/2)]^{-\lambda}.$$

Then,

$$\frac{\psi_X(t)}{\psi_X(at)} = \psi_a(t) = \left[a^\alpha + (1 - a^\alpha)\frac{1}{1 + \sigma^\alpha |t|^\alpha \exp(-i\pi\alpha\beta\,\mathrm{sign(t)}/2)}\right]^\lambda$$

where $\psi_a(t)$ is the characteristic function of the λ-fold convolutions of random variables U_n defined in (2.33). Hence GStGS distribution is self-decomposable.

2.9.2 AR(1) model with GStGS marginals

The first order GStGS autoregressive process(GStGSAR(1))is constituted by $\{X_n, n \geq 1\}$, where X_n satisfies the equation,

$$X_n = aX_{n-1} + \epsilon_n; a \in (0, 1) \text{ and } \quad \forall n > 0 \tag{2.29}$$

where $\{\epsilon_n\}$ is a sequence of independently and identically distributed random variables such that X_n is stationary Markovian with GStGS marginal distribution. In terms of characteristic function the model defined in (2.29) can be given as

$$\phi_{X_n}(t) = \phi_{X_{n-1}}(at)\phi_{\epsilon_n}(t) \tag{2.30}$$

Assuming stationarity, we have

$$\phi_{\epsilon}(t) = \frac{\phi_X(t)}{\phi_X(at)} \tag{2.31}$$

$$= \left[a^{\alpha} + (1-a^{\alpha})\frac{1}{1+\sigma^{\alpha}|t|^{\alpha}\exp(-i\pi\alpha\beta\ \mathrm{sign(t)}/2)}\right]^{\lambda} \tag{2.32}$$

Hence, we can regard $\{\epsilon_n\}$ as the η-fold convolutions of random variables U_n's such that

$$U_n = \begin{cases} 0, & \text{with probability} \quad a^{\alpha}, \\ L_n, & \text{with probability} \quad 1-a^{\alpha}, \end{cases} \tag{2.33}$$

where L_n's are independently and identically distributed strictly geometric stable random variables.

2.10 Moments of the log-transformed GGS random variable V'

Taking the logarithm of the mixture representation of the GGS distributed random variable V in (2.23), for $\mu = 0$ and $\alpha \neq 1$ gives,

$$V' = \frac{1}{\alpha}W' + X' \tag{2.34}$$

where $V' = \log|V|$, $W' = \log(W)$ and $X' = \log|X|$.

To obtain the moments of V', first we need to get the moments of W'. The characteristic function of W' can be shown as

$$\phi_{W'}(t) = \mathbf{E}\exp(itW') = \mathbf{E}W^{it} = \frac{\Gamma(\lambda + it)}{\Gamma(\lambda)}$$

where $i = \sqrt{-1}$. Using the logarithmic expansion of the gamma function, we get the cumulant generating function

$$\log\big(\phi_{W'}(t)\big) = \sum_{k=1}^{\infty} \frac{(it)^k}{k!} c_k$$

where the k^{th} cumulant is given by,

$$c_k = \psi^{(k-1)}(\lambda), \text{ where } \quad \psi^{(0)}(\lambda) = \psi(\lambda).$$

The mean and variance of W' are $\mu_1' = c_1 = \psi(\lambda)$ and $\mu_2 = c_2 = \psi^{(1)}(\lambda)$, where $\psi(\lambda)$ and $\psi^{(1)}(\lambda)$ are the digamma and trigamma functions, respectively. For $k \geq 3$, the k^{th} cumulant is the polygamma function of order $k-2$ evaluated at λ. The k^{th} order integer moment can be calculated using the formula

$$\psi^{(k-1)}(\lambda) = \mu_k' - \sum_{j=1}^{k-1} \binom{k-1}{j-1} c_j \mu_{k-j}'.$$

This implies that $\mu_1' = c_1 = \psi(\lambda)$, $\mu_2' = c_2 + c_1^2 = \psi^{(1)}(\lambda) + \psi(\lambda)^2$, $\mu_3' = c_3 + 3c_2c_1 + c_1^3 = \psi^{(2)}(\lambda) + 3\psi^{(1)}(\lambda)\psi(\lambda) + \psi(\lambda)^3$, $\mu_4' = c_4 + 4c_3c_1 + 3c_2^2 + 6c_2c_1^2 + c_1^4 = \psi^{(3)}(\lambda) + 4\psi^{(2)}(\lambda)\psi(\lambda) + 3\psi^{(1)}(\lambda)^2 + 6\psi^{(1)}(\lambda)\psi(\lambda)^2 + \psi(\lambda)^4$ and so on.

Now we can derive the moments of V′ using the moments of X' in Section 2.3 :

$$M_1' = \mathbf{E}(V') = \frac{1}{\alpha}\psi(\lambda) + \psi_0\left(1 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\log\left|\frac{\gamma}{\cos\theta}\right|, \tag{2.35}$$

$$M_2' = \mathbf{V}(V') = \frac{1}{\alpha^2}\psi^{(1)}(\lambda) + \psi_1\left(\frac{1}{2} + \frac{1}{\alpha^2}\right) - \frac{\theta^2}{\alpha^2}, \tag{2.36}$$

and $$M_3' = \mu_3(V') = \frac{1}{\alpha^3}\psi^{(2)}(\lambda) + \zeta(3)\left(1 - \frac{1}{\alpha^3}\right). \tag{2.37}$$

2.11 Weighted sums of Independent GGS variates

Let $Y_k{}' \sim GGS(\lambda, \alpha, \beta, \sigma, \mu)$ be independent GGS variates that are identically distributed. Then the distribution of a weighted sum of these variables with the weights a_k can be derived using a set of $S(\alpha, \beta, \sigma, 0)$ random variables X_k. Define $T = \sum a_k X_k$ and $Y_k{}' = \mu W + W^{\frac{1}{\alpha}} X_k$. Then

$$T = \sum_{k=1}^{n} a_k X_k \sim S\left(\alpha, \frac{\sum_{k=1}^{n} a_k{}^{\langle\alpha\rangle}}{\sum_{k=1}^{n} |a_k|^{\alpha}}\beta, \sum_{k=1}^{n} |a_k|^{\alpha}\sigma, 0\right)$$

where $x^{\langle p\rangle} = \text{sign}(x)|x|^p$ (see, Kuruoğlu(2001)) and W is gamma distributed with scale parameter 1 and shape parameter λ, and is independent of X_k. Then,

$$\begin{aligned}
\sum a_k Y_k{}' &= \sum a_k\left(\mu W + W^{\frac{1}{\alpha}} X_k\right) \\
&= \left(\sum_{k=1}^{n} a_k \mu\right) W + W^{\frac{1}{\alpha}}\left(\sum_{k=1}^{n} a_k X_k\right) \\
&\sim GGS\left(\lambda, \alpha, \frac{\sum_{k=1}^{n} a_k{}^{\langle\alpha\rangle}}{\sum_{k=1}^{n} |a_k|^{\alpha}}\beta, \sum_{k=1}^{n} |a_k|^{\alpha}\sigma, \sum_{k=1}^{n} a_k \mu\right).
\end{aligned}$$

This provides a convenient way to generate sequences of independent GGS random variables with $\mu = 0$, $\beta = 0$, or with zero values for both μ and

β(except when $\alpha = 1$). We call these the centered, deskewed and symmetrized sequences, respectively:

$$\begin{aligned} {Y_k{}'}^C &= Y_{3k}{}' + Y'_{3k-1} - 2Y_{3k-2}{}' \\ &\sim GGS\left(\lambda, \alpha, \left[\frac{2-2^\alpha}{2+2^\alpha}\right]\beta, [2+2^\alpha]\sigma, 0\right), \end{aligned} \tag{2.38}$$

$$\begin{aligned} {Y_k{}'}^D &= Y_{3k}{}' + Y_{3k-1}{}' - 2^{1/\alpha}Y_{3k-2}{}' \\ &\sim GGS\left(\lambda, \alpha, 0, 4\sigma, [2-2^{1/\alpha}]\mu\right), \end{aligned} \tag{2.39}$$

$${Y_k{}'}^S = Y_{2k}{}' - Y_{2k-1}{}' \sim GGS\left(\lambda, \alpha, 0, 2\sigma, 0\right). \tag{2.40}$$

Using such sequences, we may apply methods for symmetric variates to skewed variates and we may apply skew-estimation methods for centered variates to noncentered variates, with the effective loss of some sample.

2.12 Estimation for $GGS(\lambda, \alpha, \beta, \sigma, \mu)$ distributions

We use a similar procedure of geometric stable distributions, for parametric estimation GGS distributions. We apply the centro-symmetrization procedure; therefore, we set $\theta = 0$ and obtain α and λ solving $M_2{}'$ and $M_3{}'$. The estimates of α and λ may then be used to solve $M_2{}'$ of the skewed process for the parameter θ and hence β. Similarly, $M_1{}'$ is solved for γ and hence σ.

Logarithmic estimator for α and λ: Apply centro-symmetrization as given by equation (2.40) to the observed data. Estimate $M_2{}'$ and $M_3{}'$ for the transformed data and solve $M_2{}'$ and $M_3{}'$ for α and λ. That is the estimates $\hat{\alpha}$ and $\hat{\lambda}$ are the solutions of the equations

$$\hat{M_2}' = \frac{1}{\alpha^2}\psi^{(1)}(\lambda)+\psi_1\left(\frac{1}{2}+\frac{1}{\alpha^2}\right) \quad \text{and} \quad \hat{M_3}' = \frac{1}{\alpha^3}\psi^{(2)}(\lambda)+\zeta(3)\left(1-\frac{1}{\alpha^3}\right).$$

Here we consider an approximation based on the first few terms of the series representation of the digamma function.

Logarithmic estimator for β: By centering as (2.38) and assuming estimates of α and λ are available, we estimate M_2' for this data and hence estimate θ as follows:

$$|\theta| = \alpha\left(\frac{1}{\alpha^2}\psi^{(1)}(\lambda)+\psi_1\left(\frac{1}{2}+\frac{1}{\alpha^2}\right)-M_2'\right)^{1/2}.$$

Estimate of β is

$$\beta = \frac{\tan(\hat{\theta})}{\tan(\hat{\alpha}\pi/2)}\left(\frac{2-2^{\hat{\alpha}}}{2+2^{\hat{\alpha}}}\right).$$

Logarithmic estimator for σ: For the estimate of γ we apply the centering given by (2.38). Estimate M_1' for the transformed data and hence the estimate is

$$\hat{\gamma} = \exp\left\{\alpha\left(M_1' + \frac{1}{\alpha}(\psi_0-\psi(\lambda))-\psi_0\right)\right\}.$$

Hence, $\hat{\sigma} = \frac{\hat{\gamma}^{\frac{1}{\hat{\alpha}}}}{2+2^{\hat{\alpha}}}$

The series representation of the digamma function $\psi(\tau)$ is

$$\psi(\tau) = \log(\tau) - \frac{1}{2\tau} - \frac{1}{12\tau^2} + \frac{1}{120\tau^4} - \frac{1}{252\tau^6} + O(\frac{1}{\tau^8}).$$

Therefore, we approximate $\psi(\tau)$ as

$$\psi(\tau) = \log(\tau) - \frac{1}{2\tau} - \frac{1}{12\tau^2} + \frac{1}{120\tau^4} - \frac{1}{252\tau^6}.$$

This results approximation of $\psi^{(1)}(\tau)$ as $\psi^{(1)}(\tau) = \frac{1}{\tau} + \frac{1}{2\tau^2} + \frac{1}{6\tau^3} - \frac{1}{30\tau^5} + \frac{1}{42\tau^7}$ and the approximation of $\psi^{(2)}(\tau)$ as $\psi^{(2)}(\tau) = \frac{1}{30\tau^6} - \frac{1}{\tau^2} - \frac{1}{\tau^3} - \frac{1}{2\tau^4} - \frac{1}{6\tau^8}$.

We carry out a simulation study to obtain the estimates of of the parameters λ, α, β and σ. For different values of the parameters, we generated 10000 random samples of sizes n=30, 100, 200, 500, 20000 each from the GGS($\lambda, \alpha, \beta, \sigma, 0$) distribution, and computed the bias and the root-mean-square error (RMSE). The results obtained are given in Table 2.2. From the results, it is evident that for each values of the parameters, the values of bias and RMSEs decrease as the sample size increases.

Table 2.2: Average values of bias and RMSEs using different values of λ, α, β and σ for sample sizes n=30, 100, 200, 500, 20000 corresponding to $GGS(\lambda, \alpha, \beta, \sigma, 0)$ distribution

		Bias					RMSE				
$(\lambda, \alpha, \beta, \sigma)$	Est	n=30	100	200	500	20000	n=30	100	200	500	20000
(15,1.2,0.8,20)	$\hat{\lambda}$	6.601	4.003	3.905	1.001	0.002	8.006	6.980	4.002	2.003	0.130
	$\hat{\alpha}$	0.521	0.470	0.397	0.231	0.001	0.806	0.560	0.154	0.032	0.020
	$\hat{\beta}$	0.553	0.478	0.411	0.110	0.032	0.932	0.563	0.507	0.040	0.052
	$\hat{\sigma}$	17.087	8.098	7.654	0.924	0.003	24.062	11.098	10.022	1.002	0.102
(10,1.4,0.6,15)	$\hat{\lambda}$	5.018	3.099	2.972	.0877	0.020	13.545	7.665	7.003	1.980	0.006
	$\hat{\alpha}$	0.597	0.430	0.380	0.221	0.000	7.980	3.092	3.007	0.005	0.000
	$\hat{\beta}$	0.441	0.304	0.300	0.210	0.010	2.094	1.076	1.06	0.065	0.001
	$\hat{\sigma}$	4.679	1.891	1.003	0.670	0.000	6.055	4.345	3.990	1.093	0.023
(5,1.6,0.4,10)	$\hat{\lambda}$	4.014	3.786	3.069	1.085	0.001	18.043	10.005	8.076	2.031	0.021
	$\hat{\alpha}$	0.605	0.553	0.453	0.201	0.003	4.661	3.002	2.873	0.672	0.003
	$\hat{\beta}$	0.396	0.272	0.255	0.100	0.005	2.675	1.098	1.005	0.456	0.000
	$\hat{\sigma}$	5.700	3.081	2.756	1.090	0.000	16.007	8.091	7.900	2.001	0.040
(2,1.8,0.2,5)	$\hat{\lambda}$	2.022	1.978	1.657	1.002	0.021	4.006	2.005	1.784	0.543	0.002
	$\hat{\alpha}$	0.743	0.436	0.400	0.11	0.001	4.761	2.221	2.003	0.451	0.007
	$\hat{\beta}$	0.272	0.197	0.116	0.018	0.007	2.004	1.984	1.008	0.086	0.001
	$\hat{\sigma}$	2.341	0.789	0.456	0.002	0.000	5.008	4.031	3.902	0.871	0.001
(1,1.9,0.1,2)	$\hat{\lambda}$	0.906	0.761	0.543	0.001	0.000	2.002	0.973	0.820	0.003	0.000
	$\hat{\alpha}$	0.801	0.734	0.701	0.320	0.000	0.963	0.701	0.562	0.024	0.001
	$\hat{\beta}$	0.253	0.210	0.165	0.097	0.003	1.232	1.009	0.873	0.674	0.035
	$\hat{\sigma}$	1.603	0.765	0.564	0.450	0.001	2.006	1.008	0.701	0.036	0.004

2.13 Slash generalized geometric stable distributions

In this section, we define the slash version of the generalized geometric stable distributions.

Definition 2.13.1. *A random variable Y has a slash generalized geometric stable (SGGS) distributions, denoted by $Y \sim SGGS(\lambda, \alpha, \beta, \sigma, \mu, q)$, if $Y = \frac{X}{U^{\frac{1}{q}}}$, where $q > 0$ and X is GGS random variable with characteristic function given by $\phi_X(t) = \left[1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right]^{-\lambda}$, where $0 < \alpha \leq 2, \lambda > 0, -1 \leq \beta \leq 1, \sigma > 0$, and $\mu \in \Re$, and $U \sim U(0,1)$, which is independent of X.*

In the section below, analogous to the generalized normal-Laplace (GNL) distribution (Reed(2007)), we further generalizes GGS distributions to obtain Gaussian- non Gaussian models.

2.14 Generalized normal geometric stable distributions

Reed(2007) introduced generalized normal-Laplace distribution, which is useful in financial applications for obtaining an alternative stochastic process model to Brownian motion for logarithmic prices, in which the increments exhibit fatter tails than the normal distribution. The generalized normal Laplace (GNL) distribution is defined as that of a random variable Y with characteristic function

$$\phi_X(t) = \left[\frac{\alpha\beta \exp\{i\eta t - \frac{\tau^2 t^2}{2}\}}{(\alpha - it)(\beta + it)}\right]^\lambda \tag{2.41}$$

where α, β, λ and τ and are positive parameters and $-\infty < \eta < \infty$. It follows that Y can be represented as

$$\lambda\eta + \tau\sqrt{\lambda}Z + \frac{1}{\alpha}G_1 - \frac{1}{\beta}G_2$$

where Z, G_1, and G_2 are independent with $Z \sim N(0,1)$ and G_1, G_2 gamma random variables with scale parameter 1 and shape parameter λ. For $\lambda = 1$, GNL distribution becomes what has been called an (ordinary) normal-Laplace (NL) distribution.

Now we introduce a new class of Gaussian-non Gaussian distributions, namely generalized normal geometric stable (GNGS) distribution, which generalizes GGS distributions. The generalized normal geometric stable (GNGS) distribution is defined as follows.

Definition 2.14.1. *A random variable Y is said to have generalized normal geometric stable distribution $GNGS(\eta, \tau, \lambda, \alpha, \beta, \sigma, \mu)$ if there are parameters $0 < \alpha \leq 2, \lambda > 0, -1 \leq \beta \leq 1, \tau > 0, \sigma > 0$, and $\eta, \mu \in \Re$ such that its characteristic function, $\phi(t)$ has the following form:*

$$\phi_X(t) = \left[\frac{\exp\{i\eta t - \frac{\tau^2 t^2}{2}\}}{1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t}\right]^\lambda \tag{2.42}$$

Thus, we have

$$\Phi_X(t) = \begin{cases} e^{i\lambda\eta t - \frac{\lambda\tau^2 t^2}{2}}[1 + \sigma^\alpha |t|^\alpha (1 - i\beta\tan(\frac{\pi\alpha}{2})) - i\mu t]^{-\lambda}, & if\ \alpha \neq 1, \\ e^{i\lambda\eta t - \frac{\lambda\tau^2 t^2}{2}}[1 + \sigma^\alpha |t|^\alpha (1 + i\beta\frac{2}{\pi}\log|t|) - i\mu t]^{-\lambda}, & if\ \alpha = 1. \end{cases} \tag{2.43}$$

We shall use the notation $X \sim GNGS(\eta, \tau, \lambda, \alpha, \beta, \sigma, \mu)$ to denote that X is distributed according to the generalized normal geometric stable distribution. Detailed list of special cases of GNGS laws is presented in the Table 2.3.

Distribution	Charcteristic function	Parametric values
Normal-Geometric Stable	$e^{i\eta t-\frac{\tau^2t^2}{2}}[1+\sigma^\alpha\|t\|^\alpha\omega_{\alpha,\beta}(t)-i\mu t]^{-1}$	$\eta\in\Re,\tau>0,0<\alpha\leq 2,\lambda=1,-1\leq\beta\leq 1,$ $\sigma>0,\mu\in$ R
Generalized normal-Linnik	$e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}[1+\sigma^\alpha\|t\|^\alpha]^{-\lambda}$	$\eta\in\Re,\tau>0,0<\alpha\leq 2,\lambda>0,\beta=0,$ $\sigma>0,\mu=0$
normal-Linnik	$e^{i\eta t-\frac{\tau^2t^2}{2}}[1+\sigma^\alpha\|t\|^\alpha]^{-1}$	$\eta\in\Re,\tau>0,0<\alpha\leq 2,\lambda=1,\beta=0,$ $\sigma>0,\mu=0$
Generalized normal-Asymmmetric Laplace	$e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}[1+\sigma^2\|t\|^2-i\mu t]^{-\lambda}$	$\eta\in\Re,\tau>0,\alpha=2,\lambda>0,\beta=0,$ $\sigma>0,\mu\in$ R
Generalized normal-Symmmetric Laplace	$e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}[1+\sigma^2\|t\|^2]^{-\lambda}$	$\eta\in\Re,\tau>0,\alpha=2,\lambda>0,\beta=0,$ $\sigma>0,\mu=0$
Normal-asymmmetric Laplace	$e^{i\eta t-\frac{\tau^2t^2}{2}}[1+\sigma^2\|t\|^2-i\mu t]^{-1}$	$\eta\in\Re,\tau>0,\alpha=2,\lambda=1,\beta=0,$ $\sigma>0,\mu\in$ R
Normal-symmetric Laplace	$e^{i\eta t-\frac{\tau^2t^2}{2}}[1+\sigma^2\|t\|^2]^{-1}$	$\eta\in\Re,\tau>0,\alpha=2,\lambda=1,\beta=0,$ $\sigma>0,\mu=0$
Generalized normal-Mittag-Leffler	$e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}[1+\sigma^\alpha(-it)^\alpha]^{-\lambda}$	$\eta\in\Re,\tau>0,0<\alpha<1,\ \sigma=\sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha},$ $\beta=1$ and $\mu=0,\ \lambda>0$
Normal-Mittag-Leffler	$e^{i\eta t-\frac{\tau^2t^2}{2}}[1+\sigma^\alpha(-it)^\alpha]^{-1}$	$\eta\in\Re,\tau>0,0<\alpha<1,\ \sigma=\sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha},$ $\beta=1$ and $\mu=0,\ \lambda=1$
Normal-Gamma	$e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}[1-\mu it]^{-\lambda}$	$\eta\in\Re,\tau>0,\alpha=1,\ \sigma=0,$ $\beta=1$ and $\mu>0,\ \lambda>0$
Normal-exponential	$e^{i\eta t-\frac{\tau^2t^2}{2}}[1-\mu it]^{-1}$	$\eta\in\Re,\tau>0,\alpha=1,\ \sigma=0,$ $\beta=1$ and $\mu>0,\ \lambda=1$

Table 2.3: Special cases of GNGS laws.

Theorem 2.14.1. *GNGS is infinitely divisible.*

Proof. Let $X_1, X_2, \ldots X_n$ are identically and independently distributed random variables with $GNGS(\eta, \tau, \frac{\lambda}{n}, \alpha, \beta, \sigma, \mu)$ distribution. Define $X = X_1 + X_2 + \cdots + X_n$. Then the characteristic function of X is

$$\begin{aligned}\Phi_\Theta(p)) &= \left(\left[\frac{\exp\{i\eta t - \frac{\tau^2 t^2}{2}\}}{1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t}\right]^{\frac{\lambda}{n}}\right)^n \\ &= \left[\frac{\exp\{i\eta t - \frac{\tau^2 t^2}{2}\}}{1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t}\right]^{\lambda}\end{aligned} \tag{2.44}$$

Hence X is infinitely divisible. □

2.15 Representation

A representation of GNGS random variable similar to the representation of GGS defined in (2.23), can be derived as follows.

Proposition 2.15.1. *Let* $V \sim GNGS(\eta,\tau,\lambda,\alpha,\beta,\sigma,\mu)$ *and* $X \sim S(\alpha,\beta,\sigma,0)$. *Then*

$$V = \begin{cases} \eta\lambda + \tau\sqrt{\lambda}Z + \mu W + W^{\frac{1}{\alpha}}X, & \text{if } \alpha \neq 1, \\ \eta\lambda + \tau\sqrt{\lambda}Z + \mu W + WX + \sigma W\beta(2/\pi)\log(W), & \text{if } \alpha = 1, \end{cases} \tag{2.45}$$

where $Z \sim N(0,1)$, W *is gamma distributed with scale parameter 1 and shape parameter* λ, *and* Z, W *and* X *are independent of each other.*

Proof. Case 1: $\alpha \neq 1$

$$\begin{aligned}
\phi_V(t) &= E[e^{itV}] \\
&= E[e^{it(\eta\lambda+\tau\sqrt{\lambda}Z+\mu W+W^{\frac{1}{\alpha}}X)}] \\
&= E[e^{it(\eta\lambda+\tau\sqrt{\lambda}Z)}]E[e^{it(\mu W+W^{\frac{1}{\alpha}}X}] \\
&= e^{i\lambda\eta t}E[e^{it\tau\sqrt{\lambda}Z}]E_{\mathrm{w}}(E_X[e^{it(\mu\mathrm{w}+\mathrm{w}^{\frac{1}{\alpha}}X)}|W=w]) \\
&= e^{i\lambda\eta t}e^{-\frac{\lambda\tau^2t^2}{2}}E_{\mathrm{w}}([e^{it\mu\mathrm{w}}E_X[e^{i(t\mathrm{w}^{\frac{1}{\alpha}})X}]]) \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}E_W[e^{it\mu\mathrm{w}}e^{-\sigma^\alpha|t\mathrm{w}^{\frac{1}{\alpha}}|^\alpha\omega_{\alpha,\beta}(t\mathrm{w}^{\frac{1}{\alpha}})}] \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}E_W[e^{-\mathrm{w}\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)+it\mu\mathrm{w}}] \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}E_W[e^{-(\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)+it\mu)\mathrm{w}}] \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}[1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-it\mu]^{-\lambda} \\
&= \left[\frac{\exp\{i\eta t-\frac{\tau^2t^2}{2}\}}{1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t}\right]^\lambda
\end{aligned}$$

Case2: $\alpha = 1$

$$
\begin{aligned}
\phi_V(t) &= E[e^{itV}] \\
&= E[e^{it(\eta\lambda+\tau\sqrt{\lambda}Z+\mu W+WX+\sigma W\beta(2/\pi)\log(W))}] \\
&= E[e^{it(\eta\lambda+\tau\sqrt{\lambda}Z)}]E[e^{it(\mu W+WX+\sigma W\beta(2/\pi)\log(W))}] \\
&= e^{i\lambda\eta t}E[e^{it\tau\sqrt{\lambda}Z}]E_W[e^{it(\mu W+WX+\sigma W\beta(2/\pi)\log(W))}] \\
&= e^{i\lambda\eta t}e^{-\frac{\lambda\tau^2t^2}{2}}E_W[e^{it(\mu W+\sigma W\beta(2/\pi)\log(W))}E_X[e^{i(tW)X}]] \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}E_W[e^{it(\mu W+\sigma W\beta(2/\pi)\log(W))}e^{-|\sigma tW|\omega_{\alpha,\beta}(tW)}] \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}E_W[e^{it(\mu W+\sigma W\beta(2/\pi)\log(W))}e^{-|\sigma tW|[1+i\beta(2/\pi)\mathrm{sign}(tw)\log|tw|]}] \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}E_W[e^{it\mu W-\sigma|t|W-i\beta\sigma|t|W(2/\pi)\mathrm{sign}(t)\log|t|}] \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}E_W[e^{W\{it\mu-\sigma|t|-i\beta\sigma|t|(2/\pi)\mathrm{sign}(t)\log|t|\}}] \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}[1-it\mu+\sigma|t|+i\beta\sigma|t|(2/\pi)\mathrm{sign}(t)\log|t|]^{-\lambda} \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}[1+\sigma|t|(1+i\beta(2/\pi)\mathrm{sign}(t)\log|t|)-it\mu]^{-\lambda} \\
&= e^{i\lambda\eta t-\frac{\lambda\tau^2t^2}{2}}[1+\sigma|t|\omega_{\alpha,\beta}(t)-it\mu]^{-\lambda} \\
&= \left[\frac{\exp\{i\eta t-\frac{\tau^2t^2}{2}\}}{1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t}\right]^{\lambda}
\end{aligned}
$$

□

Analogous to the representation (2.24) of GGS distribution, we present the representation for GNGS random variable as follows:

$$
U = \begin{cases} \eta\lambda+\tau\sqrt{\lambda}Z+\mu W+(W/L)^{\frac{1}{\alpha}}L\sigma H_{\alpha\beta}(\pi(s-\frac{1}{2})), & \text{if} \quad \alpha \neq 1, \\ \eta\lambda+\tau\sqrt{\lambda}Z+\mu W+W\sigma K_{\alpha\beta}(\pi(s-\frac{1}{2}),L)+\sigma W\beta(2/\pi)\log(\sigma W), \text{if } \alpha = 1, \end{cases} \tag{2.46}
$$

where $H_{\alpha\beta}(x)$ and $K_{\alpha\beta}(x,y)$ are as defined in (2.8) and (2.9) respectively and, W, L,,Z and S are independent with $W \sim G(1,\lambda)$, $L \sim \exp(1), Z \sim N(0,1)$ and $S \sim U(0,1)$.

The above representation provides a straightforward way to generate pseudo-random deviates following a GNGS distribution.

2.16 Weighted sums of Independent GNGS variates

Let $Y_k{}' \sim GNGS(\eta, \tau, \lambda, \alpha, \beta, \sigma, \mu)$ be independent GNGS variates that are identically distributed. Then the distribution of a weighted sum of these variables with the weights a_k can be derived using a set of $S(\alpha, \beta, \sigma, 0)$ random variables X_k. Define $T = \sum a_k X_k$. Since,

$$T = \sum_{k=1}^{n} a_k X_k \sim S\left(\alpha, \frac{\sum_{k=1}^{n} a_k{}^{\langle\alpha\rangle}}{\sum_{k=1}^{n} |a_k|^{\alpha}}\beta, \sum_{k=1}^{n} |a_k|^{\alpha}\sigma, 0\right)$$

where $x^{\langle p\rangle} = sign(x)|x|^p$ (see, Kuruoğlu(2001)) and W is gamma distributed with scale parameter 1 and shape parameter λ, and is independent of X_k. For $\alpha \neq 1$, define $Y_k{}' = \eta\lambda + \tau\sqrt{\lambda}Z + \mu W + W^{\frac{1}{\alpha}} X_k$, then

$$\begin{aligned}
\sum a_k Y_k{}' &= \sum a_k \left(\eta\lambda + \tau\sqrt{\lambda}Z + \mu W + W^{\frac{1}{\alpha}} X_k\right) \\
&= \left(\sum_{k=1}^{n} a_k \eta\lambda\right) + \left(\sum_{k=1}^{n} a_k \tau\sqrt{\lambda}\right) Z + \left(\sum_{k=1}^{n} a_k \mu\right) W + W^{\frac{1}{\alpha}} \left(\sum_{k=1}^{n} a_k X_k\right) \\
&= \left(\sum_{k=1}^{n} a_k \eta\right) \lambda + \left(\sum_{k=1}^{n} a_k \tau\right) \sqrt{\lambda}Z + \left(\sum_{k=1}^{n} a_k \mu\right) W + W^{\frac{1}{\alpha}} \left(\sum_{k=1}^{n} a_k X_k\right) \\
&\sim GNGS\left(\sum_{k=1}^{n} a_k \eta, \sum_{k=1}^{n} a_k \tau, \lambda, \alpha, \frac{\sum_{k=1}^{n} a_k{}^{\langle\alpha\rangle}}{\sum_{k=1}^{n} |a_k|^{\alpha}}\beta, \sum_{k=1}^{n} |a_k|^{\alpha}\sigma, \sum_{k=1}^{n} a_k \mu\right)
\end{aligned}$$

2.17 Slash generalized normal-geometric stable distributions

Here we define the slash version of the generalized normal-geometric stable distributions.

Definition 2.17.1. *A random variable Y has a slash generalized normal-geometric stable(SGNGS) distributions, denoted by $Y \sim SGNGS(\eta, \tau, \lambda, \alpha, \beta, \sigma, \mu, q)$, if $Y = \frac{X}{U^{\frac{1}{q}}}$, where $q > 0$ and X is GNGS random variable with characteristic function given by $\phi_X(t) = \left[\frac{\exp\{i\eta t - \frac{\tau^2 t^2}{2}\}}{1+\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t}\right]^\lambda$, where $0 < \alpha \leq 2, \lambda > 0, -1 \leq \beta \leq 1, \tau > 0, \sigma > 0$, and $\eta, \mu \in \Re$, and $U \sim U(0,1)$, which is independent of X.*

CHAPTER 3

GEOMETRIC GENERALIZED GEOMETRIC STABLE DISTRIBUTION

3.1 Introduction

Pillai(1990b) introduced the concept of geometric exponential distribution. Jose and Seetha Lekshmi(1999) studied the properties and applications of geometric exponential distribution. As a generalization of geometric exponential distribution, Jayakumar and Ajitha(2003) introduced geometric Mittag-Leffler distribution and developed autoregressive process with geometric Mittag-Leffler marginals. Geometric Mittag-Leffler distribution further extended to geometric Quasi Factorial gamma distributions. Seetha Lekshmi and Jose(2004) introduced geometric Laplace and extended to geometric $\alpha-$Laplace distribution. Certain limit properties of geometric Laplace distribution are derived. Seetha Lekshmi and Jose(2006) introduced and studied Geometric Pakes generalized Linnik distribution.

In the present chapter, Geometric GGS distributions(GeoGGS) are introduced and discussed its different properties. First order autoregressive process with GeoGGS marginals are developed and are extended to k^{th} order. We have also introduced Geometric GNGS distributions(GeoGNGS) and autoregressive time series models with GeoGNGS marginals are developed.

3.2 Geometric generalized geometric stable(GeoGGS) distributions

A distribution with characteristic function $\psi(t)$is geometrically infinitely divisible if and only if

$$\phi(t) = \exp\{1 - \frac{1}{\psi(t)}\},$$

where $\phi(t)$ is an infinite divisible characteristic function (see, Klebanov *et al.* (1984)).

Now, $[1 + \sigma^{\alpha}|t|^{\alpha}\omega_{\alpha,\beta}(t) - i\mu t]^{-\lambda} = \exp\left\{1 - \frac{1}{\left[1+\lambda\log\left(1+\sigma^{\alpha}|t|^{\alpha}\omega_{\alpha,\beta}(t)-i\mu t\right)\right]^{-1}}\right\}.$

Since GGS distribution is infinite divisible, it follows that

$$\left[1 + \lambda \log\left(1 + \sigma^{\alpha}|t|^{\alpha}\omega_{\alpha,\beta}(t) - i\mu t\right)\right]^{-1}$$

is geometrically infinite divisible.

A distribution with characteristic function

$$\left[1 + \lambda \log\left(1 + \sigma^{\alpha}|t|^{\alpha}\omega_{\alpha,\beta}(t) - i\mu t\right)\right]^{-1}$$

is called GeoGGS distribution. It is denoted as $GeoGGS(\lambda, \alpha, \beta, \sigma, \mu)$

Definition 3.2.1. *A random variable X is said to follow geometric generalized geometric stable distribution and write $X \sim GeoGGS(\lambda, \alpha, \beta, \sigma, \mu)$ if it has the characteristic function*

$$\phi(t) = \Big[1 + \lambda \log\big(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\big)\Big]^{-1},$$

where $\lambda > 0$ *and*

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta sign(x) \tan(\pi\alpha/2), & \text{if} \quad \alpha \neq 1, \\ 1 + i\beta(2/\pi) sign(x) \log|x|, & \text{if} \quad \alpha = 1. \end{cases}$$

Special cases: Detailed list of special cases of GeoGGS laws is presented in the Table 3.1.

Distribution	Charcteristic function	Parametric values
GeoGS	$\Big[1 + \log\big(1 + \sigma^\alpha \lvert t\rvert^\alpha \omega_{\alpha,\beta}(t) - i\mu t\big)\Big]^{-1}$	$0 < \alpha \leq 2, \lambda = 1, -1 \leq \beta \leq 1,$ $\sigma > 0, \mu \in R$
Geometric Pakes generalized Linnik	$\Big[1 + \lambda \log\big(1 + \sigma^\alpha \lvert t\rvert^\alpha\big)\Big]^{-1}$	$0 < \alpha \leq 2, \lambda > 0, \beta = 0,$ $\sigma > 0, \mu = 0$
Geometric Linnik	$\Big[1 + \log\big(1 + \sigma^\alpha \lvert t\rvert^\alpha\big)\Big]^{-1}$	$0 < \alpha \leq 2, \lambda = 1, \beta = 0,$ $\sigma > 0, \mu = 0$
Geometric generalized asymmmetric Laplace	$\Big[1 + \lambda \log\big(1 + \sigma^2 \lvert t\rvert^2 - i\mu t\big)\Big]^{-1}$	$\alpha = 2, \lambda > 0, \beta = 0,$ $\sigma > 0, \mu \in R$
Geometric generalized symmmetric Laplace	$\Big[1 + \lambda \log\big(1 + \sigma^2 \lvert t\rvert^2\big)\Big]^{-1}$	$\alpha = 2, \lambda > 0, \beta = 0,$ $\sigma > 0, \mu = 0$
Geometric asymmmetric Laplace	$\Big[1 + \log\big(1 + \sigma^2 \lvert t\rvert^2 - i\mu t\big)\Big]^{-1}$	$\alpha = 2, \lambda = 1, \beta = 0,$ $\sigma > 0, \mu \in R$
Geometric symmetric Laplace	$\Big[1 + \log\big(1 + \sigma^2 \lvert t\rvert^2\big)\Big]^{-1}$	$\alpha = 2, \lambda = 1, \beta = 0,$ $\sigma > 0, \mu = 0$
geometric Quasi Factorial Gamma	$\Big[1 + \lambda \log\big(1 + \sigma^\alpha (-it)^\alpha\big)\Big]^{-1}$	$0 < \alpha < 1,\ \sigma = \sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha},$ $\beta = 1$ and $\mu = 0,\ \lambda > 0$
Geometric Mittag-Leffler	$\Big[1 + \log\big(1 + \sigma^\alpha (-it)^\alpha\big)\Big]^{-1}$	$0 < \alpha < 1,\ \sigma = \sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha},$ $\beta = 1$ and $\mu = 0,\ \lambda = 1$
Geometric Gamma	$\big[1 + \lambda \log(1 - \mu i t)\big]^{-1}$	$\alpha = 1,\ \sigma = 0,$ $\beta = 1$ and $\mu > 0,\ \lambda > 0$
Geometric exponential	$\big[1 + \log(1 - \mu i t)\big]^{-1}$	$\alpha = 1,\ \sigma = 0,$ $\beta = 1$ and $\mu > 0,\ \lambda = 1$

Table 3.1: Special cases of GeoGGS laws.

Theorem 3.2.1. *Let $X_1, X_2, \ldots$ be independent and identically distributed as geometric generalized geometric stable random variables with parameter*

$\lambda, \alpha, \beta, \sigma, \mu$ *and that is,* $X_i \sim GeoGGS(\lambda, \alpha, \beta, \sigma, \mu), i = 1, 2, \dots$ *and* $N(\gamma)$ *be a geometric with mean* $1/\gamma, P[N(\gamma) = k] = \gamma(1-\gamma)^{k-1}, k = 1, 2, \dots, 0 < \gamma < 1$. *Define* $Y = X_1 + X_2 + \cdots + X_{N(\gamma)}$, *then* $Y \sim GeoGGS(\frac{\lambda}{\gamma}, \alpha, \beta, \sigma, \mu)$

Proof. Since $X_i \sim GeoGGS(\lambda, \alpha, \beta, \sigma, \mu)$, then its characteristic function is ,

$$\phi_X(t) = \left[1 + \lambda \log\left(1 + \left\{\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right\}\right)\right]^{-1}$$

Then the characteristic function of Y is

$$\begin{aligned}
\phi_Y(t) &= \sum_{k=1}^{n} [\phi_X(t)]^k \gamma(1-\gamma)^{k-1} \\
&= \frac{\gamma \phi_X(t)}{1 - (1-\gamma)\phi_X(t)} \\
&= \frac{\gamma \left[1 + \lambda \log\left(1 + \left\{\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right\}\right)\right]^{-1}}{1 - (1-\gamma)\left[1 + \lambda \log\left(1 + \left\{\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right\}\right)\right]^{-1}} \qquad (3.1) \\
&= \left[1 + \frac{\lambda}{\gamma} \log\left(1 + \left\{\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right\}\right)\right]^{-1}.
\end{aligned}$$

Hence $Y \sim GeoGGS(\frac{\lambda}{\gamma}, \alpha, \beta, \sigma, \mu)$. □

Now we shall consider a limit property of the GeoGGS distribution and its relationship with the GGS distribution.

Theorem 3.2.2. *Suppose* $X_1, X_2, \dots$ *be independent and identically distributed as* $GGS(\frac{\lambda}{n}, \alpha, \beta, \sigma, \mu)$ *and N, independent of* $X_1, X_2, \dots$ *be a geometric random variables with probability of success* $1/n$. *Then* $Y = X_1 + X_2 + \cdots + X_N$ *distributed as* $GeoGGS(\lambda, \alpha, \beta, \sigma, \mu)$ *as* $n \to \infty$.

Proof.

$$\left[1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right]^{-\frac{\lambda}{n}} = \left\{1 + \left[\left[1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right]^{\frac{\lambda}{n}} - 1\right]\right\}^{-1}. \qquad (3.2)$$

Hence by Lemma 3.2 of Pillai(1990b)

$$\phi_n(t) = \left\{1 + n\left[\left[1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t\right]^{\frac{\lambda}{n}} - 1\right]\right\}^{-1}$$

is the characteristic function of Y. Taking limit as $n \to \infty$, we have

$$\begin{aligned}\phi(t) &= \lim_{n\to\infty}\phi_n(t)\\ &= \left\{1 + \lim_{n\to\infty} n\left[\left[1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t\right]^{\frac{\lambda}{n}} - 1\right]\right\}^{-1}\\ &= \left[1+\lambda\log\left(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t\right)\right]^{-1}\end{aligned} \tag{3.3}$$

□

Theorem 3.2.3. *Let $X|\lambda \sim GGS(\lambda,\alpha,\beta,\sigma,\mu)$ with random λ, where λ is exponential with mean η. Then $X \sim GeoGGS(\eta,\alpha,\beta,\sigma,\mu)$.*

Proof.

$$\begin{aligned}\phi(t) &= E\left(e^{itX_\lambda}\right)\\ &= E_\lambda\left[1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t\right]^{-\lambda}\\ &= E_\lambda\left[e^{\log\left[1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t\right]^{-\lambda}}\right]\\ &= E_\lambda\left[e^{-\lambda\log\left(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t\right)}\right]\\ &= \left[1+\eta\log\left(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t\right)\right]^{-1}\end{aligned} \tag{3.4}$$

□

Theorem 3.2.4. *Let $X_1, X_2, \dots$ be independent and identically distributed as $GeoGGS(\frac{\lambda}{n},\alpha,\beta,\sigma,\mu)$. Then $Y = X_1 + X_2 + \cdots + X_n \xrightarrow{d} GGS(\lambda,\alpha,\beta,\mu)$ as $n \to \infty$.*

Proof. The ch.f of $GeoGGS(\frac{\lambda}{n}, \alpha, \beta, \sigma, \mu)$ distribution is

$$\phi_X(t) = \left[1 + \frac{\lambda}{n} \log\left(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right)\right]^{-1}$$

Then the characteristic function of Y is

$$\phi_Y(t) = \left[1 + \frac{\lambda}{n} \log\left(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right)\right]^{-n}$$

Hence,

$$\lim_{n\to\infty} \phi_Y(t) = \left[1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right]^{-\lambda}$$

That is, $Y \xrightarrow{d} GGS(\lambda, \alpha, \beta, \sigma, \mu)$.

□

Theorem 3.2.5. *Let X be a $GeoGGS(\frac{1}{\delta}, \alpha, \beta, \delta^{\frac{1}{\alpha}}\sigma, \delta\mu)$ random variable,we write $X \sim DeGS(\delta, \alpha, \beta, \sigma, \mu)$. Then X becomes $GS(\alpha, \beta, \sigma, \mu)$ with ch.fn given in (1.13), as $\delta \to 0$.*

Proof. Since $X \sim GeoGGS(\frac{1}{\delta}, \alpha, \beta, \delta^{\frac{1}{\alpha}}\sigma, \delta\mu)$, the ch. fn of X is

$$\phi_X(t) = \left[1 + \delta^{-1} \log\left(1 + \delta\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\delta\mu t\right)\right]^{-1}$$

Therefore,

$$\begin{aligned}\lim_{\delta\to 0} \phi_X(t) &= \lim_{\delta\to 0} \left[1 + \delta^{-1} \log\left(1 + \delta\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\delta\mu t\right)\right]^{-1} \\ &= [1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t]^{-1}\end{aligned}$$

□

Note: In DeGS distributions the parameter δ act as a pathway parameter as its values varies, the distributions moves to its GS forms.

Special cases: Detailed list of special cases of DeGS distributions is presented in the Table 3.2.

Distribution	Charcteristic function	Parametric values
DeGS	$\left[1+\delta^{-1}\log\left(1+\delta\sigma^{\alpha}\|t\|^{\alpha}\omega_{\alpha,\beta}(t)-i\delta\mu t\right)\right]^{-1}$	$0<\alpha\leq 2,\delta>0,-1\leq\beta\leq 1,$ $\sigma>0,\mu\in \mathrm{R}$
DeLinnik	$\left[1+\delta^{-1}\log\left(1+\delta\sigma^{\alpha}\|t\|^{\alpha}\right)\right]^{-1}$	$0<\alpha\leq 2,\delta>0,\beta=0,$ $\sigma>0,\mu=0$
DeAsymmmetric Laplace	$\left[1+\delta^{-1}\log\left(1+\delta\sigma^{2}\|t\|^{2}-i\delta\mu t\right)\right]^{-1}$	$\alpha=2,\delta>0,\beta=0,$ $\sigma>0,\mu\in \mathrm{R}$
DeSymmmetric Laplace	$\left[1+\delta^{-1}\log\left(1+\delta\sigma^{2}\|t\|^{2}\right)\right]^{-1}$	$\alpha=2,\delta>0,\beta=0,$ $\sigma>0,\mu=0$
DeMitag-Leffler(DeML)	$\left[1+\delta^{-1}\log\left(1+\delta\sigma^{\alpha}(-it)^{\alpha}\right)\right]^{-1}$	$0<\alpha<1,\ \sigma=\sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha},$ $\beta=1$ and $\mu=0,\ \delta>0$
DeExponential	$\left[1+\delta^{-1}\log\left(1-\delta\mu it\right)\right]^{-1}$	$\alpha=1,\ \sigma=0,$ $\beta=1$ and $\mu>0,\ \delta>0$

Table 3.2: Special cases of DeGS laws.

3.3 AR(1) model with GeoGGS marginals

In this section, we develop a first order new autoregressive process with GeoGGS marginals. Consider an autoregressive structure given by,

$$X_n=\begin{cases}\epsilon_n, & \text{w.p} \quad \gamma,\\ X_{n-1}+\epsilon_n, & \text{w.p} \quad 1-\gamma,\end{cases} \tag{3.5}$$

where $0<\gamma<1$. Now we shall construct an $AR(1)$ process with stationary marginal as GeoGGS distribution.

Theorem 3.3.1. *Consider an autoregressive process $\{X_n\}$ with structure given by (3.5). Then $\{X_n\}$ is strictly stationary Markovian with GeoGGS$(\lambda,\alpha,\beta,\sigma,\mu)$ marginal if and only if $\{\epsilon_n\}$ are distributed as GeoGGS$(\gamma\lambda,\alpha,\beta,\sigma,\mu)$ provided that X_0 is distributed as GeoGGS$(\lambda,\alpha,\beta,\sigma,\mu)$.*

Proof. Let us denote the Laplace transform of $\{X_n\}$ by $\psi_{X_n}(t)$ and that of ϵ_n

by $\psi_{\epsilon_n}(t)$. Then the equation (3.5) in terms of characteristic function becomes

$$\psi_{X_n}(t) = \gamma\psi_{\epsilon_n}(t) + (1-\gamma)\psi_{X_{n-1}}(t)\psi_{\epsilon_n}(t).$$

On assuming stationarity, it reduces to the form

$$\psi_X(t) = \gamma\psi_\epsilon(t) + (1-\gamma)\psi_X(t)\psi_\epsilon(t).$$

Write

$$\psi_X(t) = \Big[1 + \lambda \log\big(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\big)\Big]^{-1}$$

and hence

$$\psi_\epsilon(t) = \frac{\psi_X(t)}{\gamma + (1-\gamma)\psi_X(t)} \tag{3.6}$$

becomes

$$\psi_\epsilon(t) = \Big[1 + \gamma\lambda \log\big(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\big)\Big]^{-1}.$$

Hence it follows that $\epsilon_n \stackrel{d}{=} GeoGGS(\gamma\lambda, \alpha, \beta, \sigma, \mu)$.

The converse can be proved by the method of mathematical induction as follows: Now assume that $X_{n-1} \stackrel{d}{=} GeoGGS(\lambda, \alpha, \beta, \sigma, \mu)$. Then

$$\begin{aligned}
\psi_{X_n}(t) &= \psi_{\epsilon_n}(t)[\gamma + (1-\gamma)\psi_{X_{n-1}}(t)] \\
&= \Big[1 + \gamma\lambda \log\big(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\big)\Big]^{-1} \\
&\text{x}[\gamma + (1-\gamma)\Big[1 + \lambda \log\big(1 + \delta\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\big)\Big]^{-1} \\
&= \Big[1 + \lambda \log\big(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\big)\Big]^{-1}
\end{aligned} \tag{3.7}$$

That is, $X_n \stackrel{d}{=} GeoGGS(\lambda, \alpha, \beta, \sigma, \mu)$ □

The joint distribution of X_n and X_{n-1}

Consider the autoregressive structure given in (3.5). It can be written as

$$X_n = I_n X_{n-1} + \epsilon_{n-1}, \text{where} P(I_n = 0) = \gamma, P(I_n = 1) = 1 - \gamma$$

Then the joint ch.f of(X_n, X_{n-1}) is given by

$$\begin{aligned} \psi_{X_{n-1},X_n}(t_1,t_2) &= E\left[e^{it_1 X_{n-1}+it_2 X_n}\right] \\ &= E\left[e^{it_1 X_{n-1}+it_2(I_n X_{n-1}+\epsilon_n)}\right] \\ &= E[e^{(it_1+it_2 I_n)X_{n-1}}]\psi_{\epsilon_n}(t_2) \\ &= [\frac{1}{1+\gamma\lambda\log\left(1+\sigma^\alpha|t_2|^\alpha\omega_{\alpha,\beta}(t_2)-i\mu t_2\right)}] \\ &\text{x}[\frac{\gamma}{1+\lambda\log\left(1+\sigma^\alpha|t_1|^\alpha\omega_{\alpha,\beta}(t_1)-i\mu t_1\right)}+\frac{(1-\gamma)}{1+\lambda\log\left(1+\sigma^\alpha|t_1+t_2|^\alpha\omega_{\alpha,\beta}(t_1+t_2)-i\mu(t_1+t_2)\right)}] \end{aligned} \tag{3.8}$$

This shows the process is not time reversible.

3.4 Generalisation to a k^{th} order GeoGGS autoregressive process

Lawrence and Lewis(1982) constructed higher order analogs of the autoregressive equation(3.5) with structure as given below.

$$X_n = \begin{cases} \epsilon_n, & \text{w.p} \quad \gamma, \\ X_{n-1}+\epsilon_n, & \text{w.p} \quad \gamma_1, \\ \vdots & \\ X_{n-k}+\epsilon_n, & \text{w.p} \quad \gamma_k, \end{cases} \tag{3.9}$$

where $\gamma_1 + \gamma_2 + \ldots + \gamma_k = 1 - \gamma, 0 \le \gamma_i, \gamma \le 1, i = 1, 2, \ldots, k$ and ϵ_n is independent of $\{X_n, X_{n-1}, \ldots\}$. In terms of characteristic function, equation

(3.9) can be written as

$$\psi_{X_n}(t) = \gamma\psi_{\epsilon_n}(t) + \gamma_1\psi_{X_{n-1}}(t)\psi_{\epsilon_n}(t) + \ldots + \gamma_k\psi_{X_{n-k}}(t)\psi_{\epsilon_n}(t)$$

Assuming stationarity, we get

$$\psi_\epsilon(t) = \frac{\psi_X(t)}{\gamma + (1-\gamma)\psi_X(t)}.$$

This establishes that the results developed in the above section are valid in this case also . This gives to the k^{th} order GeoGGS autoregressive process.

3.5 Geometric generalized normal geometric stable distributions

In this section, geometric generalized normal geometric stable(GeoGNGS) distributions is introduced and its properties are studied.

A random variable Y is said to have generalized normal geomeric stable distribuion $GNGS(\eta, \tau, \lambda, \alpha, \beta, \sigma, \mu)$ if there are parameters $0 < \alpha \leq 2, \lambda > 0, -1 \leq \beta \leq 1, \tau > 0, \sigma > 0$, and $\eta, \mu \in \Re$ such that its characteristic function, $\phi(t)$ has the following form:

$$\phi_Y(t) = \left[\frac{\exp\{i\eta t - \frac{\tau^2 t^2}{2}\}}{1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t}\right]^\lambda.$$

Now,

$$\left[\frac{\exp\{i\eta t - \frac{\tau^2 t^2}{2}\}}{1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t}\right]^\lambda = \exp\left\{1 - \frac{1}{\left[1 + \lambda\frac{\tau^2 t^2}{2} - i\lambda\eta t + \lambda\log(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t)\right]^{-1}}\right\}. \tag{3.10}$$

Since GNGS distribution is infinitely divisible, it follows that

$$\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{-1}$$

is geometrically infinite divisible.

A distribution with characteristic function

$$\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{-1}$$

is called GeoGNGS distribution. It is denoted as $GeoGNGS(\eta,\tau,\lambda,\alpha,\beta,\sigma,\mu)$.

Definition 3.5.1. *A random variable X is said to follow geometric generalized normal geometric stable distribution and write $X \sim GeoGNGS(\eta,\tau,\lambda,\alpha,\beta,\sigma,\mu)$ if it has the characteristic function*

$$\phi_X(t)=\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{-1} \tag{3.11}$$

where $\eta\in\Re,\tau>0,0<\alpha\le 2,\lambda>0,-1\le\beta\le 1,\sigma>0,\mu\in\Re$.

Special cases of GeoGNGS laws listed in the Table 3.3 below

Distribution	Charcteristic function	Parametric values
GeoNGS	$\left[1+\frac{\tau^2t^2}{2}-i\eta t+\log(1+\sigma^\alpha\|t\|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{-1}$	$\eta\in\Re,\tau>0,0<\alpha\le 2,\lambda=1,-1\le\beta\le 1,$ $\sigma>0,\mu\in\mathrm{R}$
Geometric generalized normal-Linnik	$\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log\left(1+\sigma^\alpha\|t\|^\alpha\right)\right]^{-1}$	$\eta\in\Re,\tau>0,0<\alpha\le 2,\lambda>0,\beta=0,$ $\sigma>0,\mu=0$
Geometric normal-Linnik	$\left[1+\frac{\tau^2t^2}{2}-i\eta t+\log\left(1+\sigma^\alpha\|t\|^\alpha\right)\right]^{-1}$	$\eta\in\Re,\tau>0,0<\alpha\le 2,\lambda=1,\beta=0,$ $\sigma>0,\mu=0$
Geometric generalized normal-asymmmetric Laplace	$\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log\left(1+\sigma^2\|t\|^2-i\mu t\right)\right]^{-1}$	$\eta\in\Re,\tau>0,\alpha=2,\lambda>0,\beta=0,$ $\sigma>0,\mu\in\mathrm{R}$
Geometric generalized normal-symmmetric Laplace	$\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log\left(1+\sigma^2\|t\|^2\right)\right]^{-1}$	$\eta\in\Re,\tau>0,\alpha=2,\lambda>0,\beta=0,$ $\sigma>0,\mu=0$
Geometric normal-asymmmetric Laplace	$\left[1+\frac{\tau^2t^2}{2}-i\eta t+\log\left(1+\sigma^2\|t\|^2-i\mu t\right)\right]^{-1}$	$\eta\in\Re,\tau>0,\alpha=2,\lambda=1,\beta=0,$ $\sigma>0,\mu\in\mathrm{R}$
Geometric normal-symmetric Laplace	$\left[1+\frac{\tau^2t^2}{2}-i\eta t+\log\left(1+\sigma^2\|t\|^2\right)\right]^{-1}$	$\eta\in\Re,\tau>0,\alpha=2,\lambda=1,\beta=0,$ $\sigma>0,\mu=0$
Geometric generalized normal-Mittag-Leffler	$\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log\left(1+\sigma^\alpha(-it)^\alpha\right)\right]^{-1}$	$\eta\in\Re,\tau>0,0<\alpha<1,\ \sigma=\sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha},$ $\beta=1$ and $\mu=0,\ \lambda>0$
Geometric normal-Mittag-Leffler	$\left[1+\frac{\tau^2t^2}{2}-i\eta t+\log\left(1+\sigma^\alpha(-it)^\alpha\right)\right]^{-1}$	$\eta\in\Re,\tau>0,0<\alpha<1,\ \sigma=\sigma[\cos(\frac{\pi\alpha}{2})]^{1/\alpha},$ $\beta=1$ and $\mu=0,\ \lambda=1$
Geometric normal-Gamma	$\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log\left(1-\mu it\right)\right]^{-1}$	$\eta\in\Re,\tau>0,\alpha=1,\ \sigma=0,$ $\beta=1$ and $\mu>0,\ \lambda>0$
Geometric normal-exponential	$\left[1+\frac{\tau^2t^2}{2}-i\eta t+\log\left(1-\mu it\right)\right]^{-1}$	$\eta\in\Re,\tau>0,\alpha=1,\ \sigma=0,$ $\beta=1$ and $\mu>0,\ \lambda=1$

Table 3.3: Special cases of GeoGNGS laws.

The results presented below, shows some immediate properties of GeoGNGS distributions, which are analogous to the results of GeoGGS distributions.

Theorem 3.5.1. *Let $X_1, X_2, \ldots$ be independent and identically distributed as geometric generalized normal-geometric stable random variables with parameters $\eta, \tau, \lambda,\ \alpha,\ \beta,\ \sigma,\ \mu$, that is, $X_i \sim GeoGNGS(\eta,\tau,\lambda,\alpha,\beta,\sigma,\mu), i = 1, 2, \ldots$ and $N(\gamma)$ be a geometric with mean $1/\gamma$, that is, $P[N(\gamma) = k] = \gamma(1-\gamma)^{k-1}, k = 1, 2, \ldots, 0 < \gamma < 1$. Define $Y = X_1 + X_2 + \cdots + X_{N(\gamma)}$. Then $Y \sim GeoGNGS(\eta, \tau, \frac{\lambda}{\gamma}, \alpha, \beta, \sigma, \mu)$*

Proof. Since $X_i \sim GeoGNGS(\eta,\tau,\lambda,\alpha,\beta,\sigma,\mu)$, then its characteristic function is ,

$$\phi_X(t) = \left[1 + \lambda\frac{\tau^2t^2}{2} - i\lambda\eta t + \lambda\log(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t)\right]^{-1}$$

Then the characteristic function of Y is

$$\begin{aligned}
\phi_Y(t) &= \sum_{k=1}^{n}[\phi_X(t)]^k\gamma(1-\gamma)^{k-1} \\
&= \frac{\gamma\phi_X(t)}{1-(1-\gamma)\phi_X(t)} \\
&= \frac{\gamma\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{-1}}{1-(1-\gamma)\left[1+\lambda\frac{\tau^2t^2}{2}-i\lambda\eta t+\lambda\log(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{-1}} \\
&= \left[1+\frac{\lambda}{\gamma}\frac{\tau^2t^2}{2}-i\frac{\lambda}{\gamma}\eta t+\frac{\lambda}{\gamma}\log(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{-1}
\end{aligned} \tag{3.12}$$

Hence $Y \sim GeoGNGS(\eta,\tau,\frac{\lambda}{\gamma},\alpha,\beta,\sigma,\mu)$. □

Theorem 3.5.2. *Suppose $X_1, X_2, \ldots$ be independent and identically distributed as $GNGS(\eta,\tau,\frac{\lambda}{n},\alpha,\beta,\sigma,\mu)$ and N, independent of $X_1, X_2, \ldots$ be a geometric random variables with probability of success $1/n$. Then, $Y = X_1 + X_2 + \cdots + X_N$ distributed as $GeoGNGS(\eta,\tau,\lambda,\alpha,\beta,\sigma,\mu)$ as $n \to \infty$.*

Proof.

$$\begin{aligned}
\phi_X(t) &= \left[\frac{\exp\{i\eta t-\frac{\tau^2t^2}{2}\}}{1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t}\right]^{\frac{\lambda}{n}} \\
&= \left[\exp\{-i\eta t+\frac{\tau^2t^2}{2}\}(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{-\frac{\lambda}{n}} \\
&= \left\{1+\left[\left[\exp\{-i\eta t+\frac{\tau^2t^2}{2}\}(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{\frac{\lambda}{n}}-1\right]\right\}^{-1}.
\end{aligned}$$

Hence by Lemma 3.2 of Pillai(1990b)

$$\phi_n(t) = \left\{1+n\left[\left[\exp\{-i\eta t+\frac{\tau^2t^2}{2}\}(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t)-i\mu t)\right]^{\frac{\lambda}{n}}-1\right]\right\}^{-1}$$

is the characteristic function of Y. Taking limit as $n \to \infty$, we have

$$
\begin{aligned}
\phi(t) &= \lim_{n\to\infty} \phi_n(t) \\
&= \left\{ 1 + \lim_{n\to\infty} n \left[\left[\exp\{-i\eta t + \frac{\tau^2 t^2}{2}\}(1+\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t) \right]^{\frac{\lambda}{n}} - 1 \right] \right\}^{-1} \\
&= \left[1 + \lambda\frac{\tau^2 t^2}{2} - i\lambda\eta t + \lambda \log(1+\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t) \right]^{-1}
\end{aligned}
\tag{3.13}
$$

□

Theorem 3.5.3. *Let $X_1, X_2, \ldots$ be i.i.d with $GeoGNGS(\eta, \tau, \frac{\lambda}{n}, \alpha, \beta, \sigma, \mu)$. Then, $Y = X_1 + X_2 + \cdots + X_n \xrightarrow{d} GNGS(\eta, \tau, \lambda, \alpha, \beta, \mu)$ as $n \to \infty$.*

Proof. The characteristic function of $GeoGNGS(\eta, \tau, \frac{\lambda}{n}, \alpha, \beta, \sigma, \mu)$ distribution is

$$\phi_X(t) = \left[1 + \frac{\lambda}{n}\frac{\tau^2 t^2}{2} - i\frac{\lambda}{n}\eta t + \frac{\lambda}{n}\log(1+\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t) \right]^{-1}$$

Then the characteristic function of Y is

$$\phi_Y(t) = \left[1 + \frac{\lambda}{n}\frac{\tau^2 t^2}{2} - i\frac{\lambda}{n}\eta t + \frac{\lambda}{n}\log(1+\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t) \right]^{-n}$$

Hence,

$$\lim_{n\to\infty} \phi_Y(t) = \left[\frac{\exp\{i\eta t - \frac{\tau^2 t^2}{2}\}}{1+\sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t} \right]^{\lambda}$$

That is, $Y \xrightarrow{d} GNGS(\eta, \tau, \lambda, \alpha, \beta, \sigma, \mu)$. □

3.6 AR(1) model with GeoGNGS marginals

In this section, we develop a first order new autoregressive process with GeoGNGS marginals.

Theorem 3.6.1. *Consider an autoregressive process $\{X_n\}$ with structure given by (3.5). Then $\{X_n\}$ is strictly stationary Markovian with $GeoGNGS(\eta,\tau,\lambda,\alpha,\beta,\sigma,\mu)$ marginal if and only if $\{\epsilon_n\}$ are distributed as $GeoGNGS(\eta,\tau,\gamma\lambda,\alpha,\beta,\sigma,\mu)$ provided that X_0 is distributed as $GeoGNGS(\eta,\tau,\lambda,\alpha,\beta,\sigma,\mu)$.*

Proof. Let us denote the Laplace transform of $\{X_n\}$ by $\psi_{X_n}(t)$ and that of ϵ_n by $\psi_{\epsilon_n}(t)$. Then equation (3.5) in terms of characteristic function becomes

$$\psi_{X_n}(t) = \gamma\psi_{\epsilon_n}(t) + (1-\gamma)\psi_{X_{n-1}}(t)\psi_{\epsilon_n}(t).$$

On assuming stationarity, we get

$$\psi_X(t) = \gamma\psi_\epsilon(t) + (1-\gamma)\psi_X(t)\psi_\epsilon(t).$$

Write

$$\psi_X(t) = \left[1+\lambda\frac{\tau^2t^2}{2} - i\lambda\eta t + \lambda\log(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t)\right]^{-1}$$

and hence

$$\psi_\epsilon(t) = \frac{\psi_X(t)}{\gamma+(1-\gamma)\psi_X(t)} \tag{3.14}$$

becomes

$$\psi_\epsilon(t) = \left[1+\gamma\lambda\frac{\tau^2t^2}{2} - i\gamma\lambda\eta t + \gamma\lambda\log(1+\sigma^\alpha|t|^\alpha\omega_{\alpha,\beta}(t) - i\mu t)\right]^{-1}.$$

Hence it follows that $\epsilon_n \stackrel{d}{=} GeoGNGS(\eta,\tau,\gamma\lambda,\alpha,\beta,\sigma,\mu)$.

The converse can be proved by the method of mathematical induction as follows. Now assume that $X_{n-1} \stackrel{d}{=} GeoGNGS(\eta, \tau, \lambda, \alpha, \beta, \sigma, \mu)$. Then

$$\begin{aligned}
\psi_{X_n}(t) &= \psi_{\epsilon_n}(t)[\gamma + (1-\gamma)\psi_{X_{n-1}}(t)] \\
&= \left[1 + \gamma\lambda\frac{\tau^2 t^2}{2} - i\gamma\lambda\eta t + \gamma\lambda\log(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t)\right]^{-1} \\
&\text{x}[\gamma + (1-\gamma)\left[1 + \lambda\frac{\tau^2 t^2}{2} - i\lambda\eta t + \lambda\log(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t)\right]^{-1} \\
&= \left[1 + \lambda\frac{\tau^2 t^2}{2} - i\lambda\eta t + \lambda\log(1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t)\right]^{-1}
\end{aligned} \tag{3.15}$$

That is, $X_n \stackrel{d}{=} GeoGGS(\eta, \tau, \lambda, \alpha, \beta, \sigma, \mu)$. □

CHAPTER 4

WRAPPED GENERALIZED GEOMETRIC STABLE DISTRIBUTIONS

4.1 Introduction

Researchers studied circular distributions extensively because of its application in wide variety of fields. Gatto and Jammalamadaka(2003) studied the cases of wrapped Cauchy, normal and stable distributions. Jammalamadaka and Kozubowski(2004) discussed circular distributions obtained by wrapping the classical exponential and Laplace distributions on the real line around the circle. Gatto and Jammalamadaka(2007) introduced a generalization of the von Mises distribution and studied many features of the distribution. Jacob and Jayakumar(2013) proposed a new family of circular distributions by wrapping geometric distribution. Adnan and Roy(2014) derived wrapped variance gamma distribution and showed its applicability to wind direction. Joshi and Jose(2018) explored Wrapped Lindley distribution and applied the model to a data set on orientations of turtles after laying eggs. Varghese and Jose(2018) studied Wrapped hb-skewed Laplace distribution and its application in meteorology. For more references see, Lévy(1939), Jammalamadaka and Gupta(2001)and Rao et al.(2007).

The modeling of financial data such as stock returns, commodity prices, foreign currency exchange rates, have attracted the attention of numerous researchers. The first step towards the statistical modeling of stock price changes was taken by Bachelier(1900)(see, Kozubowski(1994)). His approach was based on three assumptions: independence, identical distribution and finite variance of daily changes. Since the price change over a certain period of time can be regarded as the sum of changes over shorter periods of time(weekly change = sum of daily changes, daily change = sum of changes between of the various transactions, etc.), Bachelier(1900) arrived at a normal model. Further studies, however, showed that empirical distributions of stock returns had more kurtosis, than was predicted by the normal distribution. Mandelbrot(1963a, b) and Fama(1965) proposed the symmetric stable distribution as a model for asset returns. The family of stable distributions seemed appropriate, because they could allow independent and identically distributed returns and, at the same time, account for the observed leptokurtosis in the data. Later, studies showed that the characteristic exponent does not, as it should, remain constant as the sampling period is increased. In response to these empirical inconsistencies, alternatives to the stable laws have been proposed for asset returns models. Mittnik and Rachev (1989, 1991, 1993) have considered various probability schemes and extended the stability concept of Mandelbrot, which arises from one specific (summation) scheme. These lead to a variety of distributions, stable with respect to the underlying scheme. They also fitted these alternative stable distributions to the stock-index data and compared the appropriateness. Their findings were that the (double) Weibull distribution, which arises in the geometric summation scheme, dominates all other alternative stable laws. The theory of geometric stable(GS) distributions was studied extensively. Recently Jayakumar and Sajayan(2020) introduced the generalized geometric stable distributions(GGS) as a generalization of geometric stable distributions(GS)and discussed its application to financial data modeling.

In this chapter, a new wrapped distribution namely Wrapped generalized geometric stable(WGGS)distribution is introduced. The probability density function and cumulative density function are derived and the shapes of the probability density function for different values of the parameters are presented. Expressions for characteristic function and trigonometric moments are derived. Expressions for skewness, kurtosis etc. are derived. Maximum likelihood estimation method is used for estimating parameters and a simulation study is carried out to show the consistency of the MLEs.

4.2 Wrapped generalized geometric stable distributions

We define the circular random variable

$$\Theta = X \mod (2\pi) \in [0, 2\pi) \tag{4.1}$$

where $X \sim GGS(\lambda, \alpha, \sigma, \beta, \mu)$ with characteristic function

$$\phi(t) = [1 + \sigma^{\alpha}|t|^{\alpha}\omega_{\alpha,\beta}(t) - i\mu t]^{-\lambda}, \tag{4.2}$$

$0 < \alpha \leq 2, \lambda > 0, -1 \leq \beta \leq 1, \sigma > 0, \text{and } \mu \in R.$

Since the Fourier coefficients for a wrapped circular random variable corresponds to the characteristic function at integer values for the unwrapped random variable(see, Mardia(1972)), $\Phi_\Theta(p) : p = 0, \pm1, \pm2, \ldots$ of the characteristic function of Θ is given by

$$\Phi_\Theta(p) = \phi_X(p). \tag{4.3}$$

Therefore, using (4.2), the characteristic function corresponding to the wrapped generalized geometric stable(WGGS) angular random variable is

$$\Phi_\Theta(p) = [1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(p) - i\mu^* p]^{-\lambda} \tag{4.4}$$

where $\mu^* = \mu \mod (2\pi) \in [0, 2\pi)$. Thus for $p = 1, 2, \ldots$, we have

$$\Phi_\Theta(p) = \begin{cases} [1 + \sigma^\alpha p^\alpha (1 - i\beta \tan(\frac{\pi\alpha}{2})) - i\mu^* p]^{-\lambda}, & if\ \alpha \neq 1, \\ [1 + \sigma^\alpha p^\alpha (1 + i\beta \frac{2}{\pi} \log|p|) - i\mu^* p]^{-\lambda}, & if\ \alpha = 1. \end{cases} \tag{4.5}$$

We shall use the notation $\Theta \sim WGGS(\lambda, \alpha, \beta, \sigma, \mu^*)$ to denote that Θ is distributed according to the wrapped generalized geometric stable distribution under this parametrization.

Definition 4.2.1. *An angular random variable Θ is said to follow WGGS distribution with parameters $\lambda, \alpha, \beta, \sigma, \mu^*$ if its characteristic function is*

$$\Phi_\Theta(p) = [1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(p) - i\mu^* p]^{-\lambda} \tag{4.6}$$

where

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta\, sign(x) \tan(\pi\alpha/2), & if \quad \alpha \neq 1, \\ 1 + i\beta(2/\pi)\, sign(x) \log|x|, & if \quad \alpha = 1. \end{cases}$$

The characteristic function, $\Phi_\Theta(p)$ can be written as

$$\Phi_\Theta(p) = \rho_p e^{i\mu_p}, \quad p = 0, \pm 1, \pm 2, \ldots$$

where

$$\rho_p = \begin{cases} \left[(1+\sigma^\alpha |p|^\alpha)^2 + (\sigma^\alpha |p|^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-\lambda}{2}}, & \text{if } \alpha \neq 1, \\ \left[(1+\sigma^\alpha |p|^\alpha)^2 + (\mu^* p - \sigma^\alpha |p|^\alpha \beta \frac{2}{\pi} \log|p|)^2\right]^{\frac{-\lambda}{2}}, & \text{if } \alpha = 1, \end{cases} \tag{4.7}$$

and

$$\mu_p = \begin{cases} \lambda \arctan(\frac{\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha}) \mod (2\pi), & \text{if } \alpha \neq 1, \\ \lambda \arctan(\frac{\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi} \log|p|}{1+\sigma^\alpha p^\alpha})) \mod (2\pi), & \text{if } \alpha = 1. \end{cases}$$

The probability density function of the WGGS angular random variable $\Theta \in [0, 2\pi)$, is given by

$$\begin{aligned} f_w(\theta) &= \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \Phi_\Theta(p) \exp(-ip\theta) \\ &= \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} [1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(p) - i\mu^* p]^{-\lambda} \exp(-ip\theta). \end{aligned}$$

On simplification, we get

$$f_w(\theta) = \frac{1}{2\pi} \left[1 + 2\sum_{p=1}^{\infty} (\alpha_p \cos(p\theta) + \beta_p \sin(p\theta)) \right], \tag{4.8}$$

where

$$\alpha_p = \begin{cases} \left[(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-\lambda}{2}} \cos\left(\lambda \arctan(\frac{\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right), & \text{if } \alpha \neq 1, \\ \left[(1+\sigma^\alpha p^\alpha)^2 + (\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi} \log|p|)^2\right]^{\frac{-\lambda}{2}} \cos\left(\lambda \arctan(\frac{\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi} \log|p|}{1+\sigma^\alpha p^\alpha})\right), & \text{if } \alpha = 1, \end{cases}$$

and

$$\beta_p = \begin{cases} \left[(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-\lambda}{2}} \sin\left(\lambda \arctan(\frac{\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right), & \text{if } \alpha \neq 1, \\ \left[(1+\sigma^\alpha p^\alpha)^2 + (\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi} \log|p|)^2\right]^{\frac{-\lambda}{2}} \sin\left(\lambda \arctan(\frac{\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi} \log|p|}{1+\sigma^\alpha p^\alpha})\right), & \text{if } \alpha = 1. \end{cases}$$

Using (4.8), we get the distribution function $F_w(\theta)$ as

$$F_w(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty}\{\frac{\alpha_p}{p}\sin(p\theta) + \frac{\beta_p}{p}(1-\cos(p\theta))\}\right]. \tag{4.9}$$

Special cases:

Let $\lambda = 1$. Then (4.6) becomes $\Phi_\Theta(p) = [1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(p) - i\mu^* p]^{-1}$, which is the characteristic function of wrapped geometric stable distributions(see, Jacob(2012)). The corresponding α_p and β_p are as follows:

For $\alpha \neq 1$

$$\begin{aligned} \alpha_p &= \left[(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-1}{2}} \cos\left(\arctan(\frac{\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right) \\ &= \frac{1+\sigma^\alpha p^\alpha}{(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2}, \quad p = 1, 2, \ldots \end{aligned}$$

and

$$\begin{aligned} \beta_p &= \left[(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-1}{2}} \sin\left(\arctan(\frac{\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right) \\ &= \frac{\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p}{(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2}, \quad p = 1, 2, \ldots \end{aligned}$$

Similarly for $\alpha = 1$

$$\begin{aligned} \alpha_p &= \frac{1+\sigma^\alpha p^\alpha}{(1+\sigma^\alpha p^\alpha)^2 + (\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi} \ln|p|)^2} \\ \text{and} \qquad \beta_p &= \frac{\mu^* p - \sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2})}{(1+\sigma^\alpha p^\alpha)^2 + (\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi} \ln|p|)^2} \end{aligned}$$

Analogous to the representation (2.23) of GGS random variable, we present below, the representation of WGGS random variable as follows:
Let V$\sim$ WGGS$(\lambda, \alpha, \beta, \sigma, \mu^*)$. Then

$$V = \begin{cases} \mu\Theta_{\mathrm{W}} + (\Theta_{\mathrm{W}})^{\frac{1}{\alpha}}\Theta_S \mod (2\pi), & \text{if } \alpha \neq 1, \\ \mu\Theta_{\mathrm{W}} + \Theta_{\mathrm{W}}\Theta_S + \sigma\Theta_{\mathrm{W}}\beta(2/\pi)\log(\Theta_{\mathrm{W}}) \mod (2\pi), & \text{if } \alpha = 1, \end{cases} \tag{4.10}$$

where Θ_S $\sim$WS$(\alpha, \beta, \sigma, 0)$(see, Pewsey(2008)). Θ_{W} is wrapped gamma, $W\Gamma(\lambda, 1)$ with characteristic function $(1 - ip)^{-\lambda}$(see, Coelho(2007))and is independent of Θ_S. Note that $\Theta_S \sim WS(\alpha, \beta, \sigma, \mu^*)$ has the characteristic function $\phi(p) = \exp\{-\sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(p) + i\mu^* p\}$, where $\mu^* \in [0, 2\pi)$.

Theorem 4.2.1. $\Theta \sim WGGS(\lambda, \alpha, \beta, \sigma, \mu^*)$*is infinitely divisible*

Proof. Let $\Theta_1, \Theta_2, \ldots \Theta_n$ be identically and independently distributed random variables with $WGGS(\frac{\lambda}{n}, \alpha, \beta, \sigma, \mu^*)$ distribution. Define $\Theta = \Theta_1 + \Theta_2 + \ldots \Theta_n$ mod (2π). Then the characteristic function of Θ, is

$$\begin{aligned} \Phi_\Theta(p)) &= \left([1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(p) - i\mu^* p]^{-\frac{\lambda}{n}}\right)^n \\ &= [1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(p) - i\mu^* p]^{-\lambda} \end{aligned} \tag{4.11}$$

Hence Θ is infinitely divisible. □

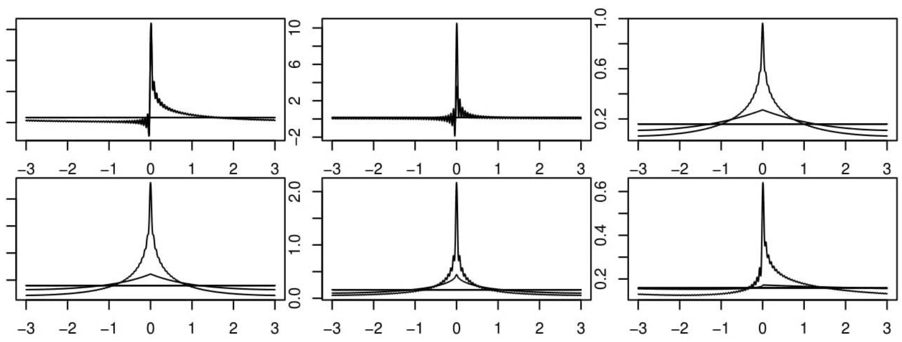

Figure 4.1: Densities of WGS($\lambda = 1$) and WGGS for various parameter values.

We draw the densities, five in each plot for λ =0.5, 1, 5, 20 and 100 while keeping the other parameters constant. The values of $(\alpha, \sigma, \beta, \mu^*)$ are (1,0,1,3),(0.5,2,1,0),(2,2,0,0),(2,2,0,6),(1.4,2,0,0),(1.4,10,-1,1) respectively.

4.3 Trigonometric moments and other parameters

By the definition of trigonometric moments, we have

$$\Phi_\Theta(p) = \alpha_p + i\beta_p, \quad p = \pm 1, \pm 2, \ldots$$

and, hence, the non-central moments of the respective distribution are given by

$$\alpha_p = \rho_p \cos(\mu_p) \quad \text{and} \quad \beta_p = \rho_p \sin(\mu_p).$$

So, we have, for $\alpha \neq 1$

$$\alpha_p = \left[(1+\sigma^\alpha |p|^\alpha)^2 + (\sigma^\alpha |p|^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-\lambda}{2}} \cos\left(\lambda \arctan(\frac{\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right)$$

$$\text{and} \quad \beta_p = \left[(1+\sigma^\alpha |p|^\alpha)^2 + (\sigma^\alpha |p|^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-\lambda}{2}} \sin\left(\lambda \arctan(\frac{\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right).$$

For $\alpha = 1$,

$$\alpha_p = \left[(1+\sigma^\alpha|p|^\alpha)^2 + (\mu^* p - \sigma^\alpha|p|^\alpha\beta\frac{2}{\pi}\log|p|)^2\right]^{\frac{-\lambda}{2}} \cos\left(\lambda \arctan(\frac{\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi}\log|p|}{1+\sigma^\alpha p^\alpha}))\right),$$

$$\text{and} \quad \beta_p = \left[(1+\sigma^\alpha|p|^\alpha)^2 + (\mu^* p - \sigma^\alpha|p|^\alpha\beta\frac{2}{\pi}\log|p|)^2\right]^{\frac{-\lambda}{2}} \sin\left(\lambda \arctan(\frac{\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi}\log|p|}{1+\sigma^\alpha p^\alpha}))\right).$$

The mean direction, $\mu = \mu_1 \in [0, 2\pi)$ is

$$\mu = \begin{cases} \lambda \arctan(\frac{\sigma^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^*}{1+\sigma^\alpha}) \mod (2\pi), & \text{if } \alpha \neq 1, \\ \lambda \arctan(\frac{\mu^*}{1+\sigma^\alpha})) \mod (2\pi), & \text{if } \alpha = 1. \end{cases}$$

By substituting ρ_p, μ_p and μ_1, we gets the central trigonometric moments

$$\bar{\alpha_p} = \rho_p \cos\left(\mu_p - p\mu_1\right),$$
$$\text{and} \quad \bar{\beta}_p = \rho_p \sin\left(\mu_p - p\mu_1\right).$$

The circular variance is given by, $V_0 = 1 - \rho$, where

$$\rho = \begin{cases} \left[(1+\sigma^\alpha)^2 + (\sigma^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^*)^2\right]^{\frac{-\lambda}{2}}, & \text{if } \alpha \neq 1, \\ \left[(1+\sigma)^2 + (\mu^*)^2\right]^{\frac{-\lambda}{2}}, & \text{if } \alpha = 1. \end{cases}$$

The circular standard deviation is given by

$$\sigma_0 = \begin{cases} \lambda \log((1+\sigma^\alpha)^2 + (\sigma^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^*)^2), & \text{if } \alpha \neq 1, \\ \lambda \log((1+\sigma)^2 + (\mu^*)^2), & \text{if } \alpha = 1. \end{cases} \tag{4.12}$$

The coefficient of skewness is

$$\zeta_1^0 = \frac{\bar{\beta}_2}{(1-\rho)^{3/2}}.$$

and the coefficient of kurtosis is,

$$\zeta_2^0 = \frac{\bar{\alpha_2} - \rho^4}{(1-\rho)^2}.$$

Table 4.1 exhibits various features of WGGS distribution for different values of the parameters $\lambda, \alpha, \beta, \sigma, \mu^*$.

Table 4.1: Values of different characteristics of WGGS distributions with parametric values $(\lambda, \alpha, \sigma, \beta, \mu^*)$ as A:(0.5,0.1,1,1,3); B:(0.5,0.5,2,1,0); C:(0.5,2,2,0,0); D:(0.5,2,2,0,6); E:(0.5,1,1,1,3); F:(0.5,1,2,1,0); G:(0.5,1,2,0,0); H:(0.5,1,2,0,6).

Properties	A	B	C	D	E	F	G	H
α_1	0.4531	0.5769	0.4472	0.3240	0.4643	0.5773	0.5773	0.3284
β_1	0.2493	0.1565	0	0.1517	0.2484	0	0	0.2029
α_2	0.3182	0.5040	0.2425	0.2089	0.3562	0.4280	0.4472	0.2307
β_2	0.2288	0.1526	0	0.0663	0.2041	-0.0733	0	0.1538
$\bar{\alpha_1}$	0.5172	0.5978	0.4472	0.3578	0.5266	0.5773	0.5773	0.3860
$\bar{\beta_1}$	0	0	0	0	0	0	0	0
$\bar{\alpha_2}$	0.3635	0.5120	0.2425	0.1847	0.3674	0.4280	0.4472	0.2408
$\bar{\beta_2}$	-0.1464	-0.1230	0	-0.1180	-0.1832	-0.0733	0	-0.1376
ρ	0.5172	0.5978	0.4472	0.3578	0.5266	0.5773	0.5773	0.3860
V_0	0.4827	0.4021	0.5527	0.6421	0.4733	0.4226	0.4226	0.6139
σ_0	1.1483	1.0143	1.2686	1.4336	1.1324	1.0481	1.0481	1.3796
ζ_1^0	-0.4365	-0.4826	0	-0.2294	-0.5625	-0.2668	0	-0.2860
ζ_2^0	1.2528	2.3761	0.6628	0.4081	1.2965	1.7741	1.8815	0.5799

4.4 Maximum likelihood estimation

In this section, we discuss the method of maximum likelihood estimation to estimate the parameters of the WGGS model. Let $\theta_1, \theta_2, \ldots, \theta_n$ be a random sample of size n from $WGGS(\lambda, \alpha, \sigma, \beta, \mu^*)$. Then, the log-likelihood function is given by,

$$\log L = -n \log 2\pi + \textstyle\sum_{i=1}^{n} \left\{ \log(1 + 2 \sum_{p=1}^{\infty} [\alpha_p \cos(p\theta_i) + \beta_p \sin(p\theta_i)]) \right\}$$

Equating the partial derivative of log-likelihood function with respect to the parameters to zero, for $\alpha \neq 1$ we get

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}[(a^2+b^2)^{-\frac{\lambda}{2}}\{B_1 \log((a^2+b^2)^{-1/2}) + A_1 \arctan(b/a)]\}}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty} \lambda(a^2+b^2)^{-\frac{\lambda}{2}-1}[\alpha_{11}\{(a\beta\tan(\frac{\pi\alpha}{2})-b)A_1 - B_1(a+b\beta\tan(\frac{\pi\alpha}{2}))\} + (aA_1 - bB_1)\alpha_{12}]}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \sigma} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty} \lambda(a^2+b^2)^{-\frac{\lambda}{2}-1}[A_1\sigma_{11}(a\beta\tan(\frac{\pi\alpha}{2})-b) - B_1\sigma_{11}(b\beta\tan(\frac{\pi\alpha}{2})+a)]}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty} \lambda(a^2+b^2)^{-\frac{\lambda}{2}-1}[(\sigma^\alpha p^\alpha \tan(\frac{\pi\alpha}{2}))(aA_1 - bB_1)]}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \mu^*} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty} \lambda(a^2+b^2)^{-\frac{\lambda}{2}-1}[p(aA_1 - bB_1)]}{f_1} = 0,$$

where

$$\alpha_{1,p} = (a^2+b^2)^{-\frac{\lambda}{2}}\cos(\lambda\arctan(b/a)), \beta_{1,p} = (a^2+b^2)^{-\frac{\lambda}{2}}\sin(\lambda\arctan(b/a)), \sigma_{11} = \alpha\sigma^{\alpha-1}p^\alpha$$

$$a = 1+\sigma^\alpha p^\alpha, b = \sigma^\alpha p^\alpha \beta\tan(\frac{\pi\alpha}{2})+\mu^* p, f_1 = 1+2\textstyle\sum_{p=1}^{\infty}[\alpha_{1,p}\cos(p\theta_i)+\beta_{1,p}\sin(p\theta_i)]$$

$$A_1 = \cos(\lambda\arctan(b/a))\sin(p\theta_i) - \sin(\lambda\arctan(b/a))\cos(p\theta_i), \alpha_{12} = \sigma^\alpha p^\alpha \beta(\pi/2)\sec^2(\pi/2),$$

$$B_1 = \cos(\lambda\arctan(b/a))\cos(p\theta_i) + \sin(\lambda\arctan(b/a))\sin(p\theta_i), \alpha_{11} = \sigma^\alpha p^\alpha \log(\sigma p)$$

Similarly, for $\alpha = 1$,

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}[(c^2+d^2)^{-\frac{\lambda}{2}}\{\log((c^2+d^2)^{-1/2})B_2 + arctan(d/c)A_2\}]}{f_2} = 0$$

$$\frac{\partial \log L}{\partial \sigma} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}[\lambda(c^2+d^2)^{-\frac{\lambda}{2}-1}\{pB_2(d\beta(2/\pi)\log(p)-c) - pA_2(d+c\beta(2/\pi)\log(p))\}]}{f_2} = 0$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}[\lambda(c^2+d^2)^{-\frac{\lambda}{2}-1}p\sigma(2/\pi)\log(p)\{dB_2 - cA_2\}]}{f_2} = 0$$

$$\frac{\partial \log L}{\partial \mu^*} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}[\lambda(c^2+d^2)^{-\frac{\lambda}{2}-1}p\{cA_2 - dB_2\}]}{f_2} = 0,$$

where,

$$\alpha_{2,p} = (c^2 + d^2)^{-\frac{\lambda}{2}} \cos(\lambda \arctan(d/c)), \beta_{2,p} = (c^2 + d^2)^{-\frac{\lambda}{2}} \sin(\lambda \arctan(d/c)), c = 1 + \sigma p,$$

$$d = \mu^* p - \sigma p \beta (2/\pi) \log(p), B_2 = \cos(\lambda \arctan(d/c)) \cos(p\theta_i) + \sin(\lambda \arctan(d/c)) \sin(p\theta_i),$$

$$A_2 = \cos(\lambda \arctan(d/c)) \sin(p\theta_i) - \sin(\lambda \arctan(d/c)) \cos(p\theta_i), f_2 = 1 + 2\sum_{p=1}^{\infty}[\alpha_{2,p} \cos(p\theta_i) + \beta_{2,p} \sin(p\theta_i)].$$

Since the above normal equations cannot be solved analytically, a numerical technique is to be adopted to get the estimates of the parameters. The log-likelihood of the WGGS$(\lambda, \alpha, \beta, \sigma, \mu^*)$ density can be computed numerically to a given level of precision for $\ell = \sum_{i=1}^{n} \log f(\theta_i)$ using finite sum approximation to (4.8) for the given set of independent observed directions $\theta^T = (\theta_1, \theta_2, ..., \theta_n)$. The *optim* function in the R *stats* package used for the numerical optimization of ℓ over the parameters. We opted the the L-BFGS-B algorithm as it allows box constraints for any or all the parameters.

We carry out a simulation study to obtain the maximum likelihood estimate of the parameters. We generate samples of size 30,50,100 and 200 and replicate the program N=1000 times to get the estimates. The results are presented in Table 4.2.

Table 4.2: Average values of bias and MSEs using different values of $\lambda, \alpha, \beta, \sigma$ and μ^*, for sample sizes n=30, 50,100,200 corresponding to $WGGS(\lambda, \alpha, \beta, \sigma, \mu*)$ distribution

		Bias				MSE			
$(\lambda, \alpha, \beta, \sigma, \mu^*)$	Est	n=30	50	100	200	n=30	50	100	200
(20,0.6,0.9,10,6)	$\hat{\lambda}$	3.006	1.034	0.790	0.004	6.000	4.060	1.011	0.011
	$\hat{\alpha}$	0.463	0.260	0.179	0.000	0.608	0.410	0.114	0.030
	$\hat{\beta}$	0.803	0.631	0.200	0.002	12.588	5.333	1.401	0.002
	$\hat{\sigma}$	6.011	3.055	1.554	0.001	16.002	7.022	2.031	0.006
	$\hat{\mu^*}$	4.002	3.909	2.555	0.011	8.005	6.075	2.011	0.110
(10,0.8,0.7,6,4)	$\hat{\lambda}$	3.002	1.202	1.072	0.010	5.545	4.505	2.805	0.003
	$\hat{\alpha}$	0.712	0.522	0.250	0.001	5.076	4.012	2.052	0.010
	$\hat{\beta}$	0.643	0.404	0.220	0.002	8.054	5.056	0.045	0.001
	$\hat{\sigma}$	3.112	1.001	0.043	0.000	12.332	6.355	2.001	0.011
	$\hat{\mu^*}$	2.044	1.011	0.874	0.001	6.062	4.033	0.022	0.000
(5,1.2,0.5,4,2)	$\hat{\lambda}$	1.021	1.002	0.065	0.000	4.023	2.025	0.701	0.010
	$\hat{\alpha}$	1.005	0.783	0.343	0.002	6.421	4.062	1.113	0.005
	$\hat{\beta}$	0.499	0.182	0.005	0.000	3.121	1.000	0.005	0.000
	$\hat{\sigma}$	1.776	1.009	0.336	0.010	8.055	2.022	1.609	0.098
	$\hat{\mu^*}$	1.0665	0.652	0.004	0.002	4.055	3.023	2.011	0.000
(1,1.5,0.3,2,1)	$\hat{\lambda}$	1.021	1.008	0.507	0.001	2.010	1.014	0.048	0.002
	$\hat{\alpha}$	1.663	0.811	0.100	0.001	8.831	4.921	1.001	0.003
	$\hat{\beta}$	0.352	0.221	0.106	0.002	3.114	1.224	0.440	0.000
	$\hat{\sigma}$	1.101	0.654	0.340	0.002	5.011	3.211	0.531	0.031
	$\hat{\mu^*}$	0.775	0.018	0.004	0.000	3.110	1.001	0.512	0.004
(0.5,1.8,0.1,2,0.5)	$\hat{\lambda}$	0.605	0.321	0.023	0.000	2.010	0.773	0.023	0.001
	$\hat{\alpha}$	1.967	1.611	0.608	0.006	12.333	7.001	1.092	0.004
	$\hat{\beta}$	0.221	0.110	0.103	0.001	2.897	1.089	0.355	0.012
	$\hat{\sigma}$	0.893	0.342	0.109	0.003	2.887	1.005	0.701	0.001
	$\hat{\mu^*}$	0.615	0.311	0.015	0.001	2.751	1.011	0.044	0.001

It can be seen that as sample size increases, the average values of bias and MSE decreases.

4.5 Wrapped generalized normal geometric stable distributions

We define the circular random variable

$$\Theta = X \mod (2\pi) \in [0, 2\pi) \tag{4.13}$$

Since the Fourier coefficients for a wrapped circular random variable corresponds to the characteristic function at integer values for the unwrapped random variable(see, Mardia(1972)), $\Phi_\Theta(p) : p = 0, \pm 1, \pm 2, \ldots$ of the characteristic function of Θ is given by

$$\Phi_\Theta(p) = \phi_X(p). \tag{4.14}$$

Therefore, from (2.42), the characteristic function corresponding to the wrapped generalized geometric stable(WGNGS) angular random variable is

$$\Phi_X(p) = \left[\frac{\exp\{i\eta^* p - \frac{\tau^2 p^2}{2}\}}{1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(t) - i\mu^* p}\right]^\lambda \tag{4.15}$$

where $\eta^* = \eta \mod (2\pi) \in [0, 2\pi), \tau > 0, 0 < \alpha \leq 2, \lambda > 0, -1 \leq \beta \leq 1, \sigma > 0, \mu^* = \mu \mod (2\pi) \in [0, 2\pi)$. Thus, for $p = 1, 2, \ldots$, we have

$$\Phi_\Theta(p) = \begin{cases} e^{i\lambda\eta^* p - \frac{\lambda\tau^2 p^2}{2}}[1 + \sigma^\alpha p^\alpha(1 - i\beta \tan(\frac{\pi\alpha}{2})) - i\mu^* p]^{-\lambda}, & if\ \alpha \neq 1, \\ e^{i\lambda\eta^* p - \frac{\lambda\tau^2 p^2}{2}}[1 + \sigma^\alpha p^\alpha(1 + i\beta\frac{2}{\pi}\log|p|) - i\mu^* p]^{-\lambda}, & if\ \alpha = 1. \end{cases} \tag{4.16}$$

We shall use the notation $\Theta \sim WGNGS(\eta^*, \tau, \lambda, \alpha, \beta, \sigma, \mu^*)$ to denote that Θ is distributed according to the wrapped generalized geometric stable distribution under this parametrization.

Definition 4.5.1. *An angular random variable Θ is said to follow WGNGS distribution with parameters $\eta^*, \tau, \lambda, \alpha, \beta, \sigma, \mu^*$ if its characteristic function is*

$$\Phi_\Theta(p) = \left[\frac{\exp\{i\eta^* p - \frac{\tau^2 p^2}{2}\}}{1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(t) - i\mu^* p}\right]^\lambda \tag{4.17}$$

where

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta sign(x)\tan(\pi\alpha/2), & \text{if} \quad \alpha \neq 1, \\ 1 + i\beta(2/\pi) sign(x)\log|x|, & \text{if} \quad \alpha = 1. \end{cases}$$

The characteristic function, $\Phi_\Theta(p)$ can be written as

$$\Phi_\Theta(p) = \rho_p e^{i\mu_p}, \quad p = 0, \pm 1, \pm 2, \ldots$$

where

$$\rho_p = \begin{cases} e^{-\frac{\lambda\tau^2p^2}{2}} \left[(1+\sigma^\alpha|p|^\alpha)^2 + (\sigma^\alpha|p|^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^*p)^2\right]^{\frac{-\lambda}{2}}, & \text{if } \alpha \neq 1, \\ e^{-\frac{\lambda\tau^2p^2}{2}} \left[(1+\sigma^\alpha|p|^\alpha)^2 + (\mu^*p - \sigma^\alpha|p|^\alpha\beta\frac{2}{\pi}\log|p|)^2\right]^{\frac{-\lambda}{2}}, & \text{if } \alpha = 1, \end{cases} \tag{4.18}$$

and

$$\mu_p = \begin{cases} \lambda\eta^*p + \lambda\arctan(\frac{\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2})+\mu^*p}{1+\sigma^\alpha p^\alpha}) \mod (2\pi), & \text{if } \alpha \neq 1, \\ \lambda\eta^*p + \lambda\arctan(\frac{\mu^*p - \sigma^\alpha p^\alpha\beta\frac{2}{\pi}\log|p|}{1+\sigma^\alpha p^\alpha})) \mod (2\pi), & \text{if } \alpha = 1. \end{cases}$$

The probability density function of the WGNGS angular random variable $\Theta \in [0, 2\pi)$, is given by

$$\begin{aligned} f_w(\theta) &= \frac{1}{2\pi}\sum_{p=-\infty}^{\infty}\Phi_\Theta(p)\exp(-ip\theta) \\ &= \frac{1}{2\pi}\sum_{p=-\infty}^{\infty}\left[\frac{\exp\{i\eta^*p - \frac{\tau^2p}{2}\}}{1+\sigma^\alpha|p|^\alpha\omega_{\alpha,\beta}(t) - i\mu^*p}\right]^\lambda \exp(-ip\theta). \end{aligned}$$

On simplification, we get

$$f_w(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty}(\alpha_p\cos(p\theta) + \beta_p\sin(p\theta))\right], \tag{4.19}$$

where

$$\alpha_p = \begin{cases} e^{-\frac{\lambda\tau^2p^2}{2}}\left[(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-\lambda}{2}} \cos\left(\lambda\eta^* p + \lambda \arctan(\frac{\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2})+\mu^* p}{1+\sigma^\alpha p^\alpha})\right), & \text{if } \alpha \neq 1, \\ e^{-\frac{\lambda\tau^2p^2}{2}}\left[(1+\sigma^\alpha p^\alpha)^2 + (\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi}\log|p|)^2\right]^{\frac{-\lambda}{2}} \cos\left(\lambda\eta^* p + \lambda \arctan(\frac{\mu^* p-\sigma^\alpha p^\alpha\beta\frac{2}{\pi}\log|p|}{1+\sigma^\alpha p^\alpha})\right), & \text{if } \alpha = 1, \end{cases}$$

and

$$\beta_p = \begin{cases} e^{-\frac{\lambda\tau^2p^2}{2}}\left[(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-\lambda}{2}} \sin\left(\lambda\eta^* p + \lambda \arctan(\frac{\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2})+\mu^* p}{1+\sigma^\alpha p^\alpha})\right), & \text{if } \alpha \neq 1, \\ e^{-\frac{\lambda\tau^2p^2}{2}}\left[(1+\sigma^\alpha p^\alpha)^2 + (\mu^* p - \sigma^\alpha p^\alpha \beta \frac{2}{\pi}\log|p|)^2\right]^{\frac{-\lambda}{2}} \sin\left(\lambda\eta^* p + \lambda \arctan(\frac{\mu^* p-\sigma^\alpha p^\alpha\beta\frac{2}{\pi}\log|p|}{1+\sigma^\alpha p^\alpha})\right), & \text{if } \alpha = 1. \end{cases}$$

Using (4.19), we get the distribution function $F_w(\theta)$ as

$$F_w(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty}\{\frac{\alpha_p}{p}\sin(p\theta) + \frac{\beta_p}{p}(1-\cos(p\theta))\}\right]. \tag{4.20}$$

Special cases:

Let $\lambda = 1, \eta^* = 0, \tau = 0$. Then (4.17) becomes $\Phi_\Theta(p) = [1 + \sigma^\alpha|p|^\alpha\omega_{\alpha,\beta}(p) - i\mu^* p]^{-1}$, which is the characteristic function of wrapped geometric stable distributions(see, Jacob(2012)). The corresponding α_p and β_p are as follows: For $\alpha \neq 1$

$$\begin{aligned}\alpha_p &= \left[(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-1}{2}} \cos\left(\arctan(\frac{\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right) \\ &= \frac{1+\sigma^\alpha p^\alpha}{(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p)^2}, \quad p = 1, 2, \ldots\end{aligned}$$

and

$$\begin{aligned}\beta_p &= \left[(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-1}{2}} \sin\left(\arctan(\frac{\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right) \\ &= \frac{\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p}{(1+\sigma^\alpha p^\alpha)^2 + (\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p)^2}, \quad p = 1, 2, \ldots\end{aligned}$$

Similarly, for $\alpha = 1$

$$\alpha_p = \frac{1 + \sigma^\alpha p^\alpha}{(1 + \sigma^\alpha p^\alpha)^2 + (\mu^* p - \sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}))^2}$$

$$\text{and} \qquad \beta_p = \frac{\mu^* p - \sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2})}{(1 + \sigma^\alpha p^\alpha)^2 + (\mu^* p - \sigma^\alpha p^\alpha \beta \tan(\frac{\pi\alpha}{2}))^2}$$

Analogous to the representation of GNGS random variable in Proposition 2.15.1, we present below, a representation of WGNGS random variable as follows:

Let V$\sim$ WGNGS$(\eta, \tau, \lambda, \alpha, \beta, \sigma, \mu^*)$. Then

$$V = \begin{cases} \eta^*\lambda + \tau\sqrt{\lambda}\Theta_Z + \mu\Theta_{\mathrm{W}} + (\Theta_{\mathrm{W}})^{\frac{1}{\alpha}}\Theta_S \mod (2\pi), & \text{if } \alpha \neq 1, \\ \eta^*\lambda + \tau\sqrt{\lambda}\Theta_Z + \mu\Theta_{\mathrm{W}} + \Theta_{\mathrm{W}}\Theta_S + \sigma\Theta_{\mathrm{W}}\beta(2/\pi)\log(\Theta_{\mathrm{W}}) \mod (2\pi), & \text{if } \alpha = 1, \end{cases} \tag{4.21}$$

where $\Theta_Z \sim WN(\eta, \tau^2)$, Θ_S $\sim$WS$(\alpha, \beta, \sigma, 0)$(see, Pewsey(2008)). Θ_{W} is wrapped gamma, $W\Gamma(\lambda, 1)$ with characteristic function$(1 - ip)^{-\lambda}$(see, Coelho(2007))and is independent of Θ_S. Note that $\Theta_S \sim WS(\alpha, \beta, \sigma, \mu^*)$ has the characteristic function $\phi(p) = \exp\{-\sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(p) + i\mu^* p\}$ where $\mu^* \in [0, 2\pi)$.

Theorem 4.5.1. $\Theta \sim WGNGS(\eta^*, \tau, \frac{\lambda}{n}, \alpha, \beta, \sigma, \mu^*)$*is infinitely divisible*

Proof. Let $\Theta_1, \Theta_2, \ldots \Theta_n$ are identically and independently distributed random variables with $WGNGS(\eta^*, \tau, \frac{\lambda}{n}, \alpha, \beta, \sigma, \mu^*)$ distribution. Define $\Theta = \Theta_1 + \Theta_2 + \ldots \Theta_n \mod (2\pi)$. Then the characteristic function of Θ, is

$$\begin{aligned} \Phi_\Theta(p)) &= \left(\left[\frac{\exp\{i\eta^* p - \frac{\tau^2 p^2}{2}\}}{1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(t) - i\mu^* p} \right]^{\frac{\lambda}{n}} \right)^n \\ &= \left[\frac{\exp\{i\eta^* p - \frac{\tau^2 p^2}{2}\}}{1 + \sigma^\alpha |p|^\alpha \omega_{\alpha,\beta}(t) - i\mu^* p} \right]^\lambda \end{aligned} \tag{4.22}$$

Hence Θ is infinitely divisible. □

4.6 Trigonometric moments and other parameters

By the definition of trigonometric moments, we have

$$\Phi_\Theta(p) = \alpha_p + i\beta_p, \quad p = \pm 1, \pm 2, \ldots$$

and, hence, the non-central moments of the respective distribution are given by

$$\alpha_p = \rho_p \cos(\mu_p) \quad \text{and} \quad \beta_p = \rho_p \sin(\mu_p).$$

So, we have, for $\alpha \neq 1$

$$\alpha_p = e^{\frac{-\lambda\tau^2 p^2}{2}} \left[(1+\sigma^\alpha|p|^\alpha)^2 + (\sigma^\alpha|p|^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-\lambda}{2}} \cos\left(\lambda\eta^* p + \lambda\arctan(\frac{\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right)$$

$$\text{and} \quad \beta_p = e^{\frac{-\lambda\tau^2 p^2}{2}} \left[(1+\sigma^\alpha|p|^\alpha)^2 + (\sigma^\alpha|p|^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p)^2\right]^{\frac{-\lambda}{2}} \sin\left(\lambda\eta^* p + \lambda\arctan(\frac{\sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^* p}{1+\sigma^\alpha p^\alpha})\right).$$

For $\alpha = 1$,

$$\alpha_p = e^{\frac{-\lambda\tau^2 p^2}{2}} \left[(1+\sigma^\alpha|p|^\alpha)^2 + (\mu^* p - \sigma^\alpha|p|^\alpha\beta\frac{2}{\pi}\log|p|)^2\right]^{\frac{-\lambda}{2}} \cos\left(\lambda\eta^* p + \lambda\arctan(\frac{\mu^* p - \sigma^\alpha p^\alpha\beta\frac{2}{\pi}\log|p|}{1+\sigma^\alpha p^\alpha}))\right),$$

$$\text{and} \quad \beta_p = e^{\frac{-\lambda\tau^2 p^2}{2}} \left[(1+\sigma^\alpha|p|^\alpha)^2 + (\mu^* p - \sigma^\alpha|p|^\alpha\beta\frac{2}{\pi}\log|p|)^2\right]^{\frac{-\lambda}{2}} \sin\left(\lambda\eta^* p + \lambda\arctan(\frac{\mu^* p - \sigma^\alpha p^\alpha\beta\frac{2}{\pi}\log|p|}{1+\sigma^\alpha p^\alpha}))\right).$$

The mean direction, $\mu = \mu_1 \in [0, 2\pi)$ is

$$\mu = \begin{cases} \lambda\left(\eta^* + \arctan(\frac{\sigma^\alpha\beta\tan(\frac{\pi\alpha}{2})+\mu^*}{1+\sigma^\alpha})\right) \mod (2\pi), & \text{if } \alpha \neq 1, \\ \lambda\left(\eta^* + \arctan(\frac{\mu^*}{1+\sigma^\alpha})\right) \mod (2\pi), & \text{if } \alpha = 1. \end{cases}$$

By substituting ρ_p, μ_p and μ_1, we gets the central trigonometric moments

$$\bar{\alpha_p} = \rho_p \cos\left(\mu_p - p\mu_1\right),$$
$$\text{and} \quad \bar{\beta}_p = \rho_p \sin\left(\mu_p - p\mu_1\right).$$

The circular variance is given by $V_0 = 1 - \rho$, where

$$\rho = \begin{cases} e^{-\frac{\lambda\tau^2}{2}} \left[(1+\sigma^\alpha)^2 + (\sigma^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^*)^2\right]^{\frac{-\lambda}{2}}, & \text{if } \alpha \neq 1, \\ e^{-\frac{\lambda\tau^2}{2}} \left[(1+\sigma)^2 + (\mu^*)^2\right]^{\frac{-\lambda}{2}}, & \text{if } \alpha = 1. \end{cases}$$

The circular standard deviation is given by

$$\sigma_0 = \begin{cases} \sqrt{\lambda\{\tau^2 + \log((1+\sigma^\alpha)^2 + (\sigma^\alpha \beta \tan(\frac{\pi\alpha}{2}) + \mu^*)^2)\}}, & \text{if } \alpha \neq 1, \\ \sqrt{\lambda\{\tau^2 + \log((1+\sigma)^2 + (\mu^*)^2)\}}, & \text{if } \alpha = 1. \end{cases} \tag{4.23}$$

The coefficient of skewness is given by,

$$\zeta_1^0 = \frac{\bar{\beta}_2}{(1-\rho)^{3/2}}.$$

The coefficient of kurtosis is given by,

$$\zeta_2^0 = \frac{\bar{\alpha_2} - \rho^4}{(1-\rho)^2}.$$

Table 4.3 exhibits various features of WGNGS distribution for different values of the parameters $\eta^*, \tau, \lambda, \alpha, \sigma, \beta, \mu^*$.

Table 4.3: Values of different characteristics of WGNGS distributions with parametric values $(\eta^*, \tau, \lambda, \alpha, \sigma, \beta, \mu^*)$ as A:(3,1,0.5,0.1,1,1,3); B:(0,2,0.5,0.5,2,1,0); C:(0,2,0.5,2,2,0,0); D:(3,2,0.5,2,2,0,6); E:(3,1,0.5,1,1,1,3); F:(0,2,0.5,1,2,1,0); G:(0,2,0.5,1,2,0,2); H:(6,2,0.5,1,2,0,3).

Properties	A	B	C	D	E	F	G	H
α_1	-0.1688	0.2122	0.1645	-0.0473	-0.1675	0.2124	0.1854	-0.1730
β_1	0.3657	0.0576	0	0.1229	0.3744	0	0.0561	-0.0444
α_2	-0.1278	0.0092	0.0044	-0.0040	-0.1403	0.0078	0.0068	0.0065
β_2	-0.0668	0.0028	0	-0.0007	-0.0558	-0.0013	0.0024	0.0010
$\bar{\alpha_1}$	0.4028	0.2199	0.1645	0.1316	0.4101	0.2124	0.1937	0.1786
$\bar{\beta_1}$	0	0	0	0	0	0	0	0
$\bar{\alpha_2}$	0.1338	0.0094	0.0044	0.0034	0.1352	0.0078	0.0070	0.0062
$\bar{\beta_2}$	-0.0539	-0.0023	0	-0.0022	-0.0674	-0.0013	-0.0018	-0.0022
ρ	0.4028	0.2199	0.1645	0.1316	0.4101	0.2124	0.1937	0.1786
V_0	0.5972	0.7801	0.8355	0.8684	0.5898	0.7876	0.8063	0.8214
σ_0	1.0154	0.7048	0.5995	0.5313	1.0275	0.6910	0.6563	0.6273
ζ_1^0	-0.1167	-0.0033	0	-0.0027	-0.1488	-0.0019	-0.0025	-0.0030
ζ_2^0	0.3012	0.0116	0.0053	0.0041	0.3072	0.0093	0.0086	0.0076

4.7 Maximum likelihood estimation

In this section, we discuss the method of maximum likelihood estimation to estimate the parameters of the model. Let $\theta_1, \theta_2, \ldots, \theta_n$ be a random sample of size n from $WGGS(\lambda, \alpha, \sigma, \beta, \mu^*)$. Then, the log-likelihood function is given by,

$$\log L = -n \log 2\pi + \sum_{i=1}^{n} \left\{ \log(1 + 2 \sum_{p=1}^{\infty} [\alpha_p \cos(p\theta_i) + \beta_p \sin(p\theta_i)]) \right\}.$$

Equating the partial derivative of log-likelihood function with respect to the parameters to zero, for $\alpha \neq 1$ we get,

$$\frac{\partial \log L}{\partial \eta^*} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}\left[\{\alpha_{1,p}\sin(p\theta_i) - \beta_{1,p}\cos(p\theta_i)\}\lambda p\right]}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \tau^2} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}\left[(\alpha_{1,p}\cos(p\theta_i) + \beta_{1,p}\sin(p\theta_i))(\frac{-\lambda p^2}{2})\right]}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}\left[e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}}\{B_1C_1 + A_1(\eta^*p + \arctan(b/a))\}\right]}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}\lambda e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}-1}[\alpha_{11}\{(a\beta\tan(\frac{\pi\alpha}{2}) - b)A_1 - B_1(a + b\beta\tan(\frac{\pi\alpha}{2}))\} + (aA_1 - bB_1)\alpha_{12}]}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \sigma} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}\lambda e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}-1}[A_1\sigma_{11}(a\beta\tan(\frac{\pi\alpha}{2}) - b) - B_1\sigma_{11}(b\beta\tan(\frac{\pi\alpha}{2}) + a)]}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}\lambda e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}-1}[(\sigma^\alpha p^\alpha\tan(\frac{\pi\alpha}{2}))(aA_1 - bB_1)]}{f_1} = 0$$

$$\frac{\partial \log L}{\partial \mu^*} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}\lambda e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}-1}[p(aA_1 - bB_1)]}{f_1} = 0$$

where, $\alpha_{1,p} = e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}}\cos(\lambda\eta^*p + \lambda\arctan(b/a))$, $\beta_{1,p} = e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}}\sin(\lambda\eta^*p + \lambda\arctan(b/a))$, $a = 1 + \sigma^\alpha p^\alpha$, $b = \sigma^\alpha p^\alpha\beta\tan(\frac{\pi\alpha}{2}) + \mu^*p$, $\alpha_{11} = \sigma^\alpha p^\alpha\log(\sigma p)$, $\alpha_{12} = \sigma^\alpha p^\alpha\beta(\pi/2)\sec^2(\pi/2), A_1 = \cos(\lambda\eta^*p + \lambda\arctan(b/a))\sin(p\theta_i) - \sin(\lambda\eta^*p + \lambda\arctan(b/a))\cos(p\theta_i)$, $B_1 = \cos(\lambda\eta^*p + \lambda\arctan(b/a))\cos(p\theta_i) + \sin(\lambda\eta^*p + \lambda\arctan(b/a))\sin(p\theta_i)$,$C_1 = \log((a^2+b^2)^{-1/2}) - \frac{\tau^2p^2}{2}$, $\sigma_{11} = \alpha\sigma^{\alpha-1}p^\alpha$ and $f_1 = 1 + 2\sum_{p=1}^{\infty}[\alpha_{1,p}\cos(p\theta_i) + \beta_{1,p}\sin(p\theta_i)]$. Similarly, for $\alpha = 1$,

$$\frac{\partial \log L}{\partial \eta^*} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}\left[\{\alpha_{2,p}\sin(p\theta_i) - \beta_{2,p}\cos(p\theta_i)\}\lambda p\right]}{f_2} = 0$$

$$\frac{\partial \log L}{\partial \tau^2} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}\left[(\alpha_{2,p}\cos(p\theta_i) + \beta_{2,p}\sin(p\theta_i))(\frac{-\lambda p^2}{2})\right]}{f_2} = 0$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}[e^{-\frac{\lambda\tau^2p^2}{2}}(c^2+d^2)^{-\frac{\lambda}{2}}\{B_2C_2 + (\eta^*p + arctan(d/c))A_2\}]}{f_2} = 0$$

$$\frac{\partial \log L}{\partial \sigma} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}[\lambda e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}-1}\{pB_2(d\beta(2/\pi)log(p) - c) - pA_2(d + c\beta(2/\pi)log(p))\}]}{f_2} = 0$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}[\lambda e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}-1}p\sigma(2/\pi)\log(p)\{dB_2 - cA_2\}]}{f_2} = 0$$

$$\frac{\partial \log L}{\partial \mu^*} = \sum_{i=1}^{n} \frac{2\sum_{p=1}^{\infty}[\lambda e^{-\frac{\lambda\tau^2p^2}{2}}(a^2+b^2)^{-\frac{\lambda}{2}-1}p\{cA_2 - dB_2\}]}{f_2} = 0$$

where, $\alpha_{2,p} = e^{-\frac{\lambda\tau^2p^2}{2}}(c^2 + d^2)^{-\frac{\lambda}{2}}\cos(\lambda\eta^* p + \lambda\arctan(d/c))$, $\beta_{2,p} = e^{-\frac{\lambda\tau^2p^2}{2}}(c^2 + d^2)^{-\frac{\lambda}{2}}\sin(\lambda\eta^* p + \lambda\arctan(d/c))$, $c = 1 + \sigma p$, $d = \mu^* p - \sigma p\beta(2/\pi)\log(p)$, $f_2 = 1 + 2\sum_{p=1}^{\infty}[\alpha_{2,p}\cos(p\theta_i) + \beta_{2,p}\sin(p\theta_i)]$, $A_2 = \cos(\lambda\eta^* p + \lambda\arctan(d/c))\sin(p\theta_i) - \sin(\lambda\eta^* p + \lambda\arctan(d/c))\cos(p\theta_i)$, $B_2 = \cos(\lambda\eta^* p + \lambda\arctan(d/c))\cos(p\theta_i) + \sin(\lambda\eta^* p + \lambda\arctan(d/c))\sin(p\theta_i))$,$C_2 = \log((c^2 + d^2)^{-1/2}) - \frac{\tau^2p^2}{2}$,

Since the above normal equations cannot be solved analytically, a numerical technique is to be adopted to get the solutions for the estimates of the parameters. The log-likelihood of the WGGS$(\lambda, \alpha, \beta, \sigma, \mu^*)$ density can be computed numerically to a given level of precision for $\log L = \sum_{i=1}^{n}\log f(\theta_i)$ using finite sum approximation to (4.19) for the given set of independent observed directions $\theta^T = (\theta_1, \theta_2, ..., \theta_n)$. The *optim* function in the R *stats* package used for the numerical optimization of ℓ over the parameters.

CHAPTER 5

MULTIVARIATE GENERALIZED GEOMETRIC STABLE DISTRIBUTIONS AND PROCESSES

5.1 Introduction

A geometric stable law is defined as a limiting distribution of appropriately normalized sums of a random number of independent identically distributed random variables, where the number of terms has a geometric distribution. Geometric stable distributions generalizes distributions like exponential and Laplace distribution. Generalized geometric stable(GGS)distributions is the univariate generalization of geometric stable distributions. Kozubowski and Panorska(1999)introduced a multivariate generalization of geometric stable distribution and used it for modeling multivariate financial portfolios of securities. The normal-Laplace distribution, which results from the convolution of independent normal and Laplace random variables is introduced by Reed and Jorgensen (2004). Manu(2013) introduced a multivariate normal-Laplace distribution, and studied its properties and applications in multivariate financial data modeling.

The geometric stable distribution has found applications in a variety of areas such as economics, insurance mathematics, reliability and queuing theories, and other fields. This distribution is often used for modeling phenomena with heavier tails. We know that a random variable V is said to follow a generalized geometric stable distribution with parameters $0 < \alpha \leq 2, \lambda > 0, -1 \leq \beta \leq 1, \sigma > 0$, and μ real, if its characteristic function, $\phi(t)$ has the following form:

$$\phi(t) = [1 + \sigma^{\alpha}|t|^{\alpha}\omega_{\alpha,\beta}(t) - i\mu t]^{-\lambda} \tag{5.1}$$

where

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \text{sign}(x)\tan(\pi\alpha/2), & \text{if} \quad \alpha \neq 1, \\ 1 + i\beta(2/\pi)\text{sign}(x)\log|x|, & \text{if} \quad \alpha = 1, \end{cases} \tag{5.2}$$

and

$$sign(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x = 0, \\ -1, & \text{if} \quad x < 0. \end{cases}$$

The special cases of GGS distributions include geometric stable, Laplace, exponential, Linnik etc. Multivariate extensions of symmetric and asymmetric Laplace distributions discussed in Kotz *et al.*(2001). Multivariate generalized asymmetric Laplace distributions and its applications are studied in Kozubowski *et al.*(2013). Many properties in the univariate laws can be extended to this class of distributions. For more details (see, Ernst(1998)).

We have, the characteristic function of GNGS distributions as

$$\phi_X(t) = \left[\frac{\exp\{i\eta t - \frac{\tau^2 t^2}{2}\}}{1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t}\right]^\lambda, \tag{5.3}$$

where $\eta \in \Re, \tau > 0, 0 < \alpha \leq 2, \lambda > 0, -1 \leq \beta \leq 1, \sigma > 0, \mu \in \Re$.

The special cases of GNGS distributions include normal-Laplace, normal-Linnik etc. Manu(2013) introduced multivariate normal-Laplace distribution and developed first order autoregressive processes with multivariate normal-Laplace marginals.

Kozubowski and Panorska(1999) introduced a multivariate extension of geometric stable distributions. As shown in Mittnik and Rachev(1991), there is a one-to-one correspondence between characteristic functions of geometric stable and α-stable distributions: $\mathbf{Y}$ is geometric stable if and only if its characteristic function $\psi(\mathbf{t})$ has the form

$$\Psi(\mathbf{t}) = (1 - \log \phi(\mathbf{t}))^{-1} = \int_0^\infty [\Phi(\mathbf{t})]^z e^{-z} dz. \tag{5.4}$$

Utilizing (5.4) and the spectral representation of α-stable laws(see, Samorodnitsky and Taqqu(1994)), a multivariate geometric stable distribution was introduced in Kozubowski and Panorska(1999): A geometric stable random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$ can be described by its characteristic function as

$$\psi_{\mathbf{Y}}(\mathbf{t}) = \left[1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s}) \boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-1},$$

where $0 < \alpha \leq 2$, $\boldsymbol{\Gamma}$ is a finite measure on the unit sphere $S_d \in \Re^d$, $\boldsymbol{\mu} \in \Re^d$ is the location vector, and $\omega_{\alpha,\beta}$ is given by (5.2). Measure $\boldsymbol{\Gamma}$ is called the spectral measure of the vector Y.

A representation of $GS_\alpha(\mathbf{\Gamma},\boldsymbol{\mu})$ random vector presented below.

Lemma 5.1.1. $\mathbf{Y} \sim GS_\alpha(\mathbf{\Gamma},\boldsymbol{\mu})$ *if and only if*

$$\mathbf{Y} \stackrel{d}{=} \begin{cases} \boldsymbol{\mu} Z + Z^{\frac{1}{\alpha}} \boldsymbol{X}, & \text{if } \alpha \neq 1, \\ \boldsymbol{\mu} Z + Z\boldsymbol{X} + \big(Z(2/\pi)\log(Z)\big)\,\mathbf{g}, & \text{if } \alpha = 1, \end{cases}$$

with

$$\mathbf{g} = (g_1, g_2, \ldots, g_d) \quad \text{and} \quad g_k = \int_{\mathbf{S}_d}^{\infty} s_k \mathbf{\Gamma}(ds),$$

where $\mathbf{X} \sim S_\alpha(\mathbf{\Gamma},\mathbf{0})$($\alpha$-stable distribution with spectral measure $\mathbf{\Gamma}$ and location parameter $\boldsymbol{\mu}$, (see, Samorodnitsky and Taqqu(1994))) , $Z \sim \exp(1)$, and $\mathbf{X}$ and Z are independent.

In the present chapter, we introduce multivariate GGS distributions, and study its properties. Also multivariate GeoGGS distributions are introduced. First order autoregressive processes with multivariate GeoGGS distributions is developed. Multivariate GNGS distributions is introduced, as an extension of multivariate normal-Laplace distribution. We introduced the GeoGNGS distributions and studied its properties.

5.2 The multivariate generalized geometric stable distributions

Here we introduce multivariate generalized geometric stable laws . Let $\{\mathbf{X}^{(n)} = (X_1^{(n)}, X_2^{(n)}, \ldots, X_d^{(n)}), n \geq 1\}$ be independently and identically distributed random vectors in $\Re^d$, and let $N_{p,\lambda}$ be an NB random variable with parameters $p \in (0,1), \lambda > 0$, independent of $\{\mathbf{X}^{(n)}\}$,

$$P(N_{p,\lambda} = k) = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)\Gamma(k+1)} p^\lambda (1-p)^{k-1},\ k = 1, 2, \ldots. \tag{5.5}$$

GGS laws are the only possible limits, and therefore, good approximations, of properly scaled and centered random sums of random vectors. Namely, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$ is GGS, if there exist $a(p) > 0$, and $\mathbf{b}(p) \in \Re^d$ such that

$$a(p) \sum_{i=1}^{N_{p,\lambda}} \left(\mathbf{X}^{(i)} + \mathbf{b}(p) \right) \xrightarrow{d} \mathbf{Y}, \qquad \text{as } p \to 0 \tag{5.6}$$

The random vectors $\mathbf{X}^{(i)}$ appearing in (5.6) are in the domain of attraction of the GGS vector $\mathbf{Y}$.

A GGS random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$ can be described by its characteristic function $\psi(\mathbf{t}) = E \exp\{i\mathbf{t}'\mathbf{Y}\}$, we can write the characteristic function of GGS vector as

$$\Psi_{\mathbf{Y}}(\mathbf{t}) = \left[1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha} \omega_{\alpha,1}(\mathbf{t}'\mathbf{s}) \mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t} \right]^{-\lambda}, \tag{5.7}$$

where $0 < \alpha < 2, \lambda > 0$, $\mathbf{\Gamma}$ is a finite measure on the unit sphere $S_d \in \Re^d$, $\boldsymbol{\mu} \in \Re^d$, and $\omega_{\alpha,\beta}$ is given by (5.2). Measure $\mathbf{\Gamma}$ is called the spectral measure of the vector $\mathbf{Y}$, and carries the information about the dependence structure between its components. We denote $\mathbf{Y} \sim GGS_{\alpha}(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$.

Similar to the univariate case, multivariate GGS distributions also possesses the infinite divisibility property.

Infinite divisibility. The characteristic function $\Phi_{\mathbf{X}}(\mathbf{t})$ of $GGS_{\alpha}(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$ can be written as

$$\Phi_{\mathbf{X}}(\mathbf{t}) = \left[\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha} \omega_{\alpha,1}(\mathbf{t}'\mathbf{s}) \mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t} \right)^{-\frac{\lambda}{n}} \right]^n,$$

for any integer $n > 0$. The term in brackets is the characteristic function of a $GGS_{\alpha}(\frac{\lambda}{n}, \mathbf{\Gamma}, \boldsymbol{\mu})$

The class of elliptical distributions. The class of elliptical distributions is a generalization of multivariate normal distributions (see, Kelker (1970)).

Definition 5.2.1. *A random vector* $\mathbf{X}$ *has a multivariate elliptical distribution, if. its characteristic function can be expressed as*

$$\mathbf{X}(\mathbf{t}) = \exp(i\boldsymbol{\nu}'\mathbf{t})\psi(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}) \tag{5.8}$$

for some column vector $\boldsymbol{\nu}$*, positive matrix* $\boldsymbol{\Sigma}$ *and for some function* $\psi(t) \in \psi_n$*, which is called the characteristic generator.*

GGS distributions with $\boldsymbol{\mu} = 0, \alpha = 2, \lambda = 1$ are elliptically contoured, as their characteristic function depends on $\mathbf{t}$ only through the quadratic form $\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}$. With a non-singular $\boldsymbol{\Sigma}$, they are also elliptically symmetric.

5.2.1 Representation

In this section, we present a useful representation of GGS random vectors analogous to geometric stable random vectors, which extends the representation of univariate GGS laws given in Proposition 2.7.2.

Theorem 5.2.1. $\mathbf{Y} \sim GGS_\alpha(\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ *if and only if*

$$\mathbf{Y} \stackrel{d}{=} \begin{cases} \boldsymbol{\mu}W + W^{\frac{1}{\alpha}}\boldsymbol{X}, & \text{if}\, \alpha \neq 1, \\ \boldsymbol{\mu}W + W\boldsymbol{X} + \big(W(2/\pi)\log(W)\big)\,\mathbf{g}, & \text{if } \alpha = 1, \end{cases} \tag{5.9}$$

with

$$\mathbf{g} = (g_1, g_2, \ldots, g_d) \quad \text{and} \quad g_k = \int_{\mathbf{S}_d}^{\infty} s_k \boldsymbol{\Gamma}(ds),$$

where $\mathbf{X} \sim S_\alpha(\boldsymbol{\Gamma}, \mathbf{0})$($\alpha$-stable distribution with spectral measure $\boldsymbol{\Gamma}$ and location parameter $\boldsymbol{\mu}$, (see, Samorodnitsky and Taqqu(1994))), $W \sim G(1, \lambda)$, and $\mathbf{X}$ and W are independent.

Proof. Case 1: $\alpha \neq 1$.Let $\mathbf{Y_1} = \boldsymbol{\mu} W + W^{\frac{1}{\alpha}}\mathbf{X}$

$$\begin{aligned}
\Psi_{\mathbf{Y_1}}(\mathbf{t}) &= E[e^{i\mathbf{t}'\mathbf{Y_1}}] \\
&= E_{\text{w}}\left[E\left[e^{i\mathbf{t}'\mathbf{Y_1}}|W\right]\right] \\
&= E_{\text{w}}\left[E_X\left[\exp\left\{i\mathbf{t}'\left(\boldsymbol{\mu} W + W^{\frac{1}{\alpha}}\mathbf{X}\right)\right\}|W\right]\right] \\
&= E_{\text{w}}\left[\exp\{i\mathbf{t}'\boldsymbol{\mu}\text{W}\}\Phi(W\mathbf{t})\right] \\
&= E_{\text{w}}\left[\exp\{i\mathbf{t}'\boldsymbol{\mu}\text{W}\}[\Phi(\mathbf{t})]^{\text{W}}\right] \\
&= E_W[\Phi(\mathbf{t})\exp\{i\mathbf{t}'\boldsymbol{\mu}\}]^{\text{W}} \\
&= E[\exp\{-\{-(\log\Phi(\mathbf{t}) + i\mathbf{t}'\boldsymbol{\mu})\}\text{W}\}]
\end{aligned}$$

This is the Laplace transform of W, that is, $Ee^{-s\text{w}}$, with $s = -(\log\Phi(\mathbf{t})+i\mathbf{t}'\boldsymbol{\mu})$. Thus, representation (5.9) holds for $\alpha \neq 1$.

Case 2: $\alpha = 1$. Let $\mathbf{Y_2} = \boldsymbol{\mu} W + W\mathbf{X} + W(2/\pi)\log(W)\mathbf{g}$, with the variables defined in the statement of the theorem. Conditioning on W produces

$$\begin{aligned}
\Psi_{\mathbf{Y_2}}(\mathbf{t}) &= E[e^{i\mathbf{t}'\mathbf{Y_2}}] \\
&= E_{\text{w}}\left[E\left[e^{i\mathbf{t}'\mathbf{Y_2}}|W\right]\right] \\
&= E_{\text{w}}\left[E_X\left[\exp\left\{i\mathbf{t}'\left(\boldsymbol{\mu} W + W\mathbf{X} + W(2/\pi)\log(W)\mathbf{g}\right)\right\}|W\right]\right] \\
&= E_{\text{w}}\left[\exp\{i\mathbf{t}'\boldsymbol{\mu}\text{W} + W(2/\pi)\log(W)i\mathbf{t}'\mathbf{g}\}\Phi(W\mathbf{t})\right]
\end{aligned}$$

where Φ is the characteristic function of $\mathbf{X} \sim S_\alpha(\boldsymbol{\Gamma}, \mathbf{0})$. Note that for any $\mathbf{t} \in \Re^d$ and $\text{w} > 0, \Phi$ satisfies

$$\Phi(\text{w}\mathbf{t}) = [\Phi(\mathbf{t})]^{\text{w}}\exp\{-\text{w}\frac{2}{\pi}\log(\text{w})i\mathbf{t}'\mathbf{g}\}.$$

Therefore,

$$\Psi_{\mathbf{Y_2}}(\mathbf{t}) = E_W[\Phi(\mathbf{t})\exp\{i\mathbf{t}'\boldsymbol{\mu}\}]^{\mathrm{w}}$$

Thus, representation (5.9) holds for $\alpha = 1$.

□

Remarks:

1. When $\alpha = 2$, the characteristic function of a GGS vector can be written as
$$\psi(\mathbf{t}) = \left[1 + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t} - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda},$$
where $\boldsymbol{\Sigma}$ is a dxd positive-definite symmetric matrix. This is the characteristic function of multivariate generalized Laplace distribution(see, Kozubowski *et al.*(2013))

2. When $\boldsymbol{\Gamma} \equiv \mathbf{0}(\boldsymbol{\Gamma}(A) = 0$, for any Borel set in $\Re^d$), th characteristic function becomes
$$\psi(\mathbf{t}) = \left[1 - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}.$$
It admits the representation $\mathbf{Y} \sim \boldsymbol{\mu}W$, where $W \sim G(1, \lambda)$.

3. Summation: Let $\mathbf{X} \sim GGS_\alpha(\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ and $\mathbf{Y} \sim GGS_\alpha(\gamma, \boldsymbol{\Gamma}, \boldsymbol{\mu})$, $\mathbf{X}$ and $\mathbf{Y}$ are independent, then $\mathbf{X} + \mathbf{Y} \sim GGS_\alpha(\lambda + \gamma, \boldsymbol{\Gamma}, \boldsymbol{\mu})$.

4. If $d = 1$, the unit sphere consists of only two points: $S_1 = \{1, -1\}$. Denoting $\Gamma_1 = \Gamma(\{1\})$ and $\Gamma_{-1} = \Gamma(\{-1\})$, in case $\alpha \neq 1$, characteristic function (5.7) becomes

$$\begin{aligned}\psi(t) &= \left[1 + |t1|^\alpha\left(1 - i\mathrm{sgn}(t1)\tan(\frac{\pi\alpha}{2})\right)\Gamma_1 + |t(-1)|^\alpha\left(1 - i\mathrm{sgn}(t(-1))\tan(\frac{\pi\alpha}{2})\right)\Gamma_{-1} - it\mu\right]^{-\lambda} \\ &= \left[1 + \left((\Gamma_1 + \Gamma_{-1})^{\frac{1}{\alpha}}\right)^\alpha |t|^\alpha\left(1 - i\mathrm{sgn}(t)\frac{\Gamma_1 - \Gamma_{-1}}{\Gamma_1 + \Gamma_{-1}}\tan(\frac{\pi\alpha}{2})\right)\Gamma_1 - it\mu\right]^{-\lambda}\end{aligned}$$

Comparing the above expression with (5.1), we see that a univariate GGS random variable with spectral representation $GGS_{\alpha}(\lambda, \Gamma, \mu)$ has parameters

$$\lambda, \alpha, \beta = \frac{\Gamma_1 - \Gamma_{-1}}{\Gamma_1 + \Gamma_{-1}}, \sigma = (\Gamma_1 + \Gamma_{-1})^{\frac{1}{\alpha}}, \mu$$

The skewness parameter β is zero if the spectral measure is symmetric. Similar result holds for $\alpha = 1$.

5.3 Multivariate slash generalized geometric stable distributions

Now let us define the slash version of the generalized geometric stable distributions.

Definition 5.3.1. *A random vector* $\mathbf{Y} \in \Re^d$ *has a d-variate slash generalized geometric stable* ($SGGS_d$) *distributions, denoted by* $\mathbf{Y} \sim SGGS_{\alpha}(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu}, q)$, *if* $\mathbf{Y} = \frac{\mathbf{X}}{U^{\frac{1}{q}}}$, *where* $q > 0$ *and* $\mathbf{X}$ *is GGS random vector with characteristic function given by* $\Psi_{\mathbf{X}}(\mathbf{t}) = \left[1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha} \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}$, *where* $0 < \alpha < 2, \lambda > 0$, $\mathbf{\Gamma}$ *is a finite measure on the unit sphere* $S_d \in \Re^d, \boldsymbol{\mu} \in \Re^d$ *and* $U \sim U(0,1)$, *which is independent of* $\mathbf{X}$.

5.4 The multivariate geometric generalized geometric stable distributions

In this section, multivariate geometric generalized normal geometric stable(GeoGGS) distributions is introduced and its properties are studied.

A $GGS_{\alpha}(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$ random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$ can be described by its characteristic function as

$$\Psi_{\mathbf{Y}}(\mathbf{t}) = \left[1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}, \tag{5.10}$$

where $0 < \alpha < 2, \lambda > 0$, $\mathbf{\Gamma}$ is a finite measure on the unit sphere $S_d \in \Re^d$, $\boldsymbol{\mu} \in \Re^d$ is the location vector, and $\omega_{\alpha,\beta}$ is given by (5.2). Measure $\mathbf{\Gamma}$ is called the spectral measure of the vector $\mathbf{Y}$,

Now, we can write,

$$\left[1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}$$

as

$$\exp\left\{1 - \frac{1}{\left[1 + \lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}}\right\}.$$

Since $GGS_{\alpha}(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$ distribution is infinitely divisible, it follows that

$$\left[1 + \lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}$$

is geometrically infinite divisible.

A distribution with characteristic function

$$\left[1 + \lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}$$

is called GeoGGS distribution. It is denoted as $GeoGGS_{\alpha}(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$

Definition 5.4.1. *A d-variate random vector* $\mathbf{X}$ *is said to follow multivariate geometric generalized geometric stable distribution and write* $\mathbf{X} \sim$

GeoGGS$_\alpha(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$ *if it has the characteristic function*

$$\phi_{\mathbf{X}}(t) = \left[1 + \lambda \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1},$$

where $0 < \alpha < 2, \lambda > 0$, $\mathbf{\Gamma}$ *is a finite measure on the unit sphere* $S_d \in \Re^d$, $\boldsymbol{\mu} \in \Re^d$ *is the location vector, and* $\omega_{\alpha,\beta}(x)$ *is given by (5.2).*

Theorem 5.4.1. *Let* $\mathbf{X}_1, \mathbf{X}_2, \ldots$ *be independent and identically distributed d-variate geometric generalized geometric stable random vectors, that is,* $\mathbf{X}_i \sim$ *GeoGGS*$_\alpha(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu}), i = 1, 2, \ldots$ *and* $N(\gamma)$ *be a geometric with mean* $1/\gamma, P[N(\gamma) = k] = \gamma(1-\gamma)^{k-1}, k = 1, 2, \ldots, 0 < \gamma < 1$. *Define* $\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_{N(\gamma)}$, *then* $\mathbf{Y} \sim$ *GeoGGS*$_\alpha(\frac{\lambda}{\gamma}, \mathbf{\Gamma}, \boldsymbol{\mu})$

Proof. Since $\mathbf{X}_i \sim$ *GeoGGS*$_\alpha(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$, then its characteristic function is ,

$$\phi_{\mathbf{X}}(\mathbf{t}) = \left[1 + \lambda \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}$$

Then the characteristic function of $\mathbf{Y}$ is

$$\begin{aligned}
\phi_{\mathbf{Y}}(\mathbf{t}) &= \sum_{k=1}^{n} [\phi_{\mathbf{X}}(\mathbf{t})]^k \gamma(1-\gamma)^{k-1} \\
&= \frac{\gamma\phi_{\mathbf{X}}(\mathbf{t})}{1 - (1-\gamma)\phi_{\mathbf{X}}(\mathbf{t})} \\
&= \frac{\gamma\left[1 + \lambda \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}}{1 - (1-\gamma)\left[1 + \lambda \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}} \\
&= \left[1 + \frac{\lambda}{\gamma} \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}.
\end{aligned} \tag{5.11}$$

Hence $\mathbf{Y} \sim$ *GeoGGS*$_\alpha(\frac{\lambda}{\gamma}, \mathbf{\Gamma}, \boldsymbol{\mu})$. □

Theorem 5.4.2. *Suppose* $\mathbf{X}_1, \mathbf{X}_2, \ldots$ *are independently and identically distributed as* $GGS_\alpha(\frac{\lambda}{n}, \mathbf{\Gamma}, \boldsymbol{\mu})$ *and N, independent of* $\mathbf{X}_1, \mathbf{X}_2, \ldots$ *be a geometric random variables with probability of success* $1/n$. *Then* $\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_N$ *distributed as* $GeoGGS_\alpha(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$ *as* $n \to \infty$.

Proof.

$$\left[1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right]^{-\frac{\lambda}{n}} = \left\{1+\left[\left[1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right]^{\frac{\lambda}{n}}-1\right]\right\}^{-1}.$$

Hence by Lemma 3.2 of Pillai(1990b)

$$\phi_n(\mathbf{t}) = \left\{1+n\left[\left[1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right]^{\frac{\lambda}{n}}-1\right]\right\}^{-1}$$

is the characteristic function of Y. Taking limit as $n \to \infty$, we have

$$\begin{aligned}\phi(\mathbf{t}) &= \lim_{n\to\infty}\phi_n(\mathbf{t})\\ &= \left\{1+\lim_{n\to\infty}n\left[\left[1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right]^{\frac{\lambda}{n}}-1\right]\right\}^{-1} \\ &= \left[1+\lambda\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}.\end{aligned} \tag{5.12}$$

□

Theorem 5.4.3. *Let* $\mathbf{X}|\lambda \sim GGS_\alpha(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$ *with random* λ, *where* λ *is exponential with mean* η. *Then* $\mathbf{X} \sim GeoGGS_\alpha(\eta, \mathbf{\Gamma}, \boldsymbol{\mu})$.

Proof.

$$
\begin{aligned}
\phi(\mathbf{t}) &= E\left(e^{i\mathbf{t}'\mathbf{X}_\lambda}\right) \\
&= E_\lambda\left[1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda} \\
&= E_\lambda\left[e^{\log\left[1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}}\right] \\
&= E_\lambda\left[e^{-\lambda\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right)}\right] \\
&= \left[1+\eta\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}.
\end{aligned}
\tag{5.13}
$$

□

Theorem 5.4.4. *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent and identically distributed with $GeoGGS_\alpha(\frac{\lambda}{n}, \boldsymbol{\Gamma}, \boldsymbol{\mu})$. Then $\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n \xrightarrow{d} GGS_\alpha(\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ as $n \to \infty$.*

Proof. The characteristic function of $GeoGGS_\alpha(\frac{\lambda}{n}, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ distribution is

$$
\phi_{\mathbf{X}}(\mathbf{t}) = \left[1+\frac{\lambda}{n}\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}
$$

Then the characteristic function of Y is

$$
\phi_{\mathbf{Y}}(\mathbf{t}) = \left[1+\frac{\lambda}{n}\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-n}.
$$

Hence,

$$
\lim_{n\to\infty}\phi_{\mathbf{Y}}(\mathbf{t}) = \left[1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^\alpha\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}
$$

That is, $\mathbf{Y} \xrightarrow{d} GGS_\alpha(\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$. □

5.4.1 AR(1) model with multivariate GeoGGS marginals

Consider the linear additive autoregreessive equation

$$\mathbf{X}_n = \rho \mathbf{X}_{n-1} + \boldsymbol{\epsilon}_n, n = 0, \pm 1, \pm 2, \ldots, |\rho| \leq 1 \tag{5.14}$$

where $\mathbf{X}_n$ and innovations $\boldsymbol{\epsilon}_n$ are independent d- variate random vectors. Lawrence(1978) derived the gamma and the Laplace solution of equation (5.14). In this section, we develop a first order new autoregressive process with multivariate GeoGGS marginals. Consider an autoregressive structure given by,

$$\mathbf{X}_n = \begin{cases} \boldsymbol{\epsilon}_n, & \text{w.p} \quad \gamma, \\ \mathbf{X}_{n-1} + \boldsymbol{\epsilon}_n, & \text{w.p} \quad 1-\gamma, \end{cases} \tag{5.15}$$

where $0 < \gamma < 1$. Now we shall construct an $AR(1)$ process with stationary marginal as multivariate GeoGGS distribution.

Theorem 5.4.5. *Consider an autoregressive process $\{\mathbf{X}_n\}$ with structure given by (5.15). Then $\{\mathbf{X}_n\}$ is strictly stationary Markovian with $GeoGGS_\alpha(\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ marginal if and only if $\{\boldsymbol{\epsilon}_n\}$ are distributed as $GeoGGS_\alpha(\gamma\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ provided that $\mathbf{X}_0$ is distributed as $GeoGGS_\alpha(\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$.*

Proof. Let us denote the Laplace transform of $\{\mathbf{X}_n\}$ by $\psi_{\mathbf{X}_n}(\mathbf{t})$ and that of $\boldsymbol{\epsilon_n}$ by $\psi_{\boldsymbol{\epsilon}_n}(\mathbf{t})$, equation (5.15) in terms of characteristic function becomes

$$\psi_{\mathbf{X}_n}(\mathbf{t}) = \gamma \psi_{\boldsymbol{\epsilon}_n}(\mathbf{t}) + (1-\gamma)\psi_{\mathbf{X}_{n-1}}(\mathbf{t})\psi_{\boldsymbol{\epsilon}_n}(\mathbf{t}).$$

On assuming stationarity, it reduces to the form

$$\psi_{\mathbf{X}}(\mathbf{t}) = \gamma \psi_{\boldsymbol{\epsilon}}(\mathbf{t}) + (1-\gamma)\psi_{\mathbf{X}}(\mathbf{t})\psi_{\boldsymbol{\epsilon}}(\mathbf{t}).$$

Write

$$\psi_{\mathbf{X}}(\mathbf{t}) = \left[1 + \lambda \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha} \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}$$

and hence

$$\psi_{\boldsymbol{\epsilon}}(\mathbf{t}) = \frac{\psi_{\mathbf{X}}(\mathbf{t})}{\gamma + (1-\gamma)\psi_{\mathbf{X}}(\mathbf{t})} \tag{5.16}$$

becomes

$$\psi_{\boldsymbol{\epsilon}}(\mathbf{t}) = \left[1 + \gamma\lambda \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha} \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}.$$

Hence it follows that $\boldsymbol{\epsilon}_n \stackrel{d}{=} GeoGGS_{\alpha}(\gamma\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$

The converse can be proved by the method of mathematical induction as follows. Now assume that $\mathbf{X}_{n-1} \stackrel{d}{=} GeoGGS_{\alpha}(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$. Then

$$\begin{aligned}
\psi_{\mathbf{X}_n}(\mathbf{t}) =& \psi_{\boldsymbol{\epsilon}_n}(\mathbf{t})[\gamma + (1-\gamma)\psi_{\mathbf{X}_{n-1}}(\mathbf{t})] \\
=& \left\{\left[1 + \gamma\lambda \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha} \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}\right\} \text{x} \\
& \left\{[\gamma + (1-\gamma)\left[1 + \lambda \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha} \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}\right\} \\
=& \left[1 + \lambda \log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha} \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\mathbf{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}
\end{aligned} \tag{5.17}$$

That is, $\mathbf{X}_n \stackrel{d}{=} GeoGGS_{\alpha}(\lambda, \mathbf{\Gamma}, \boldsymbol{\mu})$ □

The joint distribution of $\mathbf{X}_n$ and $\mathbf{X}_{n-1}$

Consider the autoregressive structure given in (5.15). It can be written as

$$\mathbf{X}_n = I_n \mathbf{X}_{n-1} + \boldsymbol{\epsilon}_{n-1}, \text{where} P(I_n = 0) = p, P(I_n = 1) = 1 - p$$

Then the joint characteristic function of$(\mathbf{X}_n, \mathbf{X}_{n-1})$ is given by

$$\begin{aligned}
\psi_{\mathbf{X}_{n-1},\mathbf{X}_n}(\mathbf{t}_1, \mathbf{t}_2) =& E\left[e^{i\mathbf{t}_1'\mathbf{X}_{n-1}+i\mathbf{t}_2'\mathbf{X}_n}\right] \\
=& E\left[e^{i\mathbf{t}_1'\mathbf{X}_{n-1}+i\mathbf{t}_2'(I_n\mathbf{X}_{n-1}+\boldsymbol{\epsilon}_n)}\right] \\
=& E[e^{(i\mathbf{t}_1+i\mathbf{t}_2 I_n)'\mathbf{X}_{n-1}}]\psi_{\boldsymbol{\epsilon}_n}(\mathbf{t}_2) \\
=& \left[\frac{1}{1+\gamma\lambda\log\left(1+\int_{S_d}|\mathbf{t_2}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t_2}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t_2}\right)}\right] \\
& \text{x}\left[\frac{p}{1+\lambda\log\left(1+\int_{S_d}|\mathbf{t_1}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t_1}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t_1}\right)} + \frac{1-p}{1+\lambda\log\left(1+\int_{S_d}|(\mathbf{t_1}+\mathbf{t_2})'\mathbf{s}|^\alpha \omega_{\alpha,1}((\mathbf{t_1}+\mathbf{t_2})'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'(\mathbf{t_1}+\mathbf{t_2})\right)}\right]
\end{aligned} \tag{5.18}$$

This shows the process is not time reversible.

5.5 Generalisation to a k^{th} order multivariate GeoGGS autoregressive process

Consider the higher order process, analogs of the autoregressive equation (5.15) with structure as given below.

$$\mathbf{X}_n = \begin{cases} \boldsymbol{\epsilon}_n, & \text{w.p} \quad \gamma, \\ \mathbf{X}_{n-1} + \boldsymbol{\epsilon}_n, & \text{w.p} \quad \gamma_1, \\ \vdots & \\ \mathbf{X}_{n-k} + \boldsymbol{\epsilon}_n, & \text{w.p} \quad \gamma_k, \end{cases} \tag{5.19}$$

where $\gamma_1 + \gamma_2 + \ldots + \gamma_k = 1 - \gamma, 0 \leq \gamma_i, \gamma \leq 1, i = 1, 2, \ldots, k$ and $\boldsymbol{\epsilon}_n$ is independent of $\{\mathbf{X}_n, \mathbf{X}_{n-1}, \ldots\}$.

In terms of characteristic function, equation (5.19) can be written as

$$\psi_{\mathbf{X}_n}(t) = \gamma\psi_{\boldsymbol{\epsilon}_n}(t) + \gamma_1\psi_{\mathbf{X}_{n-1}}(t)\psi_{\boldsymbol{\epsilon}_n}(t) + \ldots + \gamma_k\psi_{\mathbf{X}_{n-k}}(t)\psi_{\boldsymbol{\epsilon}_n}(t).$$

Assuming stationarity, we get

$$\psi_{\boldsymbol{\epsilon}}(t) = \frac{\psi_{\mathbf{X}}(t)}{\gamma + (1-\gamma)\psi_{\mathbf{X}}(t)}.$$

This establishes that the results developed in the above section are valid in this case also. This gives to the k^{th} order GeoGGS autoregressive process.

5.6 Multivariate generalized normal-geometric stable distributions

Definition 5.6.1. *A d-variate random vector* $\mathbf{X}$ *is said to follow multivariate generalized normal-geometric stable distribution and write* $\mathbf{X} \sim GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ *if it has the characteristic function*

$$\phi_{\mathbf{X}}(t) = \exp\{i\lambda\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left[1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda},$$

where $\boldsymbol{\eta} \in \Re^d, \boldsymbol{\mathcal{T}} > 0, 0 < \alpha < 2, \lambda > 0$, $\boldsymbol{\Gamma}$ *is a finite measure on the unit sphere* $S_d \in \Re^d, \boldsymbol{\mu} \in \Re^d$, *and* $\omega_{\alpha,\beta}(x)$ *is given by (5.2).*

When $\lambda = 1$, we get multivariate normal-geometric stable distributions. The characteristic function is given by

$$\phi_{\mathbf{X}}(\mathbf{t}) = \exp\{i\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left[\mathbf{1} + \int_{\mathbf{S_d}} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,\mathbf{1}}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(\mathbf{ds}) - \mathbf{i}\boldsymbol{\mu}'\mathbf{t}\right]^{-\mathbf{1}}.$$

Some properties:

Let $\mathbf{X} \sim GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$, then $\mathbf{X}$ can be expressed as

$$\mathbf{X} \stackrel{d}{=} \mathbf{Z} + \mathbf{Y}$$

where $\mathbf{Z}$ and $\mathbf{Y}$ are independent random vectors with $\mathbf{Z}$ following a d-variate normal distribution with mean vector $\lambda\boldsymbol{\eta}$ and dispersion matrix $\lambda\boldsymbol{\mathcal{T}}$ ($N_d(\lambda\boldsymbol{\eta}, \lambda\boldsymbol{\mathcal{T}})$) and $\mathbf{Y}$ following a d-variate GGS distributions $GGS_\alpha(\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$.

Remarks:

1. When $\alpha = 2$ the characteristic function of a GNGS vector can be written as

$$\psi(\mathbf{t}) = \exp\{i\lambda\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\} \left[\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t} - \mathbf{i}\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}$$

   where $\boldsymbol{\Sigma}$ is a dxd positive-definite symmetric matrix.

2. Summation: Let $\mathbf{X} \sim GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ and $\mathbf{Y} \sim GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \gamma, \boldsymbol{\Gamma}, \boldsymbol{\mu})$, $\mathbf{X}$ and $\mathbf{Y}$ are independent, then $\mathbf{X} + \mathbf{Y} \sim GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda + \gamma, \boldsymbol{\Gamma}, \boldsymbol{\mu})$.

3. If $d = 1$, the unit sphere consists of only two points: $S_1 = \{1, -1\}$. Denoting $\Gamma_1 = \Gamma(\{1\})$ and $\Gamma_{-1} = \Gamma(\{-1\})$, in case $\alpha \neq 1$, characteristic function (5.7) becomes

$$\begin{aligned}\psi(t) &= \exp\{i\lambda t_1\eta_1 - \frac{1}{2}\lambda {t_1}^2\tau_{11}\} \left[1 + |t1|^\alpha \left(1 - i\text{sgn}(t1)\tan(\frac{\pi\alpha}{2})\right)\Gamma_1 + |t(-1)|^\alpha \left(1 - i\text{sgn}(t(-1))\tan(\frac{\pi\alpha}{2})\right)\Gamma_{-1} - it\mu\right]^{-\lambda}. \\ &= \exp\{i\lambda t_1\eta_1 - \frac{1}{2}\lambda\tau_{11}{t_1}^2\} \left[1 + \left((\Gamma_1 + \Gamma_{-1})^{\frac{1}{\alpha}}\right)^\alpha |t|^\alpha \left(1 - i\text{sgn}(t)\frac{\Gamma_1 - \Gamma_{-1}}{\Gamma_1 + \Gamma_{-1}}\tan(\frac{\pi\alpha}{2})\right)\Gamma_1 - it\mu\right]^{-\lambda}.\end{aligned}$$

   Comparing the above expression with (2.42), we see that a univariate GNGS random variable with spectral representation $GNGS_\alpha(\eta, \mathcal{T}, \lambda, \Gamma, \mu)$ has parameters

$$\eta = \eta_1, \tau = \tau_{11}, \lambda, \alpha, \beta = \frac{\Gamma_1 - \Gamma_{-1}}{\Gamma_1 + \Gamma_{-1}}, \sigma = (\Gamma_1 + \Gamma_{-1})^{\frac{1}{\alpha}}, \mu$$

Similar result holds for $\alpha = 1$.

Proposition 5.6.1. $\mathbf{Y} \sim GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ *if and only if*

$$\mathbf{Y} \stackrel{d}{=} \begin{cases} \lambda\boldsymbol{\eta} + \sqrt{\lambda}\boldsymbol{\Lambda}\mathbf{Z} + \boldsymbol{\mu}W + W^{\frac{1}{\alpha}}\boldsymbol{X}, & \text{if } \alpha \neq 1, \\ \lambda\boldsymbol{\eta} + \sqrt{\lambda}\boldsymbol{\Lambda}\mathbf{Z} + \boldsymbol{\mu}W + W\boldsymbol{X} + \big(W(2/\pi)\log(W)\big)\,\mathbf{g}, & \text{if } \alpha = 1, \end{cases} \tag{5.20}$$

with

$$\mathbf{g} = (g_1, g_2, \ldots, g_d) \quad \text{and} \quad g_k = \int_{\mathbf{S}_d}^{\infty} s_k \boldsymbol{\Gamma}(ds),$$

where $\mathbf{Z} \sim N_d(\mathbf{0}, \mathbf{I})$($d$-variate standard normal random vector) and $\boldsymbol{\Lambda}$ is a dxd invertible matrix such that $\boldsymbol{\mathcal{T}} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathrm{T}}$, $\mathbf{X} \sim S_\alpha(\boldsymbol{\Gamma}, \mathbf{0})$($\alpha$-stable distribution with spectral measure $\boldsymbol{\Gamma}$ and location parameter $\boldsymbol{\mu}$, (see, Samorodnitsky and Taqqu(1994))), $W \sim G(1, \lambda)$, and $\mathbf{Z}$, $\mathbf{X}$ and W are independent.

The class of elliptical distributions: The multivariate normal and multivariate GGS distributions with $\alpha = 2, \lambda = 1, \boldsymbol{\mu} = 0$ belong to elliptical family, since their characteristic functions can be factorized as (5.8). The multivariate generalized normal-geometric distribution $\alpha = 2, \lambda = 1, \boldsymbol{\mu} = 0$ belongs to the class of elliptical distributions, as, the sum of elliptical distributions is elliptical(see, Fang et al. (1987)).

Infinite divisibility:. Multivariate GNGS distributions possesses the infinite divisibility property. Since the characteristic function $\Phi_{\mathbf{X}}(\mathbf{t})$ of $GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ can be written as

$$\Phi_{\mathbf{X}}(t) = \left[\exp\{i\frac{\lambda}{n}\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\frac{\lambda}{n}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\} \left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)^{-\frac{\lambda}{n}}\right]^n.$$

5.7 Multivariate slash generalized normal-geometric stable distributions

Now we define the slash version of the generalized normal-geometric stable distributions.

Definition 5.7.1. *A random vector* $\mathbf{Y} \in \Re^d$ *has a d-variate slash generalized normal-geometric stable* ($SGNGS_d$) *distributions, denoted by* $\mathbf{Y} \sim SGNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu}, q)$*, if* $\mathbf{Y} = \frac{\mathbf{X}}{U^{\frac{1}{q}}}$*, where* $q > 0$ *and* $\mathbf{X}$ *is GNGS random vector with characteristic function given by* $\phi_{\mathbf{X}}(t) = \exp\{i\lambda\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left[1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}$*, where* $\boldsymbol{\eta} \in \Re^d, \boldsymbol{\mathcal{T}} > 0, 0 < \alpha < 2, \lambda > 0$*,* $\boldsymbol{\Gamma}$ *is a finite measure on the unit sphere* $S_d \in \Re^d, \boldsymbol{\mu} \in \Re^d$*, and* $U \sim U(0,1)$*, which is independent of* $\mathbf{X}$*.*

5.8 Multivariate geometric generalized normal geometric stable distributions

A $GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$ can be described by its characteristic function as

$$\phi_{\mathbf{Y}}(t) = \exp\{i\lambda\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left[1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda},$$

where $\boldsymbol{\eta} \in \Re^d, \boldsymbol{\mathcal{T}} > 0, 0 < \alpha < 2, \lambda > 0$, $\boldsymbol{\Gamma}$ is a finite measure on the unit sphere $S_d \in \Re^d, \boldsymbol{\mu} \in \Re^d$, and $\omega_{\alpha,\beta}(x)$ is given by (5.2). Now, we can write

$$\exp\{i\lambda\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left[1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}$$

as

$$\exp\left\{1-\frac{1}{\left[1+\frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}-i\lambda\mathbf{t}'\boldsymbol{\eta}+\lambda\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}}\right\}$$

Since $GNGS_\alpha(\boldsymbol{\eta},\boldsymbol{\mathcal{T}},\lambda,\boldsymbol{\Gamma},\boldsymbol{\mu})$ distribution is infinitely divisible, it follows that

$$\left[1+\frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}-i\lambda\mathbf{t}'\boldsymbol{\eta}+\lambda\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}$$

is geometrically infinitely divisible.

A distribution with characteristic function

$$\left[1+\frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}-i\lambda\mathbf{t}'\boldsymbol{\eta}+\lambda\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}$$

is called GeoGNGS distribution. It is denoted as $GeoGNGS_\alpha(\boldsymbol{\eta},\boldsymbol{\mathcal{T}},\lambda,\boldsymbol{\Gamma},\boldsymbol{\mu})$.

Definition 5.8.1. *A d-variate random vector* $\mathbf{X}$ *is said to follow multivariate geometric generalized normal-geometric stable distribution and write* $\mathbf{X}\sim GeoGNGS_\alpha(\boldsymbol{\eta},\boldsymbol{\mathcal{T}},\lambda,\boldsymbol{\Gamma},\boldsymbol{\mu})$ *if it has the characteristic function*

$$\phi_{\mathbf{X}}(\mathbf{t})=\left[1+\frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}-i\lambda\mathbf{t}'\boldsymbol{\eta}+\lambda\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s})-i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1},$$

where $\boldsymbol{\eta}\in\Re^d,\boldsymbol{\mathcal{T}}>0,0<\alpha<2,\lambda>0,$ $\boldsymbol{\Gamma}$ *is a finite measure on the unit sphere* $S_d\in\Re^d,\boldsymbol{\mu}\in\Re^d$ *is the location vector, and* $\omega_{\alpha,\beta}(x)$ *is given by (5.2).*

Theorem 5.8.1. *Let* $\mathbf{X}_1,\mathbf{X}_2,\ldots$ *be independent and identically distributed as d-variate geometric generalized normal-geometric stable random vectors, that is,* $\mathbf{X}_i\sim GeoGNGS_\alpha(\boldsymbol{\eta},\boldsymbol{\mathcal{T}},\lambda,\boldsymbol{\Gamma},\boldsymbol{\mu}),i=1,2,\ldots$ *and* $N(\gamma)$ *be a geometric with mean* $1/\gamma$*, that is,* $P[N(\gamma)=k]=\gamma(1-\gamma)^{k-1},k=1,2,\ldots,0<\gamma<1$*. Define*

$\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_{N(\gamma)}$, *then* $\mathbf{Y} \sim GeoGNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \frac{\lambda}{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\mu})$

Proof. Since $\mathbf{X}_i \sim GeoGNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$, then its characteristic function is,

$$\phi_{\mathbf{X}}(\mathbf{t}) = \left[1 + \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} - i\lambda\mathbf{t}'\boldsymbol{\eta} + \lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}$$

Then the characteristic function of $\mathbf{Y}$ is

$$\begin{aligned}
\phi_{\mathbf{Y}}(\mathbf{t}) &= \sum_{k=1}^{n}[\phi_{\mathbf{X}}(\mathbf{t})]^k\gamma(1-\gamma)^{k-1} \\
&= \frac{\gamma\phi_{\mathbf{X}}(\mathbf{t})}{1-(1-\gamma)\phi_{\mathbf{X}}(\mathbf{t})} \\
&= \frac{\gamma\left[1 + \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} - i\lambda\mathbf{t}'\boldsymbol{\eta} + \lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}}{1-(1-\gamma)\left[1 + \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} - i\lambda\mathbf{t}'\boldsymbol{\eta} + \lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}} \\
&= \left[1 + \frac{1}{2}\frac{\lambda}{\gamma}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} - i\frac{\lambda}{\gamma}\mathbf{t}'\boldsymbol{\eta} + \frac{\lambda}{\gamma}\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^\alpha \omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}
\end{aligned} \tag{5.21}$$

Hence $\mathbf{Y} \sim GeoGNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \frac{\lambda}{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ □

Theorem 5.8.2. *Suppose* $\mathbf{X}_1, \mathbf{X}_2, \ldots$ *are independently and identically distributed as* $GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \frac{\lambda}{n}, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ *and* N*, independent of* $\mathbf{X}_1, \mathbf{X}_2, \ldots$ *be a geometric random variables with probability of success* $1/n$*. Then* $\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_N$ *distributed as* $GeoGNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ *as* $n \to \infty$*.*

Proof.

$$\begin{aligned}
\Phi_{\mathbf{X}}(t) &= \exp\{i\frac{\lambda}{n}\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\frac{\lambda}{n}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left[1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\frac{\lambda}{n}}\\
&= \left[\exp\{-i\mathbf{t}'\boldsymbol{\eta} + \frac{1}{2}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-\frac{\lambda}{n}}\\
&= \left\{1+\left[\left[\exp\{-i\mathbf{t}'\boldsymbol{\eta} + \frac{1}{2}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{\frac{\lambda}{n}} - 1\right]\right\}^{-1}.
\end{aligned}$$

Hence by Lemma 3.2 of Pillai(1990a)

$$\phi_n(\mathbf{t}) = \left\{1+n\left[\left[\exp\{-i\mathbf{t}'\boldsymbol{\eta} + \frac{1}{2}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{\frac{\lambda}{n}} - 1\right]\right\}^{-1}$$

is the characteristic function of Y. Taking limit as $n \to \infty$, we have

$$\begin{aligned}
\phi(\mathbf{t}) &= \lim_{n\to\infty}\phi_n(\mathbf{t})\\
&= \left\{1+\lim_{n\to\infty} n\left[\left[\exp\{-i\mathbf{t}'\boldsymbol{\eta} + \frac{1}{2}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{\frac{\lambda}{n}} - 1\right]\right\}^{-1}\\
&= \left[1 - i\lambda\mathbf{t}'\boldsymbol{\eta} + \lambda\frac{1}{2}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} + \lambda\log\left(1+\int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}
\end{aligned} \tag{5.22}$$

□

Theorem 5.8.3. *Let* $\mathbf{X}_1, \mathbf{X}_2, \ldots$ *be independent and identically distributed as* $GeoGNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \frac{\lambda}{n}, \boldsymbol{\Gamma}, \boldsymbol{\mu})$. *Then* $\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n \xrightarrow{d} GNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ *as* $n \to \infty$.

Proof. The characteristic function of $GeoGNGS_\alpha(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \frac{\lambda}{n}, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ distribution is

$$\phi_{\mathbf{X}}(\mathbf{t}) = \left[1 - i\frac{\lambda}{n}\mathbf{t}'\boldsymbol{\eta} + \frac{\lambda}{n}\frac{1}{2}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} + \frac{\lambda}{n}\log\left(1 + \int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}$$

Then the characteristic function of $\mathbf{Y}$ is

$$\phi_{\mathbf{Y}}(\mathbf{t}) = \left[1 - i\frac{\lambda}{n}\mathbf{t}'\boldsymbol{\eta} + \frac{\lambda}{n}\frac{1}{2}\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} + \frac{\lambda}{n}\log\left(1 + \int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-n}$$

Hence,

$$\lim_{n\to\infty}\phi_{\mathbf{Y}}(\mathbf{t}) = \exp\{i\lambda\mathbf{t}'\boldsymbol{\eta} - \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t}\}\left[1 + \int_{S_d}|\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right]^{-\lambda}$$

That is, $\mathbf{Y} \xrightarrow{d} GNGS_{\alpha}(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$. □

5.8.1 AR(1) model with multivariate GeoGNGS marginals

We shall now construct an AR(1) processes with stationary marginals as multivariate GeoGNGS distributions.

Theorem 5.8.4. *Consider an autoregressive process $\{\mathbf{X}_n\}$ with structure given by (5.15). Then $\{\mathbf{X}_n\}$ is strictly stationary Markovian with $GeoGNGS_{\alpha}(\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ marginal if and only if $\{\boldsymbol{\epsilon}_n\}$ are distributed as $GeoGNGS_{\alpha}(\gamma\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$ provided that $\mathbf{X}_0$ is distributed as $GeoGNGS_{\alpha}(\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$*

Proof. Let us denote the Laplace transform of $\{\mathbf{X}_n\}$ by $\psi_{\mathbf{X}_n}(\mathbf{t})$ and that of $\boldsymbol{\epsilon}_n$ by $\psi_{\boldsymbol{\epsilon}_n}(\mathbf{t})$, equation (5.15) in terms of characteristic function becomes

$$\psi_{\mathbf{X}_n}(\mathbf{t}) = \gamma\psi_{\boldsymbol{\epsilon}_n}(\mathbf{t}) + (1-\gamma)\psi_{\mathbf{X}_{n-1}}(\mathbf{t})\psi_{\boldsymbol{\epsilon}_n}(\mathbf{t}).$$

On assuming stationarity, it reduces to the form

$$\psi_{\mathbf{X}}(\mathbf{t}) = \gamma\psi_{\boldsymbol{\epsilon}}(\mathbf{t}) + (1-\gamma)\psi_{\mathbf{X}}(\mathbf{t})\psi_{\boldsymbol{\epsilon}}(\mathbf{t}).$$

Write

$$\psi_{\mathbf{X}}(\mathbf{t}) = \left[1 + \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} - i\lambda\mathbf{t}'\boldsymbol{\eta} + \lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}$$

and hence

$$\psi_{\boldsymbol{\epsilon}}(\mathbf{t}) = \frac{\psi_{\mathbf{X}}(\mathbf{t})}{\gamma + (1-\gamma)\psi_{\mathbf{X}}(\mathbf{t})} \tag{5.23}$$

becomes

$$\psi_{\boldsymbol{\epsilon}}(\mathbf{t}) = \left[1 + \frac{1}{2}\gamma\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} - i\gamma\lambda\mathbf{t}'\boldsymbol{\eta} + \gamma\lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}.$$

Hence it follows that $\boldsymbol{\epsilon}_n \stackrel{d}{=} GeoGNGS_{\alpha}(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \gamma\lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$

The converse can be proved by the method of mathematical induction as follows. Now assume that $\mathbf{X}_{n-1} \stackrel{d}{=} GeoGNGS_{\alpha}(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$. Then

$$\begin{aligned}
\psi_{\mathbf{X}_n}(\mathbf{t}) =& \psi_{\boldsymbol{\epsilon}_n}(\mathbf{t})[\gamma + (1-\gamma)\psi_{\mathbf{X}_{n-1}}(\mathbf{t})] \\
=& \left\{\left[1 + \frac{1}{2}\gamma\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} - i\gamma\lambda\mathbf{t}'\boldsymbol{\eta} + \gamma\lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}\right\} \text{x} \\
& \left\{[\gamma + (1-\gamma)\left[1 + \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} - i\lambda\mathbf{t}'\boldsymbol{\eta} + \lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}\right\} \\
=& \left[1 + \frac{1}{2}\lambda\mathbf{t}'\boldsymbol{\mathcal{T}}\mathbf{t} - i\lambda\mathbf{t}'\boldsymbol{\eta} + \lambda\log\left(1 + \int_{S_d} |\mathbf{t}'\mathbf{s}|^{\alpha}\omega_{\alpha,1}(\mathbf{t}'\mathbf{s})\boldsymbol{\Gamma}(d\mathbf{s}) - i\boldsymbol{\mu}'\mathbf{t}\right)\right]^{-1}
\end{aligned} \tag{5.24}$$

That is, $\mathbf{X}_n \stackrel{d}{=} GeoGNGS_{\alpha}(\boldsymbol{\eta}, \boldsymbol{\mathcal{T}}, \lambda, \boldsymbol{\Gamma}, \boldsymbol{\mu})$. □

CHAPTER 6

APPLICATIONS

6.1 Introduction

Kozubowski(2001) examined the S&P index data and illustrated the potential of geometric stable distributions in modeling financial data. Mittag-Leffler distribution has been used to model random phenomena in finance and economics. Jose et al.(2010) applied the generalized Mittag-Leffler (GML) distribution in astrophysics and time series modeling.

Circular data analysis, and more generally spherical data analysis, has been practiced in areas like astronomy, ornithology, demography, geology, geography, meteorology, earth sciences, oceanography, and in biology. In ornithology, the nest orientation of birds, migration direction or general flight pattern is studied(see, Bergin (1991), Squires and Ruggiero(1996), Beason (1980), Bryan and Coulter(1987), Matthews(1974), Schmidt-Koeing (1963)). In biomathematics, the idea of circular distributions in animal behavior studies on homing, migration, escape, and exploratory behavior etc are well accepted.

In demography, circular data analysis has been used to study the concepts such as geographic marital patterns, occupational relocation in the same city and settlement trends(see, Coleman and Haskey(1986), Clark and Burt(1980), Upton (1986)).

In the present chapter, we consider the applications of univariate generalized geometric stable distributions to financial data. We used the currency exchange rates for validation of GGS model over other models. We modeled the data set of ordered remission times of bladder cancer patients to the DeML distribution. Applications of wind data to the wrapped GGS distributions also discussed.

6.2 Modeling price exchange rates

Here we study the distribution of the Japanese currency (Yen) exchange rates (in relation to US dollar). The data are daily exchange rates from 1/1/80 to 12/7/90. We consider the change in the log(price) from time i to $i+1$, that is, each data point P_i equals $P_i = \log(X_{i+1}) - \log(X_i)$, where X_i represents the closing price on day i. We shall compare the fit of normal, geometric stable and generalized geometric stable models. For comparison, we shall use histograms, QQ plots ad Kolmogorov Smirnov Statistic.

We use maximum likelihood method to estimate the parameters of assumed normal model, which resulted in mean 0 and standard deviation 0.07. For the geometric stable model, we applied estimation procedure based on method moments. It results in $\alpha = 1.7$. Since distribution appears to be symmetric, β is taken as 0. The parameters of generalized geometric stable are estimated based on the estimation procedure, proposed in Chapter 2. The estimates are obtained as $\lambda = 1.23, \alpha = 1.21, \beta = 0.01, \sigma = 1.99, \mu = .02$. The solutions of the non linear equations in the estimation techniques are obtained through the R programming package '*nleqslv*'. We calculated the values of trigamma and psigamma functions using *trigamma*() and *psigamma*() functions in R. We have simulated random samples from the above distributions and compare visually the histograms (see, Figure 6.1). Compared to normal, geometric stable and GGS models are more appropriate to the data. But histograms suggest slight improvement of GGS model upon the geometric stable model. We use the empirical QQ-Plots to validate the model. The fit is measured by the closeness of the graph to the straight line(straight line shows perfect fit). Figure 6.2 and Figure 6.3 represents the results. Further, we used the Kolmogorov-Smirnov distance to measure the goodness-of-fit, and present the results in Table 6.1. It also shows the GGS model fits the data better than the other models considered.

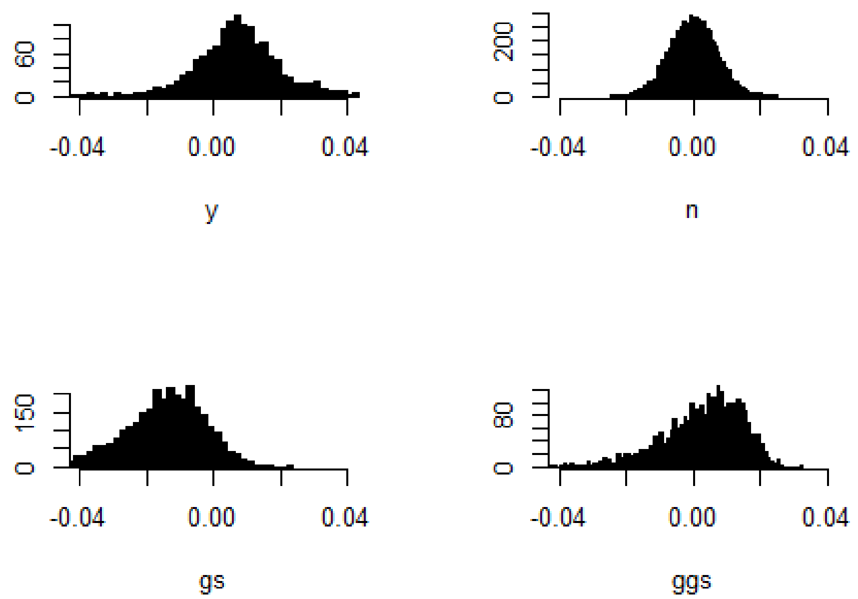

Figure 6.1: Histograms of model fit to the yen data. Clockwise from the top left: the data, the normal, the gs and ggs.

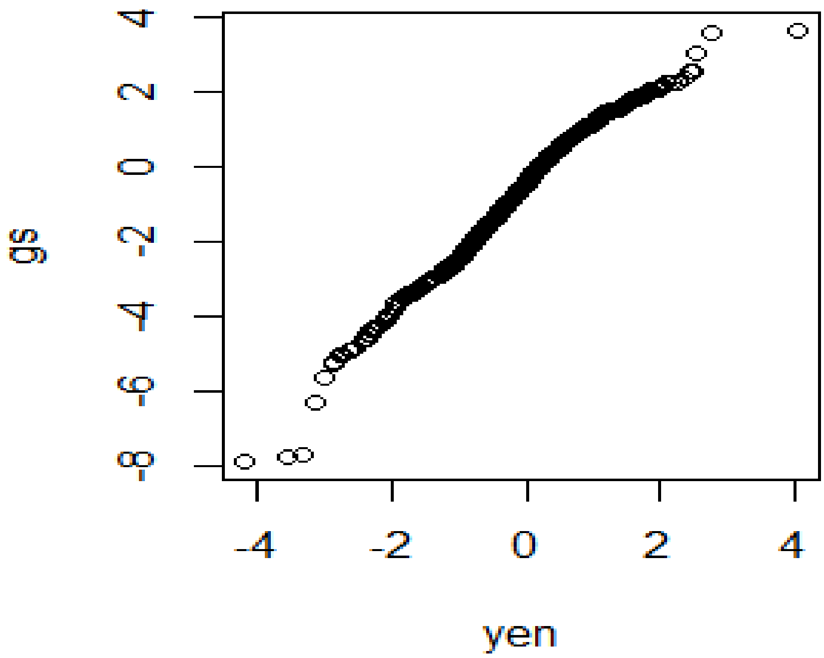

Figure 6.2: QQ plot of Yen data with geometric stable

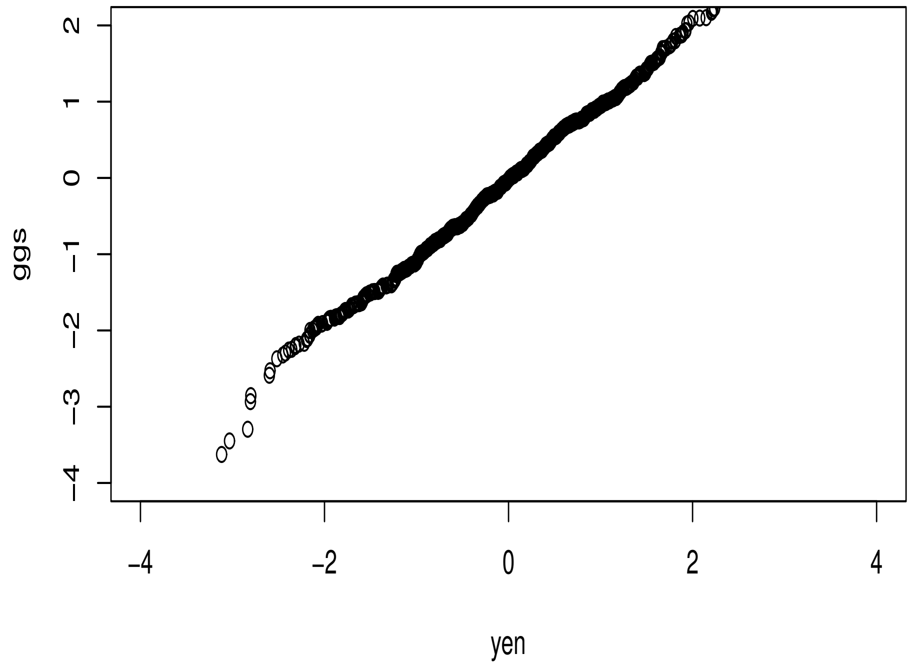

Figure 6.3: QQ plot of Yen data with generalized geometric stable.

Table 6.1: Kolmogorov distance for three models.

Normal	GS	GGS
0.39168	0.25443	0.15532

The Kolmogorov distance test numerically supports the results that GGS models dominates all other models considered.

6.3 Modeling remission times data using DeML distribution

In this section, we model the data set of ordered remission times (in months) of a random sample of 128 bladder cancer patients, reported in Lee and Wang(2003) to show the appropriateness of the proposed model to real life situations. The data set is given in Table 6.2

Table 6.2: The Remission Times (in Months) of 128 Bladder Cancer

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.2	2.23
0.52	4.98	6.97	9.02	13.29	0.4	2.26	3.57	5.06	7.09
0.22	13.8	25.74	0.5	2.46	3.46	5.09	7.26	9.47	14.24
0.82	0.51	2.54	3.7	5.17	7.28	9.74	14.76	26.31	0.81
0.62	3.28	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
0.39	10.34	14.38	34.26	0.9	2.69	4.18	5.34	7.59	10.66
0.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
0.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
0.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
0.4	3.02	4.34	5.71	7.93	11.79	18.1	1.46	4.4	5.85
0.26	11.98	19.13	1.76	3.25	4.5	6.25	8.37	12.02	2.02
0.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
0.73	2.07	3.36	6.39	8.65	12.63	22.69	5.49		

Here we compare the fit of three distributions, namely $DeML(\delta, \lambda, \alpha)$, $\mathcal{L}(\theta)$ and $E(\lambda)$ to the data set. Note that $\mathcal{L}(\theta)$ is the Lindley distribution with the probability density function

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)\exp(-\theta x), x > 0, \theta > 0,$$

and $E(\lambda)$ is exponential distribution with probability density function

$$f(x) = \lambda \exp(-\lambda x), x > 0, \lambda > 0.$$

We use maximum likelihood method to estimate the parameters of assumed models, which resulted in $\hat{\lambda} = 0.11688$ for exponential and $\hat{\theta} = 0.21322$ for Lindley distribution. For the DeML model, we applied estimation procedure based on method of moments based on empirical characteristic function. The estimates are obtained as $\hat{\alpha} = 0.857, \hat{\delta} = 3.871$ and $\hat{\lambda} = 3.923$. The solutions of the non linear equations in the estimation techniques are obtained through the R programming package '*nleqslv*'. We have simulated random samples from the above distributions and compare visually the histograms (see, Figure

6.4). Compared to exponential and Lindley, DeML model is more appropriate to the data. We use the empirical QQ-Plots to validate the model. The fit is measured by the closeness of the graph to the straight line. Figure 6.5 represents the results. Further, we used the Kolmogorov-Smirnov distance to measure the goodness-of-fit, and present the results in Table 6.3. It also shows the DeML model fits the data better than the other models considered.

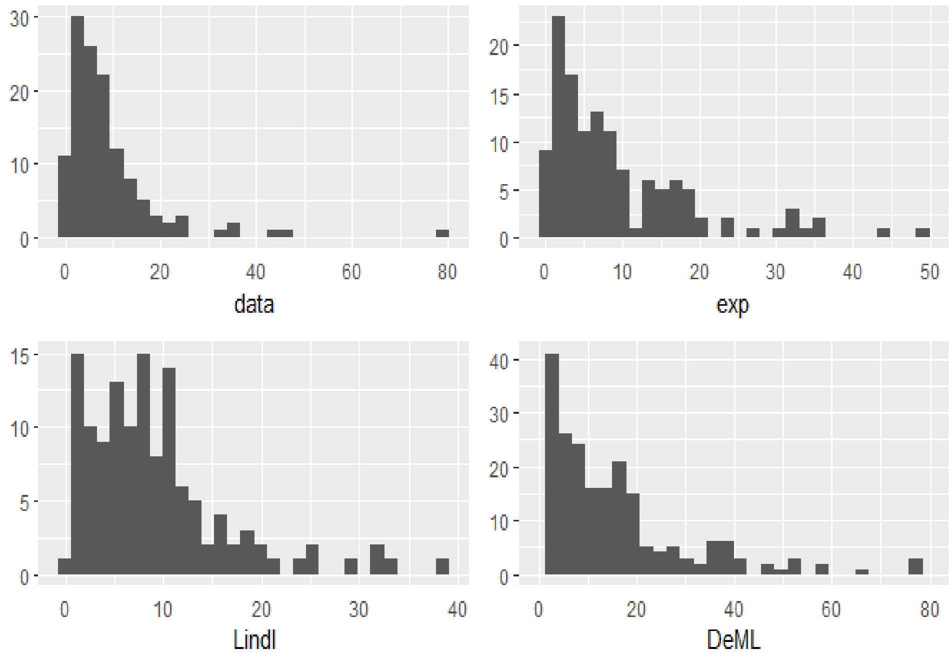

Figure 6.4: Histograms of model fit to the remission data. Clockwise from the top left: the data, the exponential, the Lindley and DeML.

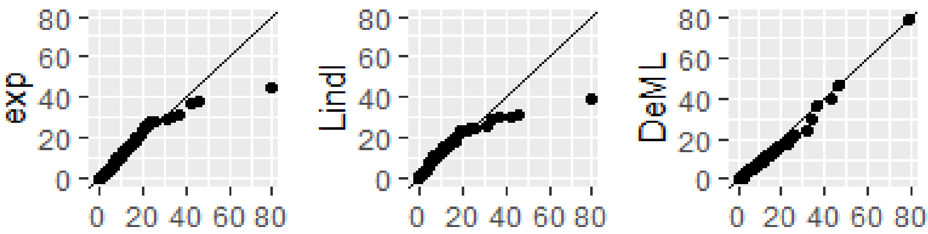

Figure 6.5: QQ plots of remission times data with the models

Table 6.3: Kolmogorov distance for three models

Exponential	Lindley	DeML
0.11719	0.0625	0.0423

The Kolmogorov distance test numerically supports the results that DeML distribution dominates all other models considered.

6.4 Applications to wind data

In this section, we study wind data set reported in Fisher(1993) to show the appropriateness of the proposed WGGS model to real life situations. The data set of directions (in degree) are given in Table 6.4. The performance of the model is compared with that of wrapped variance gamma distribution and generalized von Mises distribution using log-likelihood, AIC, and BIC.

Table 6.4: Wind data set

0	15	50	90	150	182	220	235
240	245	250	255	265	270	280	
285	300	315	330	335	340	345	

Here we compare the fit of three distributions, namely $\text{WGGS}(\lambda, \alpha, \beta, \sigma, \mu^*)$, $\text{WvG}(\mu, \lambda, \alpha, \beta, \gamma)$ and $\text{GvM}(\mu_1, \mu_2, \kappa_1, \kappa_2, \delta)$ to the data in Table 6.4. Note $\text{WvG}(\mu, \lambda, \alpha, \beta, \gamma)$ is the wrapped variance gamma distribution with pdf

$$f(\theta) = \frac{\gamma^{2\lambda} \exp\{\beta(\theta - \mu)\}}{\sqrt{\pi}\Gamma(\lambda)(2\alpha)^{\lambda-\frac{1}{2}}} \sum_{m=-\infty}^{\infty} \frac{\exp\{\beta m 2\pi\} K_{\lambda-\frac{1}{2}}(\alpha|\theta + 2m\pi - \mu|)}{|\theta + 2m\pi - \mu|^{\lambda-\frac{1}{2}}}$$

for $\theta \in [0, 2\pi), \alpha > 0, \beta > 0, 0 \leq |\beta| < \alpha, \lambda > 0, 0 \leq |\mu| < \alpha, \gamma = \sqrt{\alpha^2 - \beta^2} > 0$ where $K(.)$ is the modified Bessel function of the third kind. $\text{GvM}(\mu_1, \mu_2, \kappa_1, \kappa_2, \delta)$ is generalized von Mises distribution with pdf

$$f(\theta) = \frac{1}{2\pi G_0(\delta, \kappa_1, \kappa_2)} \exp\{\kappa_1 \cos(\theta - \mu_1) + \kappa_2 \cos 2(\theta - \mu_2)\},$$

for $0 \in [0, 2\pi), \mu_1 \in [0, 2\pi), \mu_2 = [0, \pi), \delta = \mu_1 - \mu_2 \mod (2\pi), \kappa_1, \kappa_2 > 0$, where $G_0(\delta, \kappa_1, \kappa_2) = \int_0^{2\pi} \exp\{\kappa_1 \cos\theta + \kappa_2 \cos 2(\theta + \delta)\}\mathrm{d}\theta$

Table 6.5: Summary of fits of distributions to the data

Distribution	Estimates	log L	AIC	BIC
WGGS	$\hat{\lambda} = 1.68$	-58.1171	76.448	73.160
$(\lambda, \alpha, \beta, \sigma, \mu^*)$	$\hat{\alpha} = 1.02$			
	$\hat{\beta} = 0.53$			
	$\hat{\sigma} = 0.22$			
	$\hat{\mu}^* = 2.13$			
WvG	$\hat{\mu} = 4.07$	-63.40	136.8	136.8
$(\mu, \lambda, \alpha, \beta, \gamma)$	$\hat{\lambda} = 2.00$			
	$\hat{\alpha} = 0.90$			
	$\hat{\beta} = 2.10$			
	$\hat{\gamma} = 0.50$			
GvM	$\mu_1 = 5.02$	-67.20	144.4	141.11
$(\mu_1, \mu_2, \kappa_1, \kappa_2, \delta)$	$\mu_2 = 5.70$			
	$\kappa_1 = 1.04$			
	$\kappa_2 = 0.0003$			
	$\delta = 0.68$			

The MLE's of the parameters corresponding to WGGS, WvG and GvM distributions along with the values of log-likelihood (log L), AIC and BIC are presented in Table 6.5. From the Table, it is clear that WGGS distribution has highest log-likelihood and smaller AIC and BIC values compared to the other two models. Hence WGGS is an appropriate model for modeling the data set in Table 6.4.

CHAPTER 7

RECOMMENDATIONS

Based on the works carried out in Chapters 2 to 6, we present the following recommendations:

- The most important applications of the GS laws come from the area of finance. The appropriateness of the four parameters GGS family of distributions over the GS models, to price exchange rates established in Chapter 6. We recommend GGS distribution as a flexible model in the area of heavy modeling, especially in modeling of financial data.

- The parameter estimation problem for the GS model is addressed by Kozubowski(1999), which proposes an estimation procedure based on characteristic functions. But it requires suitable constants prior to the computations of estimates. The estimation procedures for Mittag-Leffler and Linnik distributions proposed in Kozubowski(2001) also requires pre-selection of constants. But practically such pre-selection of values is infeasible. We recommend the estimation procedure based on moments of log transformed GS and GGS random variables proposed in the present work to address this drawback.

- Reed(2007)introduced an infinitely divisible distribution namely the generalized normal Laplace distribution(GNL), represented by the characteristic function defined in equation(2.41). It arises as the distribution of convolution of independent normal and generalized Laplace random variables. GNL distribution is particularly well suited for modeling logarithmic price returns which exhibit excess kurtosis with more probability mass near the origin and in the tails and less in the flanks than would occur for normally distributed data. We have introduced generalized normal geometric stable(GNGS) distribution with characteristic function(2.42) in Chapter 2 as a generalization of GNL distribution which we recommend in similar contexts, since it provides more modeling flexibility.

- The works on the concept of geometric extensions of different models such as geometric exponentials, geometric Mittag-Leffler etc. and their applications are discussed in Chpater 3. The GeoGGS distributions introduced in the present work generalizes most of the geometric extensions in the literature. It helps unified framework for future works and applications. Theorem 3.2.5 establishes how the δ parameter act as a pathway parameter between GS and GeoGGS distributions and which makes the GeoGGS distributions more siginificant. Hence it is highly recommendable for further studies to explore the full potential of GeoGGS distributions and its further extended model, GeoGNGS distributions.

- Circular models have applications in diverse field which are discussed in Chapter 4. Circular models, WGGS and WGNGS are proposed by wrapping GGS and GNGS dostributions and some data analysis conducted in Chapter 6, to justify that generalizations are recommendable models over the existing distributions.

- Multivariate extensions presented in Chapter 5 opens up new areas of research and we propose detailed study on each distributions proposed.

Based on the findings of the present study, we recommend some future works:

- Since we have developed an estimation procedure for the parameters of $GS(\lambda, \alpha, \beta, \sigma, 0)$ and $GGS(\lambda, \alpha, \beta, \sigma, 0)$ distributions based on log moments of its representations. We propose further studies for the extension of the estmation procedure, also for $\mu \neq 0$ cases of the models.

- Extensive study on generalized normal geometric stable distributions(GNGS) to explore the full potential of the model.

- Detailed study on geometric versions of GGS and GNGS models and its applications.

- A discrete analogue of Mittag–Leffler distribution was obtained in Pillai and Jayakumar (1995) as geometric sum of Sibuya random variables having probability generating function (pgf) $\delta(s) = 1 - (1-s)^{\alpha}, |s| \leq 1, 0 < \alpha < 1$. We say that a random variable X has discrete Mittag–Leffler distribution if its pgf is $\pi(s) = \frac{1}{1+c(1-s)^{\alpha}}, c > 0, 0 < \alpha \leq 1$. (see, Jayakumar *et al.*(2010)). Researchers worked on different extensions of this discrete version of Mittag-Leffler distribution. Since, Mittag-Leffler distribution is a special case of GGS distributions, we propose further studies for similar discrete extensions to GGS models.

- The properties and applications of the newly introduced multivariate models will be explored in the future works.

REFERENCES

[1] M. A. S. Adnan, and S. Roy(2014). Wrapped variance gamma distribution with an application to wind direction, *Journal of Environmental Statistics* **6**, 1-10.

[2] J. Andel(1983). Marginal distributions of autoregressive processes. *Transactions of* 9^{th} *Prague Conference on Information Theory* **9**, 127-135.

[3] Bachelier(1900). Théorie de la speculation. *Annales Scientifiques de l' École Normale Supérieure Series* 3 **17**, 21-86.

[4] R. C. Beason(1980). Orientation of Waterfowl migration in the southwestern United States, *Journal of Wildlife Management* **44**, 447-455.

[5] T. M. Bergin(1991). A comparison of goodness of fit tests for analysis of nest orientation in western Kingbirds, *The Condor* **93**, 164-171.

[6] A. L. Bryan, and M.C. Coulter(1987). Foraging flight characteristics of Wood Stoks in East-Central Georgia, U. S. A, *Colonial Waterbirds* **10**, 157-161.

[7] D. O. Cahoy(2013). Estimation of Mittag-Leffler parameters, *Communications in Statistics-Simulation and Computation* **42**, 303-315.

[8] J. M. Chambers, C. L. Mallows, and B. W. Stuck(1976). A method for simulating stable random variables, *Journal of the American Statistical Association* **71**, 340-344.

[9] W. A. V. Clark and J. E. Burt(1980). The impact of workplace on residential relocation, *Annals of the Association of American Geographers* **70**, 59-67.

[10] C. A. Coelho(2007). The Wrapped Gamma Distribution and Wrapped Sums and Linear Combinations of Independent Gamma and Laplace Distributions, *Journal of Statistical Theory and Practice* **1**, 1-29.

[11] D. A. Coleman and J. C. Haskey(1986). Marital distance and its geographical orientation in England and Wales, 1979, *Transactions of the Institute of British Geographers* **11**, 337-355.

[12] L. S. Dewald and P. A. W.Lewis(1985). A new Laplace second order autoregressive time series model-NLAR(2). *IEEE Transactions on Information Theory* **31**,645-651.

[13] M. D. Ernst(1998). A multivariate generalized Laplace distributions, *Computational Statistics* **13**, 227-232.

[14] E. Fama(1965). The behavior of stock market prices, *Journal of Business* **38**, 34-105.

[15] K. T. Fang, S. Kotz, K. W. Ng(1990). *Symmetric Multivariate and Related Distributions*, Chapman & Hall, London.

[16] V. Feller(1971). *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2^{nd} edn., Wiley, New York.

[17] T. S. Ferguson(1996). *A Course in Large Sample Theory*, Chapman&Hall, London.

[18] N. I. Fisher(1993). *Statistical Analysis of Circular Data,* Cambridge University Press, Melbourne.

[19] Y. Fujita(1993). A generalization of the results of Pillai. *Annals of the Institute of Statistical Mathematics* **45**, 361-365.

[20] R. Gatto and S. R. Jammalamadaka(2003). Inference for Wrapped Symmetric α-Stable Circular Models, *Sankhyá Ser A* **65**, 333-355.

[21] R. Gatto and S. R. Jammalamadaka(2007). The generalized von Mises distribution, *Statistical Methodology* **4**, 341-353.

[22] D. P. Gaver and P. A. W. Lewis(1980). First order autoregressive Gamma sequences and point processes, *Advances in Applied Probability* **12**, 727–745.

[23] S. Jacob(2012). Study on Circular Distributions, Ph.D. Thesis. University of Calicut: http://hdl.handle.net/10603/89283.

[24] S. Jacob and K. Jayakumar(2013). Wrapped Geometric Distribution: A new Probability Model for Circular Data, *Journal of Statistical Theory and Applications* **12**, 348 - 355.

[25] S. R. Jammalamadaka, A. S. Gupta(2001). *Topics in Circular Statistics.* World Scientific, New York.

[26] S. R. Jammalamadaka and T. J. Kozubowski(2003). A new family of circular models: The wrapped Laplace distributions, *Advances and Applications in Statistics* **3**, 77-103.

[27] S. R. Jammalamadaka and T. J. Kozubowski(2004). New families of wrapped distributions for modeling skew circular data, *Communications in Statistics-Theory and Methods* **33**, 2059-2074.

[28] K. Jayakumar (1997). First order autoregressive semi α-Laplace processes, *Statistica* **LVII**, 455-463.

[29] K. Jayakumar and B. K. Ajitha(2003). On the geometric Mittag–Leffler distributions, *Calcutta Statistical Association Bulletin* **54**, 195-208.

[30] K. Jayakumar and R. N. Pillai(1993). The first order autoregressive Mittag-Leffler process, *Journal of Applied Probability* **30**, 462-466.

[31] K. Jayakumar, M. M. Ristić and D. A. Mundassery(2010). A Generalization to Bivariate Mittag–Leffler and Bivariate Discrete Mittag–Leffler Autoregressive Processes, *Communications in Statistics - Theory and Methods* **39**, 942-955.

[32] K. Jayakumar and T. Sajayan(2020). On Estimation of Geometric Stable Distributions, *Journal of the Indian Society for Probability and Statistics* **21**, 329–347.

[33] K. K. Jose and M. T. Manu(2014). Multivariate normal-Laplace distribution and processes, *Statistica* **74**, 27-44.

[34] K. K. Jose and V. Seetha Lekshmi(1999). On Geometric Exponential distribution and its applications, *Journal of Indian Statistical Association* **37**, 51-58.

[35] K. K. Jose, P. Uma, V. Seetha Lekshmi, and H. J. Haubold(2010). Generalized Mittag-Leffler distributions and processes for applications in astrophysics and time series modeling, *Proceedings of the Third*

UN/ESA/NASA Workshop on the International Heliophysical Year 2007 and Basic Space Science, 79-92.

[36] S. Joshi and K. K. Jose(2018). Wrapped Lindley distribution, *Communications in Statistics - Theory and Methods* **47**, 1013-1021.

[37] K. Kafadar(1988). *Slash Distribution, Encyclopedia of Statistical Sciences* Vol. 8, (Johnson, N.L., Kotz, S., Read, C., Eds.), Wiley: New York, 510–511.

[38] D. Kelker(1970). Distribution theory of spherical distributions and a location-scale parameter generalization, *Sankhyá A* **32**, 419-430.

[39] L. B. Klebanov, G. M. Maniya and I. A. Melamed(1984). A problem of Zolotarev and analogs of infinitely divisible and stable distribution in a scheme for summing a random number of random variables. *Theory of Probability & Its Applications* **29**, 791-794.

[40] S. Kotz, T.J. Kozubowski, K. Podgórski(2001). *The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering and Finance*, Birkhäuser, Boston.

[41] T. J. Kozubowski and S. T. Rachev(1994). The theory of Geometric Stable Distributions and its use in modeling financial data, *European Journal of Operational Research* **74**, 310-324.

[42] T. J. Kozubowski(1994). The inner characterization of geometric stable laws, *Statistics & Decisions* **12**, 307-321.

[43] T. J. Kozubowski(1999). Geometric stable laws: Estimation and applications, *Mathematical and Computer Modelling* **29**, 241-253.

[44] T. J. Kozubowski(2001). Fractional moment estimation of Linnik and Mittag-Leffler parameters, *Mathematical and Computer Modelling* **34**, 1023-1035.

[45] T. J. Kozubowski and A.K. Panorska(1999). Multivariate geometric stable distributions in financial applications, *Mathematical and Computer Modelling* **29**, 83-92.

[46] T. J. Kozubowski and K. Podgórski(2000). A Multivariate and Asymmetric Generalization of Laplace Distribution, *Computational Statistics* **15**, 531-540.

[47] T. J. Kozubowski, K. Podgórski and I. Rychlik(2013). Multivariate generalized Laplace distribution and related random fields, *Journal of Multivariate Analysis* **113**, 59–72.

[48] E. E. Kuruoğlu(2001). Density parameter estimation of skewed α-stable distributions, *IEEE Transactions on Signal Processing* **49**, 2192-2201.

[49] R. G. Laha, and V. K. Rohatgi (1979). *Probability Theory*, Wiley, New York.

[50] A. J. Lawrance(1978). Some autoregressive models for point processes, *Proceedings of Colloquia Mathematica Societatis János Bolyai* **24**, 257–275.

[51] A. J. Lawrance and P. A. W. Lewis(1982) A new autoregressive time series model in exponential variables (NEAR (1)), *Advances in Applied Probability* **13**, 826-845.

[52] P. L. Lévy (1939). Addition des variables aléatoires définies sur une circonférence. *Bulletin de la Société Mathématique de France* **67**, 1-41.

[53] M. T. Manu(2013). Generalized Laplacian distributions and autoregressive processes, Ph.D. Thesis. Mahatma Gandhi University: http://hdl.handle.net/10603/28482.

[54] B. Mandelbrot(1963a). New methods in statistical economics, *Journal of Political Economy* **71**, 421-440.

[55] B. Mandelbrot(1963b). The variation of certain speculative prices, *Journal of Business* **36**, 394-419.

[56] K. V. Mardia(1972). *Statistics of Directional Data*, Academic Press, London.

[57] K.V. Mardia, P. E. Jupp (2000). *Directional Statistics*, 2^{nd} edn, Wiley, New York.

[58] G. V. T. Mathews(1974). On bird navigation, with some statistical undertones, *Journal of the Royal Statistical Society* **36**, 349-364.

[59] S. Mittnik and S. T. Rachev(1989). Stable distributions for asset returns, *Applied Mathematics Letters* **2**, 301-304.

[60] S. Mittnik and S. T. Rachev(1991). Alternative multivariate stable distributions and their applications to financial modeling, *Stable Processes and Related Topics, Birkhäuser, Boston, MA,* 107-119.

[61] S. Mittnik and S. T. Rachev(1993). Modeling asset returns with alternative stable distributions, *Econometric Reviews* **12**, 261-330.

[62] N. R. Mohan, R. Vasudeva and H.V. Hebbar (1993). On geometrically infinitely divisible laws and geometric domains of attraction, *Sankhya A* **55**, 171 - 179.

[63] A. G. Pakes(1998). Mixture representation for symmetric generalized Linnik laws, *Statistics and Probability Letters* **37**, 213-221.

[64] A. Pewsey(2008). The wrapped stable family of distributions as a flexible model for circular data, *Computational Statistics and Data Analysis* **52**, 1516 – 1523.

[65] R. N. Pillai(1985). Semi α-Laplace distributions, *Communications in Statistics - Theory and Methods* **14**, 991-1000.

[66] R. N. Pillai(1990a). On Mittag- Leffler functions and related distributions, *Annals of the Institute of Statistical Mathematics* **42**, 157-161.

[67] R. N. Pillai(1990b) Harmonic mixtures and geometric infinite divisibility, *Journal of The Indian Statistical Association* **28**, 87-98.

[68] A. V. D. Rao, I. R. Sarma and S. V. S. Girija(2007). On wrapped version of some life testing models, *Communications in Statistics-Theory and Methods* **36**, 2027-2035.

[69] W. J. Reed, M. Jorgensen(2004). The double Pareto lognormal distribution-a new parametric model for size distributions, *Communications in Statistics-Theory and Methods* **33**, 1733–1753.

[70] W. J. Reed(2007). Brownian Laplace motion and its use in financial modelling, *Communications in Statistics-Theory and Methods* **36**, 473–484.

[71] G. Samorodnitski and M. Taqqu(1994). *Stable Non-Guassian Random Processes*, Chapman & Hall, New York.

[72] K. Schmidt-Koeing(1963). On the role of the loft, the distance and site of release in pigeon homing, *Biological Bulletin* **125**, 154-164.

[73] V. Seetha Lekshmi and K. K. Jose(2004). An autoregressive process with geometric α-Laplace marginals, *Statistical Papers* **45**, 337-350.

[74] V. Seetha Lekshmi and K. K. Jose(2006). Autoregressive processes with Pakes and geometric Pakes generalized Linnik marginals, *Statistics and Probability Letters* **76**, 318-326.

[75] C. H. Sim(1993). First order autoregressive logistic process, *Journal of Applied Probability* **30**, 467-470.

[76] J. R. Squires and L. F. Ruggiero(1996). Nest Site Preference of Northern Goshawks in Southcentral Wyoming, *Journal of Wildlife Management* **60**, 170-177.

[77] F. Tan and H. Peng(2005). The slash and skew-slash student t distributions, Pre-print.

[78] J. Wang and M. G. Genton(2006). The multivariate skew-slash distribution, *Journal of Statistical Planning and Inference* **136**, 209-220.

[79] J. Varghese and K. K. Jose(2018). Wrapped hb-skewed Laplace distribution and its application in meteorology, *Biometrics & Biostatistics International Journal* **7**, 525–530.

[80] G. J. G. Upton(1986). Distance and directional analyses of settlement patterns, *Economic Geography* **62**, 167-179.