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A STUDY OF HYPERRIGID OPERATOR SYSTEMS IN C*-ALGEBRAS

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SHANKAR P.

KERALA SCHOOL OF MATHEMATICS

UNIVERSITY OF CALICUT

Kerala, India 673 635.

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KERALA SCHOOL OF MATHEMATICS

UNIVERSITY OF CALICUT

A. K. Vijayarajan
Associate Professor

Kerala School of Mathematics
24 January 2018

CERTIFICATE

I hereby certify that the thesis entitled "**A study of hyperrigid operator systems in C*-algebras**" is a bonafide work carried out by **Mr. Shankar P.,** under my guidance for the award of Degree of Ph.D., in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

A. K. Vijayarajan

Research Supervisor

DECLARATION

I hereby declare that the thesis, entitled "**A study of hyperrigid operator systems in C*-algebras**" is based on the original work done by me under the supervision of Dr. A. K. Vijayarajan, Associate Professor, Kerala School of Mathematics and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Kerala School of Mathematics,

24 January 2018

Shankar P.

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24 January 2018 Shankar P.

Contents

Chapter 1

Introduction

1.1 Motivation and Survey of Literature

The Classical theorems of Korovkin impressed several mathematicians since their discovery for the simplicity and the potential. Positive approximation process play a fundamental role in the approximation theory and it appears in a very natural way in several problems dealing with the approximation of continuous functions and qualitative properties such as monotonicity, convexity, shape preservation and so on.

Korovkin [28,29] in 1953 proved the most powerful and simplest criterion to decide whether a given sequence $(\phi_n)_{n\in\mathbb{N}}$ of positive linear operators on the space of continuous functions $C([0,1])$ is an approximation process, that is $\phi_n(f) \to f$ uniformly on $[0,1]$ for every $f \in C[0,1]$. In fact it is sufficient to verify that $\phi_n(f) \to f$ uniformly on $[0,1]$ only for $f \in \{1, x, x^2\}$. The set $\{1, x, x^2\}$ is called a Korovkin's set or a test set.

A considerable amount of research extended the Korovkin's theorems to the setting of different function spaces or more general abstract spaces such as Banach spaces, Banach algebras, Banach lattices, C^*-algebras and so on during last fifty years. At the same time, strong and fruitful connections of Korovkin's theory have been revealed not only with classical approximation theory but also with other fields such as functional analysis, measure theory, harmonic analysis, partial differential equations, probability theory and so on.

Another major advancement was the discovery of geometric theory of Korovkin's sets by Saskin [46] in 1966 and Wulbert [53] in 1968. A detailed survey of the most of these developments can be found in the survey article of Berens and Lorentz [11] in 1975. A selected part of the theory is already documented in the monograph of Altomare and Campiti [2] and survey article of Altomare [3].

Priestley [41] in 1976 initiated the study of Korovkin's theorem in C^*-algebras. Priestley proved that for a C^*-algebra A with identity I, if $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of positive linear maps from A into A satisfying $\phi_n(I) \leq I$ for all n, then

$$C = \{a \in A : a = a^*, \phi_n(a) \to a, \phi_n(a^2) \to a^2\}$$

is a J^*-algebra (i.e, a norm closed Jordan algebra of self adjoint elements of A). Recall that a Jordan algebra in A is a linear subspace of A closed under the Jordan product $a \circ b = (ab + ba)/2$. The theorem holds for the operator norm convergence, the weak operator convergence and the strong operator convergence. Also, Priestley established the above results in the trace norm convergence when $\{\phi_n\}$ acts on the trace class operators on $B(H)$.

Robertson [42] in 1977 generalized Priestley's results to complex C^*-algebras using ideas of Palmer [37] for large class of positive linear operators and obtained that the set C is actually a C^*-algebra. Robertson proved that if $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of Schwarz maps for a C^*-algebra A such that $\phi_n(I) \leq I$ for all n, then the set

$$D = \{a \in A : ||\phi_n(x) - x|| \to 0 \text{ for } x = a, a^*a, aa^*\}$$

is a C^*-algebra. Meanwhile, in 1979 Takahasi [50] improved Priestley's results in C^*-algebras considering norm convergence and without the assumption $a = a^*$.

Limaye and Namboodiri [30] in 1982 generalized the results of Priestley and Robertson and obtained the following result. Let A and B be complex C^*-algebras with identity, let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear maps from A into B and satisfying $\phi_n(I) \leq I$ for all n and ϕ is a C^*-homomorphism from A to B. Then

$$E = \{a \in A : \phi_n(a) \to \phi(a), \phi_n(a^* \circ a) \to \phi(a^* \circ a)\}$$

is a norm-closed $*$-subspace of A and is closed under the Jordan product. If all ϕ_n and ϕ are Schwarz maps, then E is a C^*-subalgebra of A. The theorem holds for the operator norm convergence, the weak operator convergence and the strong operator convergence. A slight modification of this theorem for the convergence in the trace norm is also proved in [30].

Arveson [4] in 1969 introduced the notion of boundary representation which is a non-commutative counterpart of a point in the Choquet boundary for function system in $C(X)$. The Choquet boundary of a function system is the set of points with unique representing measures. Let A be an abstract C^*-algebra and S be a linear subspace of A. An irreducible representation $\pi : A \to B(H)$ is called boundary representation for S if the only completely positive map from A to $B(H)$ which agrees with π on S is π itself; that is boundary representation has the unique completely positive extension from its restrictions to S.

Arveson introduced boundary representations to study to what extent does a subspace of operators on a Hilbert space determine the structure of the C^*-algebra it generates. Arveson proposed that there should be sufficiently many boundary representations, so that their direct sum recovers the norm on $M_n(A)$ for all $n \geq 1$. The C^*-algebra generated by this direct sum enjoys universal property and provides a realization of the C^*-envelop.

The existence of boundary representations has nice relation with the non-commutative Silov boundary. The first goal was achived by Arveson [5] in 1972 by giving several concrete examples and developing applications to operator theory. However, the existence of boundary representations and the Silov boundary were left open in general.

Tensor products of operator spaces (linear subspaces) of C^*-algebras and operator spaces of tensor product of C^*-algebras where explored by Hopenwasser in [23] and [24] to study boundary representations. In [23] it was shown that boundary representations of an operator subspace of a C^*-algebra $A \otimes M_n(\mathbb{C})$ under certain conditions are parametrised by the boundary representations of an operator subspace of the C^*-algebra A which is given by an operator subspace in $A \otimes M_n(\mathbb{C})$. In [24] it was proved that if one of the C^*-algebras of the tensor product is a GCR algebra, then the boundary representations of the tensor product of C^*-algebras correspond to products of boundary representations.

Hamana [21,22] in 1979 was able to establish the existence of the non-commutative Silov boundary by using his theory of injective envelopes. Hamana's work made no reference to boundary representations and left untouched the question of existence.

Muhly and Solel [34] in 1998 gave an algebraic characterization of boundary representations in terms of Hilbert modules, but used a generalized version of boundary representation by dropping the irreducibility condition. Muhly and Solel proved that boundary representations of operator algebras may be characterized as those completely contractive representations that determine modules that are simultaneously orthogonally projective and orthogonally injective. However, their arguments used Hamana's techniques and therefore the results did not lead to a new construction of the C^*-envelop.

Dritschel and McCullough [19] in 2005 took a major step forward by showing that every unital completely positive map of an operator system into $B(H)$ can be dilated to a completely positive map with the unique extension property. This provided a new proof of the existence of the non-commutative Silov boundary that makes no use of injectivity. The motivation for Dritschel and McCullough was the work of Agler [1] on a model theory for representations of non self-adjoint operator algebras. But their results seem to give no information about the existence of boundary representations.

Arveson [8] in 2008 settled the problem of the existence of boundary representations using the ideas of Dritschel and McCullough in the separable case. He used the disintegration theory of C^*-algebras and established that there exist sufficiently many boundary representations to completely norm it. That is, every separable operator system $S \subseteq C^*(S)$ has sufficiently many boundary representations in the sense that for every $n \geq 1$ and every $n \times n$ matrix $[s_{ij}]$ with components $s_{ij} \in S$, one has

$$||[s_{ij}]|| = \sup_{\pi} ||\pi([s_{ij}])|| \tag{1.1}$$

the supremum on the right hand side is taken over all boundary representations π for S.

Kleski [26] in 2014 established some closely related results in the separable case. He proved that from equality 1.1 “sup ” can be replaced by “max”. This implies that the Choquet boundary for a separable operator system is a boundary in the classical sense.

Finally, Davidson and Kennedy [16] in 2015 completely solved the existence of boundary representations using ideas of Arveson [4] and recent work of Dritschel and McCullough [19]. In particular their arguments neither require any disintegration theory nor they require separability. Therefore, every operator system in a C^*-algebra has sufficiently many boundary representations to completely norm it and hence they generate the C^*-envelop.

Saskin [46] in 1966 discovered an important geometric formulation of Korovkin’s theorem. In the classical case he explored the relation between the Korovkin sets and

Choquet boundary as follows. Let G be a subset of the continuous functions on the compact Hausdorff space $C(X)$ such that G separates points of X, contains the constant function 1. Then G is a Korovkin set if and only if the Choquet boundary of $S(=$ linear span$(G))$ is whole of X.

Arveson [10] in 2011 tried to prove the non-commutative analogue of Saskin's theorem using theory of non-commutative Choquet boundary for unital completely positive maps on C^*-algebras. For this purpose Arveson [10] introduced the non-commutative counterpart of the Korovkin's set which he named as hyperrigid set. Arveson defined a hyperrigid set as follows: Let G be a finite or countably infinite set that generates the abstract C^*-algebra $A = C^*(G)$. The set G is said to be hyperrigid if for every faithful representation of A on a Hilbert space H and every sequence of unital completely positive maps $\phi_1, \phi_2, ...$ from $B(H)$ to itself

$$\lim_{n\to\infty} ||\phi_n(g) - g|| = 0, \forall\, g \in G \Rightarrow \lim_{n\to\infty} ||\phi_n(a) - a|| = 0, \forall\, a \in A.$$

Arveson [10] proved that if the separable operator system is hyperrigid in the C^*-algebra then every irreducible representation of C^*-algebra is a boundary representation for the operator system. The converse to this result is called 'hyperrigidity conjecture': that is if every irreducible representation of a C^*-algebra is a boundary representation for a separable operator system then the operator system is hyperrigid. Arveson [10] gave partial answer to the hyperrigidity conjecture. He showed that hyperrigidity conjecture is true for C^*-algebras with countable spectrum.

Kleski [27] in 2014 established the hyperrigidity conjecture for all type-I C^*-algebras with additional assumptions on the co-domain. Kleski used the idea that every non-degenerate representation of the type-I C^*-algebras can be written as the direct integral of irreducible representations. Davidson and Kennedy [17] proved the conjecture for function systems. The hyperrigidity conjecture is still open for general C^*-algebras.

Bishop [12] in 1959 introduced the notion of peak points in the commutative case to study the generalization of the Choquet boundary based on slightly different ideas. Let S be a linear subspace of $C(X)$, a point $x \in X$ is a peak point of G if there exist a $f \in S$ such that $f(x) = ||f||$ and $|f(y)| < ||f||$, $x \neq y$. Suppose that S separates the points of X and contains the constant function 1. Then the set of peak points of S is a subset of the Choquet boundary of S and also the Choquet boundary of S is a subset of the closure of the set of peak points of S.

Arveson [10] in 2011 introduced the notion of peaking representation and strongly peaking representation which he used to improve his boundary theorem [5] which is as follows: Let S be a separable operator system in $B(H)$ and let A be the C^*-algebra generated by S. Let $K \neq 0$ be the ideal of compact operators in A and let $\hat{K}$ be the set of unitary equivalence classes of irreducible representations of A that live on K. Then $\hat{K}$ contains boundary representations for S if and only if the quotient map $x \in A \mapsto \dot{x} \in A/K$ is not completely isometric on S. Assuming that is the case, then among the irreducible representations of $\hat{K}$, the boundary representations for S are precisely the strongly peaking ones.

Peaking representations are a non-commutative generalization of peak points to operator systems. Like the classical case it is natural to enquire about the relation between the peaking representations and boundary representations. Arveson [9] proved that for a finite dimensional C^*-algebra all peaking representations are equivalent to the boundary representations. Kleski [26] established that for a separable operator system every peaking representation is a boundary representation.

Limaye and Namboodiri [32] in 1984 introduced the notion of weak Korovkin set in $B(H)$ using weak convergence of completely positive maps. Weak Korovkin set is a non-commutative analogue of the classical Korovkin set. They proved that an irreducible set in $B(H)$ is a weak Korovkin set if and only if the identity representation is a boundary representation for the irreducible set.

Namboodiri [36] in 2012, inspired by the work of Arveson [10] on hyperrigidity. He redefined the notion of weak Korovkin set as the weak hyperrigid set and explored the relation between the weak hyperrigid operator systems and boundary representations in [36]. Namboodiri gave a brief survey of the developments in non-commutative Korovkin-type theory in [35]. Uchiyama [52] proved the Korovkin type theorem for Schwarz maps using operator monotone functions in C^*-algebras.

1.2 Organisation of the Thesis

The notion of hyperrigidity introduced by Arveson [10] proved to be a very important idea connecting various directions of research in non-commutative approximation theory. Here we study the relation of hyperrigid operator systems to Hilbert modules, tensor product of hyperrigid operator systems, quasi hyperrigid operator systems, weak boundary representations and weak peak points.

In Chapter 1, we give the motivation and survey of various work about the classical Korovkin's theorem, Choquet boundary and peak points. The developments of the non-commutative analogue to these notions such as hyperrigidity, boundary representations and peaking representations are explained.

In Chapter 2, we gather the preliminary ideas that we need in our study of hyperrigid operator systems in C^*-algebras making the thesis self-contained as much as possible. In Section 2.1, as a prerequisite, we require a basic knowledge of the theory of C^*-algebras, von Neumann algebras, representations of C^*-algebras, various types of C^*-algebras, operator spaces and operator algebras. In section 2.2, we provide the classical notion of Choquet boundary, Shilov boundary and peak points. A couple of theorems relating peak points and Choquet boundary are explained. In Section 2.3, we describe the classical Korovkin's theorem, Korovkin set and Saskin's theorem relating the Korovkin set and Choquet boundary. In Section 2.4, we discuss the concept of completely positive maps on C^*-algebras and Stinespring's theorem for completely positive maps. In Section 2.5, we provide the concepts of boundary representation, unique extension property and C^*-envelope. We illustrate the developments in proving existence of boundary representations. In Section 2.6, we explain the notion of hyperrigidity and hyperrigidity conjecture. We provide various partial answers available in the literature to the hyperrigidity conjecture. In Section 2.7, we describe the notion of peaking representations and explain the relation between peaking representations and boundary representations.

In Chapter 3, we study the algebraic characterization of hyperrigid operator systems in terms of Hilbert modules. In Section 3.1, we provide the notion of Hilbert modules over the operator algebras, short exact isometric sequence, orthogonally projective Hilbert module and orthogonally injective Hilbert module. We illustrate the theorem relating boundary representations and orthogonality properties of Hilbert modules. In Section 3.2, we discuss the theorems due to Arveson [4] concerning extensions of contractive linear maps on unital subspaces of C^*-algebras. In Section 3.3, for an operator algebra $\mathcal{A}$ and the operator system $S = \mathcal{A} + \mathcal{A}^*$, we show that the unique extension property of the restriction to S of a representation of $C^*(S)$ is equivalent to the Hilbert modules over $\mathcal{A}$ corresponding to the representation being simultaneously orthogonally projective and orthogonally injective. This result leads to an algebraic characterization of hyperrigidity of the operator system $\mathcal{A} + \mathcal{A}^*$ in terms of the orthogonality properties of Hilbert modules over $\mathcal{A}$.

In Chapter 4, we study the tensor product of hyperrigid operator systems. In Section 4.1, we discuss the tensor product of C^*-algebras, tensor product of non-degenerate representations, spatial C^*-norm, maximal C^*-norm and nuclear C^*-algebras. In Section 4.2, we illustrate the work of Hopenwasser [23], [24] about the tensor product of boundary representations. In Section 4.3, we study hyperrigidity of operator systems in C^*-algebras in the context of tensor products of C^*-algebras. The question of whether tensor product of hyperrigid operator systems are hyperrigid is addressed here. By a result of Hopenwasser [24], tensor product of boundary representations of C^*-algebras for operator systems is a boundary representation if one of the constituent C^*-algebras is a GCR algebra. Since hyperrigidity implies that all irreducible representations are boundary representations, we will be able to deduce Hopenwasser's result as a special case if we can prove a similar result for hyperrigidity. We achieve this by establishing first that unique extension property for unital completely positive maps on operator systems carry over to tensor product of those maps defined on the tensor product of operator systems in the spatial tensor product of C^*-algebras.

In Chapter 5, we study the notions of qusi hyperrigidity, weak boundary representations, weak peak points and their relations. In Section 5.1, we introduce weak boundary representations and study the relation between boundary representations and weak boundary representations for operator systems of C^*-algebras. We prove that irreducible finite representations of an operator system are equivalent to weak boundary representations. We introduce quasi hyperrigid sets in C^*-algebras and observe that hyperrigid sets are quasi hyperrigid but quasi hyperrigid sets need not be hyperrigid. We prove an analogue of Saskin's theorem relating quasi hyperrigid operator systems and weak boundary representations for operator systems of C^*-algebras with countable spectrum. In Section 5.2, we introduce the notion of weak unique extension property. For type I C^*-algebras with an assumption on the co-domain of irreducible representations, we show that if an irreducible representation is a weak boundary representation for operator systems, then the operator system is quasi hyperrigid. In Section 5.3, we introduce the notion of weak peak points for operator systems in a C^*-algebra and prove that if an irreducible representation is a weak boundary and weak peak, then it is a boundary representation.

In Chapter 6, We discuss some problems for further research. The problems are described briefly.

Chapter 2

Preliminaries

We devote this chapter to introduce the terminologies and to list the preliminary definitions and basic results in the theory of classical Korovkin theory, Choquet boundary, peak points, non-commutative Korovkin theory, boundary representations and peaking representations.

2.1 C*-algebras and their representations

Let A be a vector space over the complex numbers $\mathbb{C}$. If A is closed with respect to the multiplication operation then A is called an *algebra*. The algebra A is called a *normed algebra* if there is associated to each element a a non-negative real number $||a||$, called the *norm* of a, with the following properties:

i. $||a|| \geq 0 \; \forall a \in A$ and $||a|| = 0$ if and only if $a = 0$;

ii. $||\lambda a|| = |\lambda|||a|| \; \forall \lambda \in \mathbb{C}, a \in A$;

iii. $||a + b|| \leq ||a|| + ||b|| \; \forall a, b \in A$;

iv. $||ab|| \leq ||a||||b|| \; \forall a, b \in A$.

A is called *Banach algebra* if A is complete with respect to the norm (if A is also a Banach space).

A mapping $a \to a^*$ of an algebra A into itself is called involution if it satisfies the following conditions

i. $(a^*)^* = a \; \forall a \in A$;

ii. $(a + b)^* = a^* + b^* \; \forall a, b \in A$;

iii. $(ab)^* = b^* a^* \; \forall a, b \in A$;

iv. $(\lambda a)^* = \overline{\lambda} a^* \; \forall \lambda \in \mathbb{C}, a \in A$.

An algebra A with an involution * is called a **-algebra*.

A Banach *-algebra A is said to be a C^**-algebra* if it satisfies the condition $||a^*a|| = ||a||^2$; $\forall a \in A$.

We give some examples of C^*-algebras.

Let X be a compact Hausdorff space and $C(X)$ denote the set of all continuous complex valued functions on X. The involution is defined by $f^*(x) = \overline{f(x)}$ for $f \in C(X)$ and $x \in X$. $C(X)$ is C^*-algebra and $C(X)$ is commutative.

Let H be a complex Hilbert space and $B(H)$ denote the bounded linear operators on the Hilbert space H. The involution is defined by A^*= the adjoint of A. Then $B(H)$ is a C^*-algebra. If the dimension of the Hilbert space is at least two then $B(H)$ is a non-commutative C^*-algebra.

Let S be a subset of a C^*-algebra A and define $S^* = \{a^* : a \in S\}$. S is said to be self-adjoint if $S = S^*$.

Let $S \subset B(H)$ be a set of operators on the Hilbert space H. The *commutant* of S denoted by S' is defined as

$$S' = \{a' \in B(H) : a'a = aa' \; \forall \; a \in S\}$$

We can see that S' is a unital subalgebra of $B(H)$ and closed in the weak operator topology on $B(H)$. It is easy to see that $S \subseteq S''$.

The celebrated *Double commutant theorem* of von Neumann says that, if M is a unital self-adjoint subalgebra of $B(H)$ then M is weakly closed if and only if M is strongly closed if and only if $M = M''$.

A *von Neumann algebra* is an unital self-adjoint subalgebra M of $B(H)$ which is closed in the weak operator topology.

Let M be a von Neumann algebra. M is said to be a factor if the center $Z = M \cap M'$ consists only of scalars. A projection $p \in M$ is *abelian* if pMp is commutative. In a von Neumann algebra M with center Z, the center of pMp is Zp. If p is abelian then $pMp = Zp$.

A von Neumann algebra M is said to be *Type I* if each non-zero projection in M dominates a non-zero abelian projection.

A *representation* of a C^*-algebra A is a *-homomorphism of A into the C^*-algebra $B(H)$ of all bounded linear operators on some Hilbert space H. A representation $\pi : A \to B(H)$ is said to be *faithful* if π is injective. A representation $\pi : A \to B(H)$ is said to be *non-degenerate* if the closed linear span $[\pi(A)H]$ of all vectors of the form $\pi(a)\xi$, $a \in A$, $\xi \in H$ is all of H. A representation $\pi : A \to B(H)$ is said to be a *cyclic representation* if there exist a vector $\xi \in H$ such that $[\pi(A)\xi] = H$. (Here the vector ξ is said to be the *cyclic vector* for the representation π).

Let $\pi : A \to B(H)$ be a representation and let K be a subspace of H. K is said to be *invariant subspace* for $\pi(a), a \in A$ if $\pi(a)$ maps K into itself. If both K and $K^\perp$ are invariant for $\pi(a), a \in A$ then K is a *reducing subspace* for $\pi(a)$.

Let $\pi : A \to B(H)$ and $\sigma : A \to B(K)$ be two irreducible representations of A. π and σ are said to be *unitarly equivalent* if there is a unitary operator $U : H \to K$ such that $\sigma(a) = U\pi(a)U^*$ for all $a \in A$. It is denoted by $\pi \sim \sigma$. A representation $\pi : A \to B(H)$ is said to be *irreducible* if $\pi(A)$ has no nontrivial closed invariant subspaces. This is same as saying that the only closed subspaces of H that are invariant for $\pi(A)$ are 0 and H.

The *spectrum* $\hat{A}$ of a C^*-algebra A is the set of all unitary equivalence classes of irreducible representations of A on a Hilbert space. Let π be a non-degenerate representation of C^*-algebra A on a Hilbert space H. Let H_0 be a subspace of H invariant under $\pi(A)$, then $\pi_0(a) = \pi(a)_{|H_0}$ defines a non-degenerate representation of A on H_0. Such a π_0 is called a *subrepresentation* of π.

An operator $T \in B(H)$ is said to be *compact* if the image of the unit ball of H under T has compact closure in the norm topology of Hilbert space H. The set of all compact operators on H is denoted by $K(H)$. Which is a closed two-sided ideal in $B(H)$.

A C^*-algebra A is said to be a *CCR* algebra if for every irreducible representation π of A, $\pi(A)$ consists of compact operators.

A C^*-algebra A is said to be a *GCR* algebra if for every irreducible representation $\pi : A \to B(H)$, $\pi(A)$ contains $K(H)$.

Let π be a universal representation of C^*-algebra A. The *enveloping von Neumann algebra* of A is the strong closure of $\pi(A)$. It will be denoted by A''. The enveloping von Neumann algebra A'' of a C^*-algebra A is isomorphic, as a Banach space to the second dual of A. Therefore, $A^{**} = A''$.

A C^*-algebra A is called *Type I C^*-algebra* if A^{**} is a Type I von Neumann algebra: i.e. if $\pi(A)''$ is a Type I von Neumann algebra for every representation π of A.

It is non-trivial that a Type I C^*-algebra is a GCR C^*-algebra. The Type I C^*-algebras have a nice representation theory, in the sense that any two irreducible representations are unitarily equivalent if and only if they have the same kernel.

A representation π from a C^*-algebra A into $B(H)$ is said to be a factor representation if $\pi(A)''$ is a factor. We can see that an irreducible representation is a factor representation.

An *operator system* S is an unital self-adjoint subspace of a C^*-algebra A. We will view S as a subspace of the C^*-algebra it generates, namely $A = C^*(S)$. If an operator system $S \subseteq B(H)$, then S is called a concrete operator system. There is a theory of abstract operator systems given by an axiomatic definition due to Choi and Effros [15] as opposed to the so called concrete operator system defined above. This distinction is irrelevant due to the representation theorem for abstract operator systems established in [15]. The representation theorem shows that all abstract operator systems can be represented as concrete operator systems.

An *operator algebra* $\mathcal{A}$ is an unital subalgebra of a C^*-algebra A. Similar to operator systems we will view $\mathcal{A}$ as a subalgebra of the C^*-algebra it generates, namely $A = C^*(\mathcal{A})$. If an operator algebra $\mathcal{A} \subseteq B(H)$, then $\mathcal{A}$ is called concrete operator algebra. There is a theory of abstract operator algebras given by an axiomatic definition due to Blecher, Ruan and Sinclair [14] as opposed to the so called concrete operator algebras defined above. This distinction is irrelevant due to the representation theorem for abstract operator algebras established in [14]. The representation theorem shows that all abstract operator algebras can be represented as concrete operator algebras.

2.2 Choquet boundary and peak points

Let X be a compact Hausdorff topological space, and let $C(X)$ be the set of continuous complex valued functions on X. A subset S of $C(X)$ is said to *separates* points of X if for each pair of points $x_1, x_2 \in X$, $x_1 \neq x_2$ there is a function $f \in S$ such that $f(x_1) \neq f(x_2)$. We say that S does not vanish on X if for each $x \in X$ there exists a

$f \in S$ such that $f(x) \neq 0$.

Let S be a subset of $C(X)$ that separates points and contains the constant function 1_X. Let $G =$ linear span(S), $\overline{G} = \overline{\text{linear span}(S)}$ and let G^* be the dual space of G. For each $x \in X$, l_x is the linear functional on G defined by $l_x(g) = g(x), g \in G$ and $l_x \in G^*$.

Define a map $\Phi : X \to G^*$ given by $\Phi(x) = l_x, x \in X$, which sends points x of X into a linear functionals $l_x \in G^*$. Since we assumed that S is separating, this will imply that the mapping Φ is one-to-one. Let X^* denote the image of X under Φ in G^*. If G^* is equipped with the weak* topology, Φ is continuous, hence we see that X^* is weak* compact set as the image of the compact set X under the mapping Φ. Therefore Φ is a homeomorphism from X onto X^*.

We define $K(G) = \overline{\text{co}}^* X^*$ be the weak* closed convex hull of X^* in G^*. We can see that $K(G)$ is weak* compact.

Theorem 2.2.1. *[11] $K(G) = \{l \in G^* : l(1_x) = 1 = ||l||\}$.*

The Krein-Milman theorem says that the set of extremal points of $K(G)$ is not empty.

Definition 2.2.1. *[11] The Choquet boundary of G consists of those points $x \in X$ for which l_x is an extremal point of $K(G)$. The Choquet boundary of G is denoted by ∂G.*

Let X be a compact metric space and $\mathcal{M}(X)$ denote the space of regular Borel measures μ on X. The norm of $\mu \in \mathcal{M}$ is its total variation $\int |d\mu|$ on X. The Riesz representation theorem says that $\mathcal{M}(X)$ is isometrically isomorphic to the dual space of $C(X)$. We use the notations $\mu(f)$ and $\int f d\mu$ for the duality relation between $C(X)$ and $\mathcal{M}(X)$. We define $\mathcal{L}(X)$ to be the cone of positive measures in $\mathcal{M}$. Let $\mathcal{P}(X)$ denote the space of probability measures μ on X that are positive and satisfy $\mu(X) = 1$. Equivalently the probability measures are characterized as $\mu(1_x) = 1 = ||\mu||$.

Let S be a subset of $C(X)$, G be the linear span of S and $\overline{G}$ be the closed linear span of S. Let G^* denotes the dual space of G. For each $x \in X$ we define

$$\mathcal{L}_x(S) = \{\mu \in \mathcal{L} : \mu(g) = g(x), g \in S\}.$$

The evaluation functional $\varepsilon_x(f)$ defined by $\varepsilon_x(f) = f(x)$ belongs to $\mathcal{L}_x$ and this set may contain further functionals. Let l_x be the restriction of ε_x to G, therefore $\mathcal{L}_x$ consists exactly of all those functionals $\mu \in \mathcal{L}$ that are extensions of l_x. Assume that $1_x \in S$. Then we see that $\mathcal{L}_x = \mathcal{P}_x$, where $\mathcal{P}_x$ is given by

$$\mathcal{P}_x(S) = \{\mu \in \mathcal{P} : \mu(g) = g(x), g \in S\}.$$

Theorem 2.2.2. *[11] Let S be a subset of $C(X)$ that separates points and contains the constant function 1_X. Let $x \in X$, then the linear functional l_x is an extremal point of $K(G)$ if and only if $\mathcal{P}_x(S) = \{\varepsilon_x\}$.*

$\mathcal{P}_x(S) = \{\varepsilon_x\}$ is same as saying that $\varepsilon_{x|_S}$ has a unique positive linear extension to $C(X)$. Using equivalent condition of the above theorem, we can redefine the definition of Choquet boundary. The following definition will help us to understand the non-commutative analogue of Choquet boundary in a batter way.

Definition 2.2.2. *Let $S \subset C(X)$ that separates points and contains the constant function 1_X and $G =$ linear span(S). The Choquet boundary ∂G of G is defined as*

$$\partial G = \{x \in X : \varepsilon_{x|_G} \text{ has a unique positive linear extension to } C(X)\}.$$

Now we justify the name boundary for ∂G. Let G be a closed subspace of $C(X)$ that separates points and contains the constant function 1_X. A subset Y of X is said to be a *boundary* for G if for each $g \in G$ there is a point $x \in Y$ such that $|g(x)| = ||g||$. We can see that the set ∂G in X is a boundary for G. The smallest closed boundary for G is called the *Silov boundary*. Silov boundary is identical with the closure of the Choquet boundary for G [40, Proposition 6.4].

The notion of *peak points* introduced by Bishop [12] for generalizations of the Choquet boundary is based on slightly different ideas.

Definition 2.2.3. *[11] Let G be a closed subspace of $C(X)$, separating points and containing the identity 1_X of $C(X)$. A point $x_0 \in X$ is a peak point of G if there exists a $g \in G$ for which $g(x_0) = \|g\|, |g(x)| < \|g\|,\ x \neq x_0$. The set of peak points of G is denoted by $P(G)$.*

The following two theorems express the relation between peak points and Choquet boundary in the classical case.

Theorem 2.2.3. *[11] Let G be a closed subspace of $C(X)$, separating points and containing the identity 1_X of $C(X)$; then $P(G) \subset \partial G$.*

Theorem 2.2.4. *[11] Let G be a closed subspace of $C(X)$, separating points and containing the identity 1_X of $C(X)$; then $\partial G \subset \overline{P(G)}$.*

2.3 Classical Korovkin theorem and Saskin theorem

The classical approximation theorem due to Korovkin [29] in 1953 unified many existing approximation processes such as Bernstein polynomial approximation of continuous functions and Weierstrass polynomial approximation of continuous functions.

Theorem 2.3.1. *[11] **(Korovkin's Theorem)** Let $\{\phi_n : n = 1, 2, 3, ...\}$ be a sequence of positive linear maps from $C([a, b])$ to itself. For each function $g_k(x) = x^k, x \in [a, b]$, $k = 0, 1, 2$, if*

$$\lim_{n\to\infty} \phi_n(g_k) = g_k \text{ uniformly on } [a, b], k = 0, 1, 2.$$

Then

$$\lim_{n\to\infty} \phi_n(f) = f \text{ uniformly on } [a, b], \text{ for all } f \text{ in } C[a, b].$$

Definition 2.3.1. *A set G in $C([a, b])$ is called a Korovkin set or test set, if for every sequence $\{\phi_n\}$, $n = 1, 2, 3, ...$ of positive linear maps form $C([a, b])$ to itself $\lim_{n\to\infty} \phi_n(g) = g$ uniformly on $[a, b]$ for every $g \in G$ implies that $\lim_{n\to\infty} \phi_n(f) = f$ uniformly on $[a, b]$ for every $f \in C([a, b])$.*

Korovkin theorem says that $\{1, x, x^2\}$ is a Korovkin set for $C([a, b])$.

Here we give the most remarkable and well celebrated theorem proved by Saskin in [46] relating Korovkin sets and Choquet boundary.

Theorem 2.3.2. *[11] **(Saskin's Theorem)** Let S be a subset of $C(X)$ that separates points of X and contains constant function 1_X. Then S is a Korovkin set in $C(X)$ if and only if the Choquet boundary $\partial G = X$, $G =$ linear span(S).*

2.4 Completely positive maps

An element in a C^*-algebra is *positive* if and only if it is self-adjoint and its spectrum is contained in the non-negative reals or equivalently if it is of the form a^*a for some element a in the C^*-algebra. We write $a \geq 0$ to denote that a is a positive element. The positive elements in a C^*-algebra A are a norm-closed, convex cone in the C^*-algebra and the positive elements in a C^*-algebra is denoted by A^+.

Let A be a C^*-algebra, let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices over the complex numbers $\mathbb{C}$, and let $M_n(A)$ denote the set of $n \times n$ matrices with entries from the C^*-algebra A. We will denote a typical element of $M_n(A)$ by $[a_{ij}]$. Let $[a_{ij}]$ and $[b_{ij}]$ in $M_n(A)$, set $[a_{ij}] \cdot [b_{ij}] = \left[\sum\limits_{k=1}^{n} a_{ik}b_{kj}\right]$ and $[a_{ij}]^* = [a_{ji}^*]$, using these operations we see that $M_n(A)$ is a *-algebra. It is not so obvious that there is a unique way to introduce a norm on $M_n(A)$ such that $M_n(A)$ becomes a C^*-algebra.

Consider the C^*-algebra $B(H)$, the set of bounded linear operators on a Hilbert space H. The identification $M_n(B(H)) = B(H^{(n)})$ (where $H^{(n)}$ denote the direct sum of n copies of the Hilbert space H), gives us a norm that makes $M_n(B(H))$ a C^*-algebra. The details of the identification can be found in [38]. Let A be a given

C^*-algebra. Now we can view $M_n(A)$ as a C^*-algebra by the following way. First choose a one-to-one $*$-representation of A on to some Hilbert space H, so that A can be identified as a C^*-subalgebra of $B(H)$. From this we can identify $M_n(A)$ as a $*$-subalgebra of $M_n(B(H))$. It is easy to see that image of $M_n(A)$ under this representation is closed, hence $M_n(A)$ is a C^*-algebra. But since norm on a C^*-algebra is unique, we see that the norm on $M_n(A)$ is independent of the particular representation of C^*-algebra A that we chose. Since positive elements remain positive under $*$-isomorphisms, we see that the positive elements of $M_n(A)$ are also uniquely determined.

Let A and B be two C^*-algebras. A linear map $\phi : A \to B$ determines a family of maps $\phi_n : M_n(A) \to M_n(B)$, $n \in \mathbb{N}$ given by the formula $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. In general the adverb *completely* means that all the maps $\{\phi_n\}$ enjoy some property. The map ϕ is called *positive* if ϕ maps positive elements of A to positive elements B. The map ϕ is called *completely positive* (CP) if ϕ_n is positive for all $n \geq 1$. The map ϕ is called *completely bounded* (CB) if $||\phi||_{\mathrm{CB}} = \sup\limits_{n\geq 1} ||\phi_n|| < \infty$. The map ϕ is said to be *completely contractive* (CC) if $||\phi||_{CB} \leq 1$.

The map ϕ is *unital completely positive* (UCP) if ϕ is completely positive and $\phi(1) = 1$. Since $||\phi||_{CB} = ||\phi(1)||$ for CP maps, we see that UCP maps are always completely contractive. The map ϕ is said to be *completely isometric* if ϕ_n is isometric for all $n \geq 1$.

We use the notation $CP(A, H)$ to denote the set of all completely positive (CP) maps from the C^*-algebra A to $B(H)$. By $UCP(A, H)$ we denote the subset of completely positive maps that are unital (UCP). When H is finite dimensional, elements of

$UCP(A, H)$ are called matrix states.

Now, we will state the Stinespring's dilation theorem and it's consequences, which will be useful for us to prove our results.

Theorem 2.4.1. *[38] **(Stinespring's dilation theorem)** Let A be a C^*-algebra with identity and let H be a Hilbert space. Then every completely positive linear map $\phi : A \to B(H)$ has the form*

$$\phi(a) = V^*\pi(a)V, \ \ a \in A,$$

where π is a representation of A on some Hilbert space K and V is a bounded operator from H to K.

If ϕ is unital, that is $\phi(1_A) = I$, then $I = \phi(1_A) = V^*\pi(1_A)V = V^*V$ therefore V is isometry.

Let $\phi(a) = V^*\pi(a)V, \ a \in A$ be as in the above theorem. Let $K_0 = [\pi(A)VH]$ and we can restrict π to K_0. Let $\pi_0 = \pi_{|K_0}$, π_0 also satisfies $\phi(a) = V^*\pi_0(a)V, \ a \in A$. Therefore $\phi(a) = V^*\pi_0(a)V, \ a \in A$ is called the minimal Stinespring's dilation of ϕ. So there is no essential loss if we assume that $[\pi(A)VH] = K$.

Now we describe the BW-topology on the space of all operator valued linear maps of a subspace of a C^*-algebra. Let S be a linear subspace of C^*-algebra A. Let $\mathcal{B}(S, H)$ denote the linear space of all bounded linear maps from S into $B(H)$. Observe that $\mathcal{B}(S, H)$ is a Banach space with operator norm. We will provide $\mathcal{B}(S, H)$ with a certain weak topology, relative to which it becomes dual of another Banach space.

Let $\mathcal{B}_r(S,H)$ denote the closed ball of radius $r>0$ i.e.,

$$\mathcal{B}_r(S,H)=\{\varphi\in\mathcal{B}(S,H):||\varphi(a)||\leq r||a|| \text{ for all } a\in S\}.$$

First, we topologize $\mathcal{B}_r$ as follows. A net $\{\varphi_\alpha\}_{\alpha\in\wedge}$ in $\mathcal{B}_r(S,H)$ converges to φ in $\mathcal{B}_r(S,H)$ if $\varphi_\alpha(a)\to\varphi(a)$ in the weak operator topology for every $a\in S$. A convex subset U of $\mathcal{B}(S,H)$ is open if $U\cap\mathcal{B}_r(S,H)$ is an open subset of $\mathcal{B}_r(S,H)$ for every $r>0$. The convex open sets form a base for a locally convex Hausdorff topology on $\mathcal{B}(S,H)$. This topology is called BW-topology. The BW-topology is the strongest locally convex topology on $\mathcal{B}(S,H)$.

The immediate consequence of a general theorem of Kadison [25] is as follows: for every $r>0$, $\mathcal{B}_r(S,H)$ is compact in the relative BW-topology.

2.5 Boundary representations

Arveson [4] introduced the notion of *boundary representation*, which is the non-commutative analogue of points in the Choquet boundary.

Definition 2.5.1. *[10] Let S be an operator system in a C^*-algebra A such that $A=C^*(S)$. A representation $\pi:A\to B(H)$ of A is said to have unique extension property (UEP) for S, if the only unital completely positive (UCP) map $\phi:A\to B(H)$ that satisfies $\phi_{|S}=\pi_{|S}$ is $\phi=\pi$ itself.*

Definition 2.5.2. *[10] Let S be an operator system in a C^*-algebra A such that $A = C^*(S)$. An irreducible representation $\pi : A \to B(H)$ of A is said to be a boundary representation for S if π has unique extension property (UEP) for S.*

The set of boundary representations for S is denoted by ∂S. The set of boundary representations is a non-commutative analogue of the Choquet boundary of a function algebra which is the set of points with unique representing measures.

A map $\phi \in UCP(A, H)$ is called *pure*, if whenever $\phi - \xi$ is completely positive for some $\xi \in CP(A, H)$, there exists $0 \leq t \leq 1$ such that $\xi = t\phi$.

Definition 2.5.3. *[19] The C^*-envelope of an operator system S (operator algebra $\mathcal{A}$), denoted by $C_e^*(S)$ ($C_e^*(\mathcal{A})$), is the essentially unique smallest C^*-algebra amongst those C^*-algebras $\mathcal{C}$ for which there is a completely isometric homomorphism $\phi : S \to \mathcal{C}$ ($\phi : \mathcal{A} \to \mathcal{C}$).*

Let S be an operator system generating the C^*-algebra $A = C^*(S)$. Let $\{\pi_x : x \in I\}$ be a set of irreducible representations of A. We say that $\{\pi_x : x \in I\}$ is sufficient for S if

$$||s|| = \sup_{x \in I} ||\pi_x(s)||, \quad s \in S$$

with similar formulas holding throughout the matrix hierarchy over S in the sense that for every $n \geq 2$ and every $n \times n$ matrix $[s_{ij}] \in M_n(S)$, we have

$$||[s_{ij}]|| = \sup_{x \in I} ||\pi_x([s_{ij}])||. \tag{2.1}$$

If the set of all boundary representations for S is sufficient in this sense, we say that S has *sufficiently many boundary representations*.

Arveson [4] proposed that there should be sufficiently many boundary representations for an operator algebra $\mathcal{A}$, so that their direct sum recovers the norm on $M_n(\mathcal{A})$ for all $n \geq 1$. In this case, Arveson showed that the C^*-algebra generated by this direct sum enjoys an important universal property, and provides a realization of the C^*-envelope of $\mathcal{A}$. Arveson was not able to prove the existence of boundary representations in general, although boundary representations in various concrete cases were exhibited. Consequently, he was also unable to prove the existence of the C^*-envelope in general.

Hamana [22] proved the existence of the C^*-envelope using other methods without using boundary representations. Hamana constructed the minimal injective operator system containing $\mathcal{A} + \mathcal{A}^*$, but it answered little regarding questions about boundary representations. Nevertheless, it did lead to a variety of cases in which the C^*-envelope can be explicitly described.

Muhly and Solel [34] gave an algebraic characterization of boundary representations in terms of Hilbert modules, but used a generalized version of boundary representation by dropping the irreducibility condition. Muhly and Solel proved that boundary representations of operator algebras may be characterized as those completely contractive representations that determine modules that are simultaneously orthogonally projective and orthogonally injective.

Theorem 2.5.1. *[34] Let H be a contractive Hilbert module over an operator algebra $\mathcal{A}$ and let ρ be the associated representation. Then ρ is the restriction to $\mathcal{A}$ of a boundary representation of $C_e^*(\mathcal{A})$ for $\mathcal{A}$ if and only if H is both orthogonally projective and orthogonally injective.*

Dritschel and McCullough [19] came up with a new proof of the existence of the C^*-envelope. It was a bonafide dilation theory argument, building on ideas of Agler [1]. Dritschel and McCullough introduced the idea of maximal dilations. The consequence of this direct dilation approach is the following: if you begin with a completely isometric representation of $\mathcal{A}$, and find a maximal dilation, then the C^*-algebra generated by the image of this dilation is the C^*-envelope. Consequently, there has been considerable interest in maximal dilations.

Arveson [8] revisited the problem of the existence of boundary representations. By using the ideas of Dritschel and McCullough and disintegration theory of representations of C^*-algebras, Arveson proved that, in the separable case, sufficiently many boundary representations exist.

Theorem 2.5.2. *[8] Every separable operator system $S \subseteq C^*(S)$ has sufficiently many boundary representations.*

Kleski [26] showed that in equality 2.1 “sup” can be replaced by “max” in the separable case. This implies that the Choquet boundary for a separable operator system is a boundary in the classical sense.

Theorem 2.5.3. *[26] Let S be a concrete separable operator system. For each $s \in S$, there exists a boundary representation π for S such that $||\pi(s)|| = ||s||$.*

Davidson and Kennedy [16] completely solved the problem of existence of boundary representations by proving that every operator system (and hence every operator algebra) has sufficiently many boundary representations to generate the C^*-envelope. Davidson and Kennedy used direct dilation-theoretic argument, building on ideas from Arveson's 1969 paper [4], and the more recent work of Dritschel and McCullough [19]. In particular, their arguments do not require any disintegration theory nor do they require separability.

2.6 Arveson's hyperrigidity conjecture

In connection with the fundamental work related to the non-commutative approximation theory, Arveson [10] introduced the notion of non-commutative analogue of Korovkin sets which he called hyperrigid sets.

Definition 2.6.1. *[10] A finite or countably infinite set G of generators of a C^*-algebra A is said to be hyperrigid if for every faithful representation $A \subseteq B(H)$ of A on a Hilbert space H and every sequence of unital completely positive (UCP) maps $\phi_n : B(H) \to B(H)$, $n = 1, 2, ...$,*

$$\lim_{n\to\infty} ||\phi_n(g) - g|| = 0, \forall\, g \in G \Rightarrow \lim_{n\to\infty} ||\phi_n(a) - a|| = 0, \forall\, a \in A.$$

It is easy to see that a set G is hyperrigid if and only if the linear span of $G \cup G^*$ is hyperrigid, so that hyperrigidity is properly thought of as a property of self-adjoint operator subspaces of a C^*-algebra.

Arveson gave the general characterization of hyperrigid sets as follows:

Theorem 2.6.1. *[10] Let S be a separable operator system that generates the C^*-algebra A such that $A = C^*(S)$. The following are equivalent:*

i. *S is hyperrigid.*

ii. *For every non-degenerate representation $\pi : A \to B(H)$ on a separable Hilbert space and every sequence $\phi_n : A \to B(H)$ of UCP maps,*

$$\lim_{n\to\infty} ||\phi_n(s) - s|| = 0, \forall\, s \in S \Rightarrow \lim_{n\to\infty} ||\phi_n(a) - a|| = 0, \forall\, a \in A.$$

iii. *For every non-degenerate representation $\pi : A \to B(H)$ on a separable Hilbert space, $\pi_{|_S}$ has the unique extension property.*

iv. *For every unital C^*-algebra B, every unital homomorphism of C^*-algebras $\theta : A \to B$ and every UCP map $\phi : B \to B$,*

$$\phi(x) = x \;\; \forall\, x \in \theta(S) \Rightarrow \phi(x) = x \;\; \forall\, x \in \theta(A).$$

Using the above theorem, Arveson [10] gave many examples of hyperrigid generators. Here, we mention some of the examples of hyperrigid generators.

Theorem 2.6.2. *[10] Consider the Volterra integration operator V acting on the Hilbert space $H = L^2[0,1]$,*

$$Vf(x) = \int_0^x f(t)dt, \quad f \in L^2[0,1].$$

It is well-known that V is irreducible, generating the C^*-algebra $K(H)$ of all compact operators. This operator has the following properties:

(i) $G = \{V, V^2\}$ is hyperrigid; for every sequence of unital completely positive maps $\phi_n : B(H) \to B(H)$ for which

$$\lim_{n\to\infty} ||\phi_n(V) - V|| = \lim_{n\to\infty} ||\phi_n(V^2) - V^2|| = 0,$$

one has

$$\lim_{n\to\infty} ||\phi_n(K) - K|| = 0$$

for every compact operator $K \in B(H)$.

(ii) The smaller generating set $G_0 = \{V\}$ of $K(H)$ is not hyperrigid.

Theorem 2.6.3. *[10] Let $V_1, V_2, ..., V_n \in B(H)$ be an arbitrary set of isometries that generates a C^*-algebra A. Then $G = \{V_1, ..., V_n, V_1V_1^*, ..., V_nV_n^*\}$ is hyperrigid generator for A.*

Arveson introduced the notion hyperrigid set to examine the relation between the hyperrigid operator systems and boundary representations, resulting in the following analogue of Saskin's Theorem 2.3.2.

Corollary 2.6.1. *[10] Let S be a separable operator system generating a C^*- algebra A such that $A = C^*(S)$. If S is hyperrigid, then every irreducible representation of A is a boundary representation for S.*

The converse of the above corollary is called hyperrigidity conjecture.

Conjecture 2.6.1. *[10] Let S be a separable operator system generating a C^*- algebra A such that $A = C^*(S)$. If every irreducible representation of A is a boundary representation for a separable operator system $S \subseteq A$, then S is hyperrigid.*

Still, hyperrigidity conjecture is not completely resolved. But it is proved for certain classes of C^*-algebras. Arveson [10] proved the conjecture for C^*-algebras having a countable spectrum.

Theorem 2.6.4. *[10] Let S be a separable operator system whose generated C^*-algebra $A = C^*(S)$ has countable spectrum, such that every irreducible representation of A is a boundary representation for S. Then S is hyperrigid.*

Kleski [27] proved the hyperrigidity conjecture for all type-I C^*-algebras with some additional assumptions. Here we will give the main results of Kleski.

Theorem 2.6.5. *[27] Let S be a separable operator system in $B(H)$ generating a C^*-algebra A, and suppose A'' is injective. Suppose every factor representation of A has the UEP relative to S. Let ρ be a faithful representation of A on $B(K)$ and let $\gamma : \rho(A) \to B(K)$ be an UCP map extending $id_{|_{\rho(S)}}$. Then for every conditional expectation $E : B(K) \to \rho(A)''$, we have $E\gamma\rho(a) = \rho(a)$ for all $a \in A$.*

Corollary 2.6.2. *[27] Let S be a separable operator system generating a Type-I C^*-algebra A. If every irreducible representation of A is a boundary representation for S, then for any representation π of A on $B(K)$ and any UCP map $\psi : \pi(A) \to B(K)$ extending $id_{|_{\pi(S)}}$ and any conditional expectation $E : B(K) \to \pi(A)''$, $E\psi\pi = \pi$.*

Corollary 2.6.3. *[27] Let S be a separable operator system generating a Type-I C^*-algebra A. If every irreducible representation of A is a boundary representation for S, then for any UCP map $\psi : A \to A''$ such that $\psi(s) = s$, we have $\psi(a) = a$.*

Davidson and Kennedy [17] proved the hyperrigidity conjecture for function systems.

Theorem 2.6.6. *[17] Let S be a concrete function system that generates a commutative C^*-algebra $C(X)$. Then S is hyperrigid if and only if every irreducible representation of $C(X)$ is a boundary representation for S.*

2.7 Peaking representstions

Arveson [10] introduced the notion of *peaking representation*, which is the non-commutative analogue of classical peak points.

Definition 2.7.1. *[10] Let S be a separable operator system and let $A = C^*(S)$ is the C^*-algebra generated by S. An irreducible representation $\pi : A \to B(H)$ is said to be a peaking representation for S if there is an $n \geq 1$ and an $n \times n$ matrix $[s_{ij}]$ over S such that*

$$|| \ (\pi[s_{ij}]) \ || > || \ (\sigma[s_{ij}]) \ ||$$

for every irreducible representation σ not unitarily equivalent to π.

Arveson [9] tried to investigate the non-commutative analogue of theorem 2.2.3 and theorem 2.2.4 which relates the peaking representations and boundary representations of the operator systems. Arveson [9] proved it in the finite dimensional case.

Theorem 2.7.1. *[9] Let S be an operator system that generates a finite dimensional C^*-algebra $C^*(S)$. An irreducible representation of $C^*(S)$ is a boundary representation for S if and only if it is peaking for S.*

Kleski [26] proved the relation between peaking representations and boundary representation for separable C^*-algebras. The main results of Kleski is as follows.

Theorem 2.7.2. *[26] Let S be a concrete separable operator system. For each $s \in S$, there exists a boundary representation π for S such that $||\pi(s)|| = ||s||$.*

Corollary 2.7.1. *[26] Let S be a separable operator system that generates a C^*-algebra $C^*(S)$. Then every peaking representation for S is a boundary representation for S.*

Chapter 3

Hyperrigidity and Hilbert modules

In this chapter, we show that for an operator algebra $\mathcal{A}$, the operator system $S = \mathcal{A}+\mathcal{A}^*$ in the C^*-algebra $C^*(S)$ and any representation ρ of $C^*(S)$ on a Hilbert space H, the restriction $\rho_{|S}$ has unique extension property if and only if the Hilbert module H over $\mathcal{A}$ is both orthogonally projective and orthogonally injective. As a corollary we deduce that when S is separable, the hyperrigidity of S is equivalent to the Hilbert modules over $\mathcal{A}$ being both orthogonally projective and orthogonally injective.

3.1 Hilbert modules over operator algebras

In this section, we recall basic definitions of Hilbert modules and related concepts relevant to our discussion.

A representation of an operator algebra $\mathcal{A}$ is a homomorphism π from $\mathcal{A}$ to the algebra $B(H)$ of all bounded operators on a Hilbert space H. The representation π is continuous with respect to the norm topologies on $\mathcal{A}$ and $B(H)$. The continuity of π as a linear map means that there is a positive constant K such that $||\pi(a)\xi|| \leq K||a||||\xi||, a \in \mathcal{A}, \xi \in H$. If $K = 1$, and then π is called contractive representation. As far as representations of operator algebras are concerned, we are interested in contractive representations here. One reason for focusing on contractive representations is that they coincide with C^*-representations when the operator algebra is a C^*-algebra. We will assume that all given representations are non-degenerate.

A (left) *Hilbert module* over an operator algebra $\mathcal{A}$ is simply a Hilbert space H which is an algebraic (left) module over $\mathcal{A}$ such that the module product is continuous ($||a \cdot \xi|| \leq ||a||||\xi||, a \in \mathcal{A}, \xi \in H$). Representations of algebras on Hilbert spaces give rise to Hilbert modules over algebras and vice versa. Since the representation of operator algebra is continuous, we will define module action using the representation of operator algebras.

Let $\pi : \mathcal{A} \to B(H)$ be a representation for an operator algebra $\mathcal{A}$ on a Hilbert space H. A (left) *Hilbert module* over $\mathcal{A}$ is simply the Hilbert space H viewed as an algebraic (left) module over A via the module action $a\xi := \pi(a)\xi$. The advantage of using the language of Hilbert modules over operator algebras and their representations is that we can pass from one to the other when it is convenient. If $\pi : \mathcal{A} \to B(H)$ is a representation, the associated module will be written as ${}_{\mathcal{A}}H$ or H_π. If H is a Hilbert module, the representation associated will be written as π_H. *Right* Hilbert modules are defined in the similar way and correspond to *anti-representations* of operator algebras

$\mathcal{A}$ ($\pi(ab) = \pi(b)\pi(a), a, b \in \mathcal{A}$).

A *contractive Hilbert module* is one where the associated representation is contractive. A Hilbert module is called *completely bounded* (*completely contractive*) if the associated representation is completely bounded (completely contractive) as a linear operator-valued map on the operator algebra. Here we assume all representations and Hilbert modules are completely contractive.

A *module map* from the Hilbert module H to the Hilbert module K is a module map in the algebraic sense: that is, continuous as a linear map from H to K and module maps are just intertwining operators for the representations. We write Hom(H, K) for the module maps from H to K and if $H = K$, we write End(H) for Hom(H, H). *Hilbert module isomorphisms* are unitary module maps.

A sequence of Hilbert modules over an operator algebra $\mathcal{A}$,

$$...H_{i-1} \xrightarrow{\phi_{i-1}} H_i \xrightarrow{\phi_i} H_{i+1} \to ...$$

where the ϕ_i's are module maps is called *exact* at H_i if the kernel of ϕ_i coincides with the range of ϕ_{i-1}. It is called *isometric* if each of the Φ_i's is a partial isometry as a Hilbert space map.

A *submodule* of a Hilbert module H is a closed subspace K of H which is a submodule in the usual algebraic sense. Thus, a submodule of H is just a closed subspace of H that is invariant for the algebra $\pi_H(\mathcal{A})$. Let K be a submodule of H, the Hilbert space orthogonal complement $K^\perp$ in H need not be a submodule of H. However, $K^\perp$

does carry an $\mathcal{A}$-module structure by the compressed action:

$$\pi_{K^\perp}(a)\xi := P\pi_H(a)\xi, a \in \mathcal{A}, \xi \in K^\perp.$$

where P is the orthogonal projection of H onto $K^\perp$. We write $K^\perp$ for a orthogonal complement of submodule K of a Hilbert module H and we always take the Hilbert space $K^\perp$ with this compressed action. We can see that $K^\perp$ is the Hilbert space realization of the quotient module H/K. That is, if H/K is given its quotient Hilbert space structure and quotient $\mathcal{A}$-module structure $(a(\xi + K) = a\xi + K, a \in \mathcal{A}$, $\xi + K \in H/K)$, then H/K is isomorphic to $K^\perp$. If $K^\perp$ is a submodule of H, we will write $H = K \oplus K^\perp$ and we say that H is the direct sum of K and $K^\perp$.

Let H, K and M are given Hilbert modules over an operator algebra $\mathcal{A}$ with K isomorphic to a submodule of H having quotient isomorphic to M, then we shall refer to this situation by saying that

$$0 \to K \to H \to M \to 0$$

is a *short exact isometric sequence*.

Now we define short exact isometric sequence in a more formal way so that it will be useful for further study.

Definition 3.1.1. *[33] A sequence of Hilbert modules over an operator algebra $\mathcal{A}$*

$$0 \to K \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$$

is said to be a short exact isometric sequence, if the map ψ has zero kernel, the range of ψ is the kernel of ϕ, the range of ϕ is all of N, ψ is isometry and ϕ is co-isometry (ϕ^* is isometry).

To say that the short exact sequence is isometric is to say that, as a Hilbert space, M is the orthogonal direct sum $K \oplus N$, and that matricially we may write π_M as

$$\begin{bmatrix} \pi_K & D \\ 0 & \pi_N \end{bmatrix}$$

where the map D carries $\mathcal{A}$ into the bounded operators mapping N into K and satisfies the equation $D(ab) = D(a)\pi_N(b) + \pi_K(a)D(b)$. That is, D is a *derivation*.

In pure algebra, a short exact sequence is said to *split* if there is a module map $\phi' : N \to M$ with the property that $\phi \circ \phi'$ is identity on N. In this event, M is isomorphic to the algebraic direct sum $K \oplus N$. In our theory, being at the Hilbert space level, we want direct sums to be orthogonal direct sums.

A short exact isometric sequence

$$0 \to K \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$$

is *orthogonally split* if there is a contractive module map $\phi' : N \to M$ such that $\phi \circ \phi'$ is identity on N.

The following proposition gives the equivalent criteria for orthogonally split condition. We can use one of the equivalent assertions to prove the orthogonally split condition.

Proposition 3.1.1. *[33] Let $0 \to K \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$ be a short exact isometric sequence of contractive Hilbert modules. Then the following are equivalent:*

i. *The sequence is split by a contraction.*

ii. *The adjoint maps, ϕ^* and ψ^* are module maps.*

iii. *The initial space of ϕ is a submodule.*

iv. *M is isomorphic to the direct sum of K and P.*

v. *If π_M is unitarily equivalent to the representation*

$$\begin{bmatrix} \pi_K & D \\ 0 & \pi_N \end{bmatrix}$$

Then the derivation D is zero.

Definition 3.1.2. *[33, 34] A Hilbert module ${}_{\mathcal{A}}P$ over an operator algebra $\mathcal{A}$ is called orthogonally projective (or orthoprojective) in case every short exact isometric sequence*

$$0 \to_{\mathcal{A}} K \to_{\mathcal{A}} M \to_{\mathcal{A}} P \to 0$$

is orthogonally split.

A Hilbert module ${}_{\mathcal{A}}I$ is called orthogonally injective (or orthoinjective) in case every short exact isometric sequence

$$0 \to_{\mathcal{A}} I \to_{\mathcal{A}} M \to_{\mathcal{A}} N \to 0$$

is orthogonally split.

Just as isometries and co-isometries are adjoints of one another, the same, essentially, is true of orthogonally projective Hilbert modules and orthogonally injective Hilbert modules. If H is a Hilbert module over operator algebra $\mathcal{A}$ with associated representation π_H, then defining ρ by the formula $\rho(a) = (\pi_H(a^*))^*, \quad a \in \mathcal{A}^*$, where the adjoint on elements of $\mathcal{A}$ is calculated in $C^*(\mathcal{A})$, yields a representation of $\mathcal{A}^*$ and a Hilbert module over $\mathcal{A}^*$. It is easy to see that, H is orthogonally projective if and only if the Hilbert space associated with ρ is orthogonally injective and vice versa.

The algebraic characterization of boundary representations (here Muhly and solel dropped the irreduciblity condition for boundary representations) by Muhly and Solel [34], characterizes boundary representations of a C^*-algebra for an operator algebra in terms of orthogonally projective and orthogonally injective modules over the operator algebra. It is as follows.

Theorem 3.1.1. *[34] Let H be a contractive Hilbert module over an operator algebra $\mathcal{A}$ and let ρ be the associated representation. Then ρ is the restriction to $\mathcal{A}$ of a boundary representation of $C_e^*(\mathcal{A})$ for $\mathcal{A}$ if and only if H is both orthogonally projective and orthogonally injective.*

3.2 Completely positive extensions

In this section, we give the theorems due to Arveson [4] concerning extensions of contractive linear maps on unital subspaces of C^*-algebras that are crucial to the proof of our main result in the next section. For the sake of completion we state the results with proof.

The following theorem shows that every unital completely contractive linear map from an unital subspace V of a C^*-algebra can be extended uniquely in a completely positive way to the operator system $V + V^*$.

Theorem 3.2.1. *[4] Let V be a linear subspace of a C^*-algebra A such that identity $e \in V$, and let S be the norm-closure of $V + V^*$. Then, every contractive linear map φ of V into $B(H)$, for which $\varphi(e) = I$ has a unique bounded self-adjoint linear extension φ_1 to S. φ_1 is positive and it is completely positive if φ is completely contractive.*

Proof. It is clear that, if a bounded self-adjoint linear extension to S exists at all, it must be unique. Using [4, lemma 1.2.7], we have $||\varphi(u) + \varphi(v)^*|| \leq 2||u + v^*||, u, v \in V$. There is a bounded linear map φ_1 of S such that $\varphi_1(u+v^*) = \varphi(u)+\varphi(v)^*, u, v \in V$. Clearly φ_1 is a self-adjoint extension of φ to S.

Now we show that φ_1 is positive. Choose a unit vector $\xi \in H$. There is a state τ of A such that $\tau(v) = \langle \varphi(v)\xi, \xi\rangle, v \in V$. Since τ and φ_1 are both self-adjoint, we have $\tau(s) = \langle \varphi_1(s)\xi, \xi\rangle, \forall\, s \in S$. Note that $\langle \varphi_1(s)\xi, \xi\rangle = \tau(s)$ is positive if $s \in S$ is positive. Hence φ_1 is positive.

Now we assume that φ is completely contractive. For each $n \geq 1$, we have $V \otimes M_n + (V \otimes M_n)^*$ is dense in $S \otimes M_n$, so that the same argument in the above paragraph shows that φ_{1n} is positive for each $n \geq 1$. Hence, φ_1 is completely positive. □

The following theorem shows that every unital completely contractive linear map from an unital subspace V of a C^*-algebra can be extended to a completely positive map on the C^*-algebra.

Theorem 3.2.2. *[4] Let V be a linear subspace of a C^*-algebra A such that identity $e \in V$, and let H be a Hilbert space. Let φ be a completely contractive linear map V into $B(H)$ such that $\varphi(e) = I$. Then φ has a completely positive extension to A.*

Proof. Using theorem 3.2.1, φ has a unique completely positive extension to the norm-closure of $V + V^*$. Now we conclude the result using [4, theorem 1.2.3]. Let S be a norm-closed operator system of A, and let H be a Hilbert space. Then for every completely positive linear map $\varphi_1 : S \to B(H)$, there is a completely positive linear map $\varphi_1' : A \to B(H)$ such that $\varphi'_{1|_S} = \varphi_1$. $\square$

Now, we will discuss the semi-invariant subspace of a Hilbert space, which is crucial to the proof of our main result in the next section.

Let K be a closed subspace of a Hilbert space H. K is said to be *semi-invariant* under a subalgebra $\mathcal{A}$ of $B(H)$, if the map $\varphi(T) = P_K T_K$ is multiplicative on $\mathcal{A}$, where P is a projection on K. This definition is due to Sarason [45].

Sarason [45] characterized the semi-invariant subspaces as follows:

Theorem 3.2.3. *[45] Let $\mathcal{A}$ be a subalgebra of $B(H)$ then subspace K of H is semi-invariant under $\mathcal{A}$ if and only if K has the form $K = M \ominus N$, where M and N are invariant subspaces of $\mathcal{A}$ such that $N \subset M$.*

Note that, if $\mathcal{A}$ is self-adjoint then semi-invariant subspaces are reducing. But in general semi-invariant subspaces need not be invariant.

3.3 Unique extension property and Hilbert modules

In this section, we establish a characterization of unique extension property for representations in the context of a C^*-algebra generated by an operator system in terms of the orthogonal projectivity and orthogonal injectivity of the Hilbert modules over the operator algebra underlying the operator system. In the proof of the theorem below we crucially make use of two extension theorems by Arveson in the context of operator systems and generated C^*-algebras given in the previous section. The theorem leads to a corollary characterizing hyperrigidity of operator systems in terms of orthogonality properties of Hilbert modules.

Theorem 3.3.1. *Let $\mathcal{A}$ be an operator algebra and consider the operator system $S = \mathcal{A} + \mathcal{A}^*$. Let $C^*(S)$ be the C^*-algebra generated by S. For any representation ρ of $C^*(S)$ on a Hilbert space H, the restriction $\rho_{|S}$ has unique extension property (UEP) if and only if H as a Hilbert module over $\mathcal{A}$ is both orthogonally projective and orthogonally injective.*

Proof. Assume that the Hilbert module H over $\mathcal{A}$ is both orthogonally projective and orthogonally injective. To show that $\rho_{|S}$ has UEP to $C^*(S)$, let σ be an unital completely positive extension of $\rho_{|S}$ to all of $C^*(S)$, and $\sigma(\cdot) = \phi^*\pi(\cdot)\phi$ be the minimal Stinespring dilation of σ. Thus, π is a representation of $C^*(S)$ on a Hilbert space K, and $\phi : H \to K$ is a Hilbert space isometry such that $\sigma(a) = \phi^*\pi(a)\phi$ for all $a \in C^*(S)$, with the minimality assumption implying that the smallest reducing subspace for $\pi(C^*(S))$ containing ϕH is all of K. In particular, for $a \in S$,

$$\rho(a) = \sigma(a) = \phi^*\pi(a)\phi.$$

We will establish the UEP of $\rho_{|S}$ by showing that σ is unitarily equivalent to the restriction of π to the range of ϕ where the equivalence implementing unitary map is ϕ. To prove that ϕ is unitary, it is enough to prove that $\phi H = K$ for which it is sufficient to show that ϕH is invariant under $\pi(S)$. For then the self-adjointness of S will imply that ϕ is reducing for $\pi(S)$ and hence for $\pi(C^*(S))$. Now, the minimality assumption above will show that $\phi H = K$.

In any case, ρ being a representation of $\mathcal{A}$, the range of ϕ is a semi-invariant subspace for $\pi(\mathcal{A})$. We will use Sarason's theorem 3.2.3 for the semi-invariant subspace ϕH to proceed.

Let P be the projection of K onto ϕH. We have $P = \phi\phi^*$. Let K_1 be the smallest invariant subspace for $\pi(\mathcal{A})$ containing ϕH. Then $K_1 = \overline{\pi(\mathcal{A})\phi H}$ as we assume π to be non-degenerate. Let P_1 be the orthogonal projection of K onto K_1. We write π_1 for the representation of $\mathcal{A}$ obtained by restricting $\pi(\mathcal{A})$ to K_1, so that $\pi_1(a) = \pi(a)_{|K_1}$ for all a in A. In other words, we may think of $\pi_1(a) = \pi(a)P_1$ for $a \in \mathcal{A}$. Also, we set $\phi_1 = P_1\phi$; that is, ϕ_1 is in fact ϕ viewed as a map from H to K_1 and gives $\phi_1^* = \phi^* P_1$. Further, let $K_2 = K_1 \ominus (\phi H)$, and let P_2 be the orthogonal projection of K onto K_2. Sarason's theorem 3.2.3 of semi-invariant subspaces gives that K_2 is invariant for $\pi_1(\mathcal{A})$ (and for $\pi(\mathcal{A})$). By construction, $P_1 = P + P_2$.

We will show that $\phi_1^* : K_1 \to H$ is a module map; that is, $\rho(a)\phi_1^* = \phi_1^*\pi_1(a)$ for all $a \in \mathcal{A}$. Indeed, for $a \in \mathcal{A}$,

$$\rho(a)\phi_1^* = \rho(a)\phi^* P_1 = \phi^*\pi(a)\phi\phi^* P_1 = \phi^*\pi(a)PP_1.$$

Now, K_2 is invariant for every $\pi(a)$, and so $\pi(a)P_2 = P_2\pi(a)P_2$ for all $a \in \mathcal{A}$. Furthermore, $\phi^* P_2 = 0$, since the initial projection of ϕ^*, namely P, is orthogonal to P_2. Thus, we find that, for $a \in \mathcal{A}$,

$$\phi^*\pi(a)PP_1 = \phi^*\pi(a)(P_1 - P_2)P_1 = \phi^*\pi(a)P_1 = \phi^*P_1\pi(a)P_1$$

as K_2 is invariant for $\pi(\mathcal{A})$ and $\phi^*P_2 = 0$. But $(\phi^*P_1)(\pi(a)P_1) = \phi_1^*\pi_1(a)$ for all $a \in \mathcal{A}$. Thus, $\rho(a)\phi_1^* = \phi_1^*\pi_1(a)$ for all $a \in \mathcal{A}$. Hence ϕ_1^* is a module map.

Since H is orthogonally projective and ϕ_1^* is co-isometric, we get that ϕ_1 is a module map too; that is, $\phi_1\rho(a) = \pi_1(a)\phi_1$ for all $a \in \mathcal{A}$, which can be rewritten as $P_1\phi\rho(a) = \pi(a)P_1\phi$. Then $\phi\rho(a) = \pi(a)\phi$ for all $a \in A$ by dropping P_1 as the range of ϕ is contained in $K_1 = \text{range}(P_1)$.

For all $a \in \mathcal{A}$,

$$\begin{aligned}
\pi(a)P &= \pi(a)PP_1 \\
&= \pi(a)\phi\phi^*P_1 \\
&= \phi\rho(a)\phi^*P_1 \\
&= \phi\rho(a)\phi_1^* \\
&= \phi\phi^*\pi(a)PP_1 \\
&= P\pi(a)P.
\end{aligned}$$

This shows that ϕH is invariant for $\pi(\mathcal{A})$.

Since ρ is a C^*-representation, using the fact that H is orthogonally injective module for $\mathcal{A}$ if and only if H is orthogonally projective for $\mathcal{A}^*$, arguing as above we can show that ϕH is invariant for $\pi(\mathcal{A}^*)$.

As ϕH is invariant for $\pi(\mathcal{A})$ and $\pi(\mathcal{A}^*)$, we have, ϕH is invariant for $\pi(S)$. Thus, $\rho_{|S}$ has UEP.

To prove the converse, suppose that $\rho_{|S}$ has UEP. We will show that H is both orthogonally projective and orthogonally injective over $\mathcal{A}$. Let

$$0 \to N \to M \to H \to 0$$

be a short exact isometric sequence determined by Hilbert modules N and M where $\phi : M \to H$ is a co-isometric module map.

Since M is a completely contractive module over $\mathcal{A}$, let ρ_M be the completely contractive representation of $\mathcal{A}$ corresponding to M. By theorem 3.2.2, ρ_M has a completely positive linear extension to $C^*(S)$. Let η be the completely positive linear extension of ρ_M of $\mathcal{A}$ to $C^*(S)$; that is $\rho_M = \eta_{|\mathcal{A}}$.

By Stinespring dilation, there is a representation π of $C^*(S)$ on a Hilbert space K and a co-isometry $\psi : K \to M$ such that $\eta(a) = \psi\pi(a)\psi^* \;\; \forall \;\; a \in C^*(S)$. In particular, $\rho_M(a) = \eta(a) = \psi\pi(a)\psi^* \;\; \forall \;\; a \in \mathcal{A}$. By theorem 3.2.1, there exists a unique completely positive extension $\widetilde{\rho}_M$ of ρ_M to S so that

$$\widetilde{\rho}_M(s) = \psi\pi(s)\psi^* \;\; \forall \;\; s \in S.$$

But then, since $\phi\rho_M(a) = \rho(a)\phi \;\; \forall \;\; a \in \mathcal{A}$, we find that $\phi\rho_M(a)\phi^* = \rho(a) \;\; \forall \;\; a \in \mathcal{A}$ and hence $\phi\widetilde{\rho}_M(s)\phi^* = \rho(s) \;\; \forall \;\; s \in S$.

Substituting for $\widetilde{\rho}_M$, we get,

$$\phi\psi\pi(s)(\phi\psi)^* = \rho(s) \;\; \forall \;\; s \in S,$$

where $\phi\psi$ is a co-isometry. On $C^*(S)$, $\phi\psi\pi(\cdot)(\phi\psi)^*$ is a completely positive map that agrees with ρ on S. Since $\rho_{|S}$ has UEP, we conclude that

$$\phi\psi\pi(a)(\phi\psi)^* = \rho(a) \;\; \forall \;\; a \in C^*(S).$$

Thus the initial space of $\phi\psi$ reduces π and $\phi\psi$ implements an equivalence between ρ and π restricted to this initial space. Let P and Q be the initial projections of ϕ and $\phi\psi$ respectively.

Then for $s \in S$, we have

$$\begin{aligned} \widetilde{\rho}_M(s)P &= (\psi\pi(s)\psi^*)(\phi^*\phi) \\ &= (\psi\pi(s)\psi^*)(\phi^*\phi)(\psi\psi^*) \end{aligned}$$

$$\begin{aligned}
&= \psi\pi(s)(\psi^*\phi^*\phi\psi)\psi^* \\
&= \psi\pi(s)Q\psi^* \\
&= \psi Q\pi(s)\psi^* \\
&= \psi(\psi^*\phi^*\phi\psi)\pi(s)\psi^* \\
&= (\psi\psi^*)(\phi^*\phi)(\psi\pi(s)\psi^*) \\
&= P\widetilde{\rho}_M(s)
\end{aligned}$$

crucially using the fact that ψ is a co-isometry with range M.

From above, in particular for $a \in \mathcal{A}$

$$\rho_M(a)P = P\rho_M(a).$$

This will imply by Propostion 3.1.1 that $\mathcal{H}$ is orthogonally projective over $\mathcal{A}$.

Also, for $a \in \mathcal{A}^*$

$$\rho_M(a)P = P\rho_M(a)$$

which shows that H is orthogonally projective over $\mathcal{A}^*$, from this we conclude that H is orthogonally injective over $\mathcal{A}$. □

The following corollary of the above theorem characterizes hyperrigidity of separable operator systems of the form $\mathcal{A} + \mathcal{A}^*$ where $\mathcal{A}$ is an operator algebra in terms of orthogonality properties of Hilbert modules over $\mathcal{A}$.

Corollary 3.3.1. *For a separable operator algebra $\mathcal{A}$, the operator system $S = \mathcal{A}+\mathcal{A}^*$ and the C^*-algebra $C^*(S)$, the following are equivalent:*

(i) *S is hyperrigid.*

(ii) *For every non-degenerate representation $\pi : C^*(S) \to B(H_\pi)$ on a separable Hilbert space, $\pi_{|S}$ has unique extension property.*

(iii) *The Hilbert module H_π over $\mathcal{A}$ is both orthogonally projective and orthogonally injective.*

Proof. The equivalence of (i) and (ii) follows from the theorem 2.6.1 and the equivalence of (ii) and (iii) follows from the above theorem. $\square$

Chapter 4

Tensor products of hyperrigid operator systems

In this chapter, we prove that tensor product of two hyperrigid operator systems is hyperrigid in the spatial tensor product of C^*-algebras. We deduce this by establishing that unique extension property for unital completely positive maps on operator systems carry over to tensor product of such maps defined on the tensor product of operator systems. Hopenwasser's result [24] about tensor product of boundary representations follows as a special case. We also provide examples to illustrate the hyperrigidity property of tensor product of operator systems.

4.1 Tensor products of C*-algebras

In this section, we define the tensor products of C^*-algebras and show the existence of a C^*-norm on the tensor products of C^*-algebras and determine to which extent it is unique.

The theory of tensor products of C^*-algebras is full of surprising number of technical problems, but the theory ends up with satisfactory form. The theory behaves very nicely for a large class of C^*-algebras called “nuclear C*-algebras”.

Let A and B are C^*-algebras, the algebraic tensor product of A and B is denoted by $A \otimes B$ over $\mathbb{C}$. $A \otimes B$ has a natural structure as a *-algebra with multiplication and involution defined as follows

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2.$$

$$(a \otimes b)^* = a^* \otimes b^*.$$

If γ is a C^*-norm on $A \otimes B$ then we write $A \otimes_\gamma B$ for the completion.

As an algebra $A \otimes B$ has the universal property for bilinear maps. The universal property is that whenever $\pi_A : A \to C$ and $\pi_B : B \to C$ are $*$-homomorphisms, where C is a complex *-algebra and $\pi_A(A)$ and $\pi_B(B)$ commute, then there is a unique $*$-homomorphism $\pi : A \otimes B \to C$ such that $\pi(a \otimes b) = \pi_A(a) \otimes \pi_B(b)$ for all $a \in A, b \in B$. If we take $C = B(H)$, then we get $*$-representations of $A \otimes B$ and hence induced C^* -seminorms.

Let H_1 and H_2 be Hilbert spaces. The tensor product of H_1 and H_2 is defined as follows. $H_1 \otimes^h H_2$ is a pre-Hilbert space with repect to the inner product

$$\left\langle \sum_i h_{1i} \otimes h_{2i}, \sum_j h_{1j} \otimes h_{2j} \right\rangle = \sum_{i,j} \langle h_{1i}, h_{1j} \rangle \langle h_{2i}, h_{2j} \rangle .$$

The Hilbert space completion will be denoted by $H_1 \otimes H_2$, which is the Hilbert tensor product of Hilbert spaces H_1 and H_2.

A standard way to define the tensor products of representations is via tensor products of Hilbert spaces. Let $\pi_A : A \to B(H_1)$ and $\pi_B : B \to B(H_2)$ be representations of C^*-algebras A and B respectively. We can form the representation $\pi = \pi_A \otimes \pi_B$ of $A \otimes B$ on $H_1 \otimes H_2$ defined by

$$\pi(a \otimes b) = (\pi_A \otimes \pi_B)(a \otimes b) = \pi_A(a) \otimes \pi_B(b).$$

If π_A and π_B are faithful, then $\pi_A \otimes \pi_B$ is faithful on $A \otimes B$, so $A \otimes B$ has at least one C^*-norm. For every π_A of A and π_B of B we have $||(\pi_A \otimes \pi_B) \sum\limits_{i=1}^{n} a_i \otimes b_i|| \leq \sum\limits_{i=1}^{n} ||a_i||||b_i||$. We can define the norm as follows

$$\left|\left|\sum_{i=1}^{n} a_i \otimes b_i\right|\right|_{\min} = \sup \left|\left|(\pi_A \otimes \pi_B)\left(\sum_{i=1}^{n} a_i \otimes b_i\right)\right|\right|$$

where the supremum is taken over all representations π_A of A and π_B of B. This norm is finite and it is a C^*-norm. This C^*-norm is called the *spatial* norm on $A \otimes B$ and it is also called the *minimal C^*-norm* because it is the smallest C^*-norm on $A \otimes B$. If π_A

and π_B are any faithful representations of A and B respectively, and for $x \in A \otimes B$, then $||(\pi_A \otimes \pi_B)(x)|| = ||x||_{\min}$. That is, the spatial norm is independent of the choice of faithful representations. This is the consequence of the minimality of this C^*-norm. The completion of $A \otimes B$ with respect to this C^*-norm is denoted by $A \otimes_s B$ and is called the minimal or spatial tensor product of C^*-algebras A and B. Spatial norm was introduced by Turumaru [51].

Theorem 4.1.1. *[13] Let A and B be C^*-algebras, and π be a non-degenerate representation of $A\otimes B$ on a Hilbert space H. Then there are unique non-degenerate representations π_A of A and π_B of B on H such that $\pi(a\otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$ for all a, b. If π is a factor representation, then π_A and π_B are also factor representations.*

For any representation π of $A \otimes B$, we have $||\pi(\sum\limits_{i=1}^{n} a_i \otimes b_i)|| \leq \sum\limits_{i=1}^{n} ||a_i||||b_i||$. We can define the norm

$$\left|\left|\sum_{i=1}^{n} a_i \otimes b_i\right|\right|_{\max} = \sup \left|\left|\pi\left(\sum_{i=1}^{n} a_i \otimes b_i\right)\right|\right|$$

where the supremum is taken over all representations. This norm is a finite C^*-norm on $A \otimes B$. It is the largest possible C^*-norm on $A \otimes B$. The completion is denoted by $A \otimes_m B$. It is called the *maximal C^*-norm* of tensor product of C^*-algebras A and B. Maximal C^*-norm was introduced by Guichardet [20].

A C^*-algebra A is called *nuclear* if, for every C^*-algebra B, there is a unique C^*-norm on $A \otimes B$. If A is nuclear, then $A \otimes_m B = A \otimes_s B$. Among the nuclear C^*-algebras are all finite-dimensional C^*-algebras, all commutative ones, all GCR-algebras, inductive limits of nuclear C^*-algebras, type I C^*-algebras, etc. But not all C^*-algebras are nuclear. The first example of a non-nuclear C^*-algebra is due to Takesaki [49]. The C^*-algebra generated by the left regular representation on $l_2(G)$ of a free group G with two generators is not nuclear.

4.2 Tensor product of boundary representations

In this section, we describe the results of Hopenwasser [23], [24] regarding the tensor product of boundary representations of the C^*-algebras for linear subspaces. Hopenwasser's results are motivation for our results about tensor product of hyperrigid operator systems in the next section.

The notoin of boundary representation of the C^*-algebra is relative to the generating linear subspace of the C^*-algebra. The boundary representation for linear subspace gives information about to which extent the subspace determines the structure of the C^*-algebra.

Let A be an unital C^*-algebra and let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices over complex numbers $\mathbb{C}$. Here we discuss the problem of finding boundary representation of C^*-algebras of the form $A \otimes M_n(\mathbb{C})$. Let B be an unital C^*-algebra, the set of elements $\{b_{ij}\}, i, j = 1, 2, ..., n$ in B are said to be *matrix units* if the following conditions are satisfied

1. $b_{ij} = b_{ij}^*$ for all i, j;

2. $b_{ij}b_{kl} = \delta_{jk}b_{il}$ for all i, j, k, l (where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise);

3. $\sum\limits_{i=1}^{n} b_{ii} = I.$

If B possesses a set of $n \times n$ matrix units $\{b_{ij}\}$ then B is $*$-isomorphic to $C \otimes M_n(\mathbb{C})$, where for C we may take the C^*-algebra $b_{11}Bb_{11}$ with unit b_{11}.

Let A be an unital C^*-algebra and let $A^{(n)} = A \otimes M_n(\mathbb{C})$. We know that if ρ is a representation of $A \otimes M_n(\mathbb{C})$ then there is a representation π of A such that ρ is unitarily equivalent to $\pi^{(n)}$. Therefore, the unitary equivalence classes of representations of $A^{(n)}$ is in one-to-one correspondence with the unitary equivalence classes of representations of A. Since our interest is in representations up to unitary equivalence, we always take representations of $A^{(n)}$ of the form $\pi^{(n)}$. Also, note that $\pi^{(n)}$ is irreducible if and only if π is irreducible.

The following theorem establish the relation between boundary representations of the C^*-algebra A and boundary representation of $A \otimes M_n(\mathbb{C})$ with respect to suitable linear subspaces.

Theorem 4.2.1. *[23] Let A be a C^*-algebra with unit e. Let $A^{(n)} = A \otimes M_n(\mathbb{C})$ and let S be a linear subspace of $A^{(n)}$ which generates $A^{(n)}$ and which contains the set of matrix units $E_{ij}(e), i, j = 1, 2, ...n$. Let J be the set operators in A which appear as a matrix entry in some elements of S. Then an irreducible representation π of A is a boundary representation for J if and only if $\pi^{(n)}$ is a boundary representation for S.*

Let S_1 be the unital generating linear subspace of a C^*-algebra A_1. Let bd(S_1) denote the set of boundary representations for S_1. Let S_2 be the unital generating linear subspace of a C^*-algebra A_2. Let $A_1 \otimes_\gamma A_2$ be a tensor product of C^*-algebras A_1 and A_2 provided with the C^*-cross norm γ. Now we discuss the relation between bd$(S_1 \otimes S_2)$ and the two sets bd(S_1) and bd(S_2).

In the commutative case, If we take $A_1 = C(X)$ and $A_2 = C(Y)$ then $A_1 \otimes_\gamma A_2 = C(X \times Y)$ and bd$(S_1 \otimes S_2)$ =bd$(S_1)\times$bd(S_2). We will see that the same result holds for any C^*-algebras A_1 and A_2 provided either A_1 or A_2 is a GCR algebra.

Theorem 4.2.2. *[24] Let S_1 and S_2 be unital generating linear subspaces of C^*-algebras A_1 and A_2 respectively. Assume that either A_1 or A_2 is a GCR algebra. Then bd$(S_1 \otimes S_2)$ =bd$(S_1)\times$bd(S_2).*

We can observe that theorem 4.2.1 will follow as a corollary of the above theorem if we take $A_2 = M_n(\mathbb{C})$.

The following lemma is crucial to the proof of the theorem in the next section.

Lemma 4.2.1. *[24] Let A be an unital C^*-algebra contained in the another C^*-algebra B. Let ϕ be an unital completely positive map on B, and let π be a representation of A such that $\phi_{|A} = \pi$. Then $\phi(ba) = \phi(b)\pi(a)$ and $\phi(ab) = \pi(a)\phi(b)$, for all $a \in A$ and $b \in B$.*

4.3 Tensor product and unique extension property

In this section, we investigate the relation between hyperrigidity of the tensor product of two operator system in the tensor product C^*-algebra and the hyperrigidity of the individual operator systems in the respective C^*-algebras. The following result shows that unique extension property of completely positive maps on operator systems carry over to tensor product of those maps defined on the tensor product of operator systems.

Theorem 4.3.1. *Let S_1 and S_2 be operator systems generating C^*-algebras A_1 and A_2 respectively. Let $\pi_i : S_i \to B(H_i), i = 1, 2$ be unital completely positive maps. Then π_1 and π_2 have unique extension property if and only if the unital completely positive map $\pi_1 \otimes \pi_2 : S_1 \otimes S_2 \to B(H_1 \otimes H_2)$ has unique extension property for $S_1 \otimes S_2 \subseteq A_1 \otimes_s A_2$.*

Proof. Assume that $\pi_1 \otimes \pi_2$ has unique extension property, that is $\pi_1 \otimes \pi_2$ has unique completely positive extension $\widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2 : A_1 \otimes_s A_2 \to B(H_1 \otimes H_2)$ which is a representation of $A_1 \otimes_s A_2$. We will show that π_1 and π_2 have unique extension property. On the contrary assume that one of the factors, say π_1 does not have unique extension property. This means that there exist atleast two extensions of π_1, a completely positive map $\phi_1 : A_1 \to B(H_1)$ and the representation $\widetilde{\pi}_1 : A_1 \to B(H_1)$ such that $\phi_1 \neq \widetilde{\pi}_1$ on A_1, but $\phi_1 = \widetilde{\pi}_1 = \pi_1$ on S_1. Using Stinespring's dilation theorem we can see that tensor product of two completely positive maps is completely positive. We have $\phi_1 \otimes_s \widetilde{\pi}_2$ is a completely positive extension of $\pi_1 \otimes \pi_2$ on $S_1 \otimes S_2$, where $\widetilde{\pi}_2$ is a unique completely positive extension of π_2 on S_2. Hence, $\phi_1 \otimes_s \widetilde{\pi}_2 \neq \widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2$ on $A_1 \otimes_s A_2$. This contradicts our assumption.

Conversely, assume that π_1 and π_2 have unique extension property, that is π_1 and π_2 have unique completely positive extensions $\widetilde{\pi}_1 : A_1 \to B(H_1)$ and $\widetilde{\pi}_2 : A_2 \to B(H_2)$ respectively where $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ are representations of A_1 and A_2 respectively. We will show that $\pi_1 \otimes \pi_2$ has the unique extension property. We have $\widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2 : A_1 \otimes_s A_2 \to B(H_1 \otimes H_2)$ is a representation and an extension of $\pi_1 \otimes \pi_2$ on $S_1 \otimes S_2$. It is enough to show that if $\phi : A_1 \otimes_s A_2 \to B(H_1 \otimes H_2)$ is a completely positive extension of $\pi_1 \otimes \pi_2$ on $S_1 \otimes S_2$ then $\phi = \widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2$ on $A_1 \otimes A_2$.

Let P be any rank one projection in $B(H_2)$. The map $a \to (1 \otimes P)\phi(a \otimes 1)(1 \otimes P)$ is completely positive on A_1, since the map is a composition of three completely positive maps. Let v be a unit vector in the range of P and let K be the range of $1 \otimes P$. Define $U : H_1 \to K$ by $U(x) = x \otimes v,\ x \in H_1$, U is a unitary map. Let $\hat{\pi} = U\widetilde{\pi}_1(a)U^*$, $a \in A_1$ and $\hat{\pi}(a)$ is the restriction to K of $\widetilde{\pi}_1(a) \otimes P = (1 \otimes P)(\widetilde{\pi}_1(a) \otimes 1)(1 \otimes P)$. Since $\hat{\pi}$ is unitarily equivalent to $\widetilde{\pi}_1$, the representation $\hat{\pi}_{|S_1}$ has unique extension property. Let $\psi(a)$ be the restriction to K of $(1 \otimes P)\phi(a \otimes 1)(1 \otimes P)$ which implies that ψ is a completely positive map that agrees with $\hat{\pi}$ on S_1, hence on all of A_1.

Let $x, y \in H_1$ and $r \in H_2$. From the above paragraph we have, for any $a \in A_1$, $\langle \phi(a \otimes 1)(x \otimes r), y \otimes r \rangle = \langle (\widetilde{\pi}_1(a) \otimes 1)(x \otimes r), y \otimes r \rangle$. (Letting P to be the rank one projection on the subspace spanned by r.) Let $D = \phi(a \otimes 1) - \widetilde{\pi}_1 \otimes 1$. Then we have $\langle D(x \otimes r), y \otimes r\rangle = 0$, for all $x, y \in H_1$, $r \in H_2$. Using polarization formula

$$\begin{aligned} 4\langle D(x\otimes r), y\otimes s\rangle &= \langle D(x\otimes(r+s)), y\otimes(r+s)\rangle \\ &\quad -\langle D(x\otimes(r-s)), y\otimes(r-s)\rangle \\ &\quad +i\langle D(x\otimes(r+is)), y\otimes(r+is)\rangle \\ &\quad -i\langle D(x\otimes(r-is)), y\otimes(r-is)\rangle . \end{aligned}$$

We have $\langle D(x \otimes r), y \otimes s\rangle = 0$, for all $x, y \in H_1$ and for all $r, s \in H_2$. Consequently, if $z_1 = \sum_{i=1}^{n} x_i \otimes r_i$ and $z_2 = \sum_{i=1}^{m} y_i \otimes s_i$, then $\langle Dz_1, z_2\rangle = 0$. Since z_1, z_2 run through a dense subset of $H_1 \otimes H_2$ and D is bounded, $D = 0$. Therefore, $\phi(a \otimes 1) = \widetilde{\pi}_1(a) \otimes 1$, for all $a \in A_1$. In the same way we can obtain $\phi(1 \otimes b) = 1 \otimes \widetilde{\pi}_2(b)$, for all $b \in A_2$. Since ϕ is a completely positive map on $A_1 \otimes A_2$ and $\phi(1 \otimes b) = 1 \otimes \widetilde{\pi}_2(b)$, for all $b \in A_2$, using a multiplicative domain argument, e.g., see lemma 4.2.1, we have

$$\phi(a \otimes b) = \phi(a \otimes 1)(1 \otimes \widetilde{\pi}_2(b)) = (1 \otimes \widetilde{\pi}_2(b))\phi(a \otimes 1)$$

for all $a \in A_1, b \in A_2$. Also, $\phi(a \otimes 1) = \widetilde{\pi}_1(a) \otimes 1$, for all $a \in A_1$. Hence, $\phi = \widetilde{\pi}_1 \otimes_s \widetilde{\pi}_2$ on $A_1 \otimes_s A_2$. □

Let A_1 and A_2 be C^*-algebras and γ is any C^*-cross norm on $A_1 \otimes A_2$. If π_1 and π_2 are irreducible representations of A_1 and A_2 respectively, then $\pi_1 \otimes_\gamma \pi_2$ is an irreducible representation of $A_1 \otimes_\gamma A_2$. Conversely, every irreducible representation

π on $A_1 \otimes_\gamma A_2$ need not factor as a product $\pi_1 \otimes_\gamma \pi_2$ of irreducible representations.

Theorem 4.3.2. *[20] If π_1 or π_2 is a type I factor representation then π is equivalent to the tensor product of π_1 and π_2.*

If we assume that one of the C^*-algebra is a GCR algebra, then by above theorem, every irreducible representation does factor. Since GCR algebras are nuclear, there is a unique C^*-cross norm on $A_1 \otimes A_2$, which we denote by $A_1 \otimes_\alpha A_2$.

Using the above facts, the theorem 4.2.2 by Hopenwasser relating boundary representations of tensor product of C^*-algebras will become a corollary to our theorem 4.3.1.

Corollary 4.3.1. *Let S_1 and S_2 be unital operator subspaces of generating C^*-algebras A_1 and A_2 respectively. Assume that either A_1 or A_2 is a GCR algebra. Then the representation $\pi_1 \otimes_\alpha \pi_2$ of $A_1 \otimes_\alpha A_2$ is a boundary representation for $S_1 \otimes S_2$ if and only if the representations π_1 of A_1 and π_2 of A_2 are boundary representations for S_1 and S_2 respectively.*

The following corollary investigates the relation between the hyperrigidity of the tensor product of two operator systems in the tensor product C^*-algebras and the hyperrigidity of the individual operator system in the respective C^*-algebras.

Corollary 4.3.2. *Let S_1 and S_2 be separable operator systems generating C^*-algebras A_1 and A_2 respectively. Assume that either A_1 or A_2 is a GCR algebra. Then S_1 and S_2 are hyperrigid in A_1 and A_2 respectively if and only if $S_1 \otimes S_2$ is hyperrigid in*

$A_1 \otimes_s A_2$.

Proof. Assume that $S_1 \otimes S_2$ is hyperrigid in the C^*-algebra $A_1 \otimes_s A_2$. By theorem 2.6.1, every unital representation $\pi : A_1 \otimes_s A_2 \to B(H_1 \otimes H_2)$, $\pi_{|_{S_1 \otimes S_2}}$ has unique extension property. We have if π is an unital representation of $A_1 \otimes_s A_2$ and since one of the C^*-algebra is GCR, then by theorem 4.3.2 every unital representation of a GCR algebra is type I. There are unique unital representations π_1 of A_1 and π_2 of A_2 such that $\pi = \pi_1 \otimes_s \pi_2$. Using theorem 4.3.1, we can see that $\pi_{1|_{S_1}}$ and $\pi_{2|_{S_2}}$ have unique extension property. This implies that S_1 and S_2 are hyperrigid in A_1 and A_2 respectively again by theorem 2.6.1.

Conversely, assume that S_1 is hyperrigid in A_1 and S_2 is hyperrigid in A_2. By theorem 2.6.1, for every unital representations $\pi_1 : A_1 \to B(H_1)$ and $\pi_2 : A_2 \to B(H_2)$, $\pi_{1|_{S_1}}$ and $\pi_{2|_{S_2}}$ have unique extension property. We have, if π_1 and π_2 are unital representations of A_1 and A_2 respectively, then $\pi_1 \otimes_s \pi_2$ is an unital representation of $A_1 \otimes_s A_2$. Using theorem 4.3.1, we can see that $\pi_1 \otimes_s \pi_{2|_{S_1 \otimes S_2}}$ has unique extension property. Now, by theorem 2.6.1 $S_1 \otimes S_2$ is hyperrigid in $A_1 \otimes_s A_2$. $\square$

Clearly, the spatial norm assumption in the above results is redundant if the C^*-algebras are nuclear. But general C^*-algebras with the lack of injectivity associated with other C^*-norms including the maximal one will require additional assumptions.

Now, we will provide some examples which illustrate the results above.

Example 4.3.1. *Let $G =$ linear span(I, S, S^*), where S is the unilateral right shift in $B(H)$ and I is the identity operator. Let $A = C^*(G)$ be the C^*-algebra generated by G. We have $K(H) \subseteq A$, $A/K(H) \cong C(\mathbb{T})$ is commutative, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. Let Id denotes the identity representation of the C^*-algebra A. Let $S^*Id(\cdot)S$ be a completely positive map on the C^*-algebra A such that $S^*IdS_{|_G} = Id_{|_G}$, it is easy to see that $S^*IdS_{|_A} \neq Id_{|_A}$. Therefore, the unital representation $Id_{|_G}$ does not have unique extension property. Using theorem 2.6.1, we conclude that G is not a hyperrigid operator system in a C^*-algebra A.*

Let $G_1 = G$, $A_1 = A$ and Id_1 denotes the identity representation of A_1. Let $G_2 = A_2 = M_n(\mathbb{C})$ and Id_2 denotes the identity representation of the C^*-algebra A_2. The completely positive map $S^*Id_1S \otimes Id_2$ on the C^*-algebra $A_1 \otimes A_2$ is such that $S^*Id_1S \otimes Id_2 = Id_1 \otimes Id_2$ on operator system $G_1 \otimes G_2$. By the above conclusion we see that $S^*Id_1S \otimes Id_2 \neq Id_1 \otimes Id_2$ on the C^*-algebra $A_1 \otimes A_2$. Therefore, the unital representation $Id_1 \otimes Id_2$ does not have unique extension property for $G_1 \otimes G_2$. Hence, by theorem 2.6.1, $G_1 \otimes G_2$ is not a hyperrigid operator system in a C^*-algebra $A_1 \otimes A_2$.

Example 4.3.2. *Let the Volterra integration operator V acting on the Hilbert space $H = L^2[0,1]$ be given by*

$$Vf(x) = \int_0^x f(t)dt, \quad f \in L^2[0,1].$$

V generates the C^*-algebra $K = K(H)$ of all compact operators. Let S = linear span(V, V^*, V^2, V^{2*}) and S is hyperrigid [Theorem 2.6.2]. Then $\tilde{S} = S + \mathbb{C} \cdot \mathbf{1}$ is a hyperrigid operator system generating the C^*-algebra $\tilde{A} = K + \mathbb{C} \cdot \mathbf{1}$. $\tilde{A}$ is a GCR algebra. Let $S_1 = S_2 = \tilde{S}$ and $A_1 = A_2 = \tilde{A}$. We know that S_1 and S_2 are hyperrigid operator systems in the C^*-algebra A_1 and A_2 respectively. By corollary 4.3.2 we conclude that $S_1 \otimes S_2$ is hyperrigid operator system in the C^*-algebra $A_1 \otimes A_2$.

Example 4.3.3. *Let G = linear span(I, S, S^*, SS^*), where S is the unilateral right shift in $B(H)$ and I is the identity operator. Let $A = C^*(G)$ be the C^*-algebra generated by the operator system G. We have, $K(H) \subseteq A$. $A/K(H) \cong C(\mathbb{T})$ is commutative, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. Since S is an isometry, G is a hyperrigid operator system [Theorem 2.6.3] in the C^*-algebra A. Let $G_1 = G$, $A_1 = A$ and $G_2 = A_2 = M_n(\mathbb{C})$. It is clear that G_2 is a hyperrigid operator system in the C^*-algebra $A_2 = C^*(G_2)$. By corollary 4.3.2 $G \otimes M_n(\mathbb{C})$ is a hyperrigid operator system in $A \otimes M_n(\mathbb{C})$.*

Chapter 5

Quasi hyperrigidity and weak peak points

In this chapter, we introduce the notions of weak boundary representation, quasi hyperrigidity and weak peak points in the non-commutative setting for operator systems in C^*-algebras. An analogue of Saskin's theorem relating quasi hyperrigidity and weak Choquet boundary for particular classes of C^*-algebras is proved. We also show that, if an irreducible representation is a weak boundary representation and a weak peak point then it is a boundary representation. Several examples are provided to illustrate these notions.

5.1 Weak Choquet boundary and quasi hyperrigidity

In this section, we introduce the notion of a weak boundary representation and discuss the nature and properties of it. We introduce the notion of quasi hyperrigid sets and discuss the relation between quasi hyperrigidity and hyperrigidity. We explore the relation between quasi hyperrigidity and weak Choquet boundary.

Definition 5.1.1. *Let A be an unital C^*-algebra and S be an operator system of A such that $A = C^*(S)$-the C^*-algebra generated by S. An irreducible representation $\pi : A \to B(H_\pi)$ is called weak boundary representation for S of A if $\pi_{|S}$ has a unique UCP map extension of the form $V^*\pi V$, namely π itself, where $V : H_\pi \to H_\pi$ is an isometry.*

The set of all weak boundary representations for S of A is called *weak Choquet boundary* of S and denoted by $\partial_W S$. We can observe that all the boundary representations are weak boundary representations for S. Thus, $\partial S \subseteq \partial_W S$.

Example 5.1.1. *Consider the classical case $A = C(X)$, where X is a compact Hausdorff space. The irreducible representations up to unitary equivalence are one dimensional representations of $C(X)$ which correspond to point evaluation functionals and thereby precisely to the points of X. Let S be a subspace of $C(X)$ containing identity such that $C^*(S) = C(X)$. Let $x \in X$, $\varepsilon_x : C(X) \to \mathbb{C}$ be the one dimensional irreducible representation given by $\varepsilon_x(f) = f(x),$ for all $f \in C(X)$. Let $V : \mathbb{C} \to \mathbb{C}$ be an isometry such that $V^*\varepsilon_x(f)V = \varepsilon_x(f)$ for all $f \in S$. Since $\mathbb{C}$ is one dimensional,*

V is unitary and hence $V^*\varepsilon_x(f)V = \varepsilon_x(f)$ for all $f \in C(X)$. Therefore, ε_x is a weak boundary representation for all $x \in X$. In the classical case, spectrum of a C^*-algebra and weak Choquet boundary are the same irrespective of the choice of the subspace S of $C(X)$. Hence, we conclude that $\partial S \subseteq \partial_W S = X$. By Saskin's theorem 2.3.2, we conclude that a subspace S is Korovkin in $C(X)$ if and only if $\partial S = \partial_W S = X$. Thus, weak Choquet boundary fails to recognise hyperrigidity even in the commutative case since $\partial_W S = X$ for all $S \subseteq C(X)$.

Example 5.1.2. *Let A be a C^*-algebra and S be an operator system in A such that $A = C^*(S)$. When A is finite dimensional it is easy to see that $\partial_W S = \hat{A}$. The same can be deduced for infinite dimensional C^*-algebras for which all the irreducible representations are finite dimensional as in the cases of infinite direct sums of matrix algebras and infinite direct sums of the form $\oplus(C(X_i) \otimes M_n(\mathbb{C}))$, where X_i is a compact Hausdorff space for each i.*

The notion of weak boundary representation is interesting in the infinite dimensional C^*-algebras. The following example shows that spectrum of a C^*-algebra is not always equal to weak Choquet boundary in infinite dimensional cases.

Example 5.1.3. *Let G = linear span(I, S, S^*), where S is the unilateral right shift in $B(H)$ and I is the identity operator. Let $A = C^*(G)$ be the C^*-algebra generated by G. We have $K(H) \subseteq A$, $A/K(H) \cong C(\mathbb{T})$ is commutative, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ and the spectrum $\hat{A}$ of A can be identified with $\{Id\} \cup \mathbb{T}$. We know that ε_t is a one dimensional irreducible representation of A for all $t \in \mathbb{T}$, therefore ε_t is a* *weak boundary representation for G of A for all $t \in \mathbb{T}$. Note that $Id_{|_G}$ has more than one UCP extension from the class $CP(A, Id, H_{Id})$. Observe that $S^*Id(\cdot)S$ is also an extension of $Id_{|_G}$. Therefore, Id is not a weak boundary representation.*

Arveson [4] introduced the notion of finite representation in the setting of subalgebras of C^*-algebras. He [4, Proposition 2.3.2] further proved that any representation π of a subalgebra $\mathcal{A}$ of a C^*-algebra A (where $\mathcal{A}$ contains the identity of the C^*-algebra A) on a Hilbert space H is *finite representation* if and only if for every isometry V in $B(H)$, the condition $V^*\pi(a)V = \pi(a)$ for all a in $\mathcal{A}$ implies that V is unitary. Arveson has remarked that using this result one can define finite representation of linear subspace (or operator system) of a C^*-algebra. We have the following result which proves that Arveson's finite representations of operator systems with the additional assumption of irreducibility will coincide with our notion of weak boundary representations for operator systems.

Proposition 5.1.1. *Let A be a C^*-algebra and S be an operator system in A such that $A = C^*(S)$. Let π be an irreducible representation of A on a Hilbert space H. Then π is a finite representation of S if and only if π is a weak boundary representation for S of A.*

Proof. Assume that π is finite. If $V^*\pi(s)V = \pi(s)$ for every $s \in S$, then V is unitary. Therefore, $\pi(S)' = \pi(A)'$. Then $V \in \pi(A)'$. Since π is irreducible, we have $\pi(A)' = \mathbb{C}1$. Therefore, V is the identity and this in turn will imply that π is a weak boundary representation.

Conversely, assume that π is a weak boundary representation. Then $\pi : A \to B(H_\pi)$, $\pi_{|S}$ has a unique UCP map extension of the form $V^*\pi V$, namely π itself, where $V : H_\pi \to H_\pi$. If $V^*\pi(s)V = \pi(s)$ for all $s \in S$ then $V^*\pi(a)V = \pi(a)$ for all $a \in A$. This $V^*\pi V$ is a representation of the C^*-algebra A. Hence it follows that VH_π is an invariant subspace of H_π and thus reducing subspace of H_π for $\pi(A)$. Since $\pi(A)$ is irreducible representation we must have $VH_\pi = H_\pi$ that is V is unitary. Therefore, π is a finite representation. $\square$

Definition 5.1.2. *A set S of generators of a C^*-algebra A is said to be quasi hyperrigid, if for every non-degenerate representation π of A on a Hilbert space H_π and for every isometry $V : H_\pi \to H_\pi$ the condition $V^*\pi(s)V = \pi(s)$ for all s in S implies that $V^*\pi(a)V = \pi(a)$ for all a in A.*

Note that a set S is quasi hyperrigid if and only if the linear span of $S \cup S^*$ is quasi hyperrigid and hence the notion extends naturally to operator systems.

Here we explore the relation between hyperrigidity and quasi hyperrigidity. It is trivial to see that hyperrigid sets are quasi hyperrigid. However, the converse is not true and hence the notion is strictly weaker. We illustrate this using several examples. The following one is a modified version from [31].

Example 5.1.4. *Let $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over $\mathbb{C}$, where $n \geq 3$. Define a unital completely positive map Φ on $M_n(\mathbb{C})$ as given below. Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & a_{nn} \end{bmatrix}$$

be arbitrary. Now define Φ on $M_n(\mathbb{C})$

$$\Phi(M) = \begin{bmatrix} a_{11} & a_{12} & 0 & . & . & . & 0 \\ a_{21} & a_{22} & 0 & . & . & . & 0 \\ 0 & 0 & a_{22} & . & . & . & 0 \\ . & . & . & a_{22} & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & a_{22} \end{bmatrix}$$

Now let $M = T$, where $a_{21} = 1$ and all other entries equal to 0. Let $S = span\{I, T, T^*\}$ and $A = C^*(S)$. Consider the sequence of unital completely positive maps $\{\Phi_n\}$ on $C^*(S)$ where $\Phi_n = \Phi$ for all n. Note that for all n, $\Phi_n(s) = s \; \forall \; s \in S$, but $\Phi_n(TT^*) \neq TT^*$. This implies that S is not a hyperrigid set. However, if V is any *isometry such that $V^*V = I$, then $VV^* = I$, since A is finite dimensional. Thus, S is quasi hyperrigid, but fails to be a hyperrigid set.*

Now we give an infinite dimensional example of a quasi hyperrigid operator system which is not hyperrigid. This example is inspired by Robertson [43]. In fact a slight modification of Robertson's construction of the CP map is made so as to make it unital. We choose an operator system in such a way that the example fit into our settings.

Example 5.1.5. *Let A be a non commutative infinite dimensional C^*-algebra with only finite dimensional irreducible representations, there exists an element $x \in A$ of norm one satisfying $x^2 = 0$ [18, 2.12.21]. Let $sp(x)$ denote the spectrum of x, we have* $\{0, 1\} \subseteq sp(x^*x) = sp(xx^*) \subseteq [0, 1]$. *There are two cases to consider.*

Case 1. Let $sp(x^*x)$ has at least three points. Choose $\lambda \in sp(x^*x)$ with $0 < \lambda < 1$. *Define continuous functions f, g, h on $[0, 1]$ which vanish at 0 and satisfy* $0 \leq f \leq g \leq 1$, $0 \leq h \leq 1$, $fg = f$, $hg = 0$ *and* $f(\lambda) = g(\lambda) = h(1) = 1$. *Thus,* $g = 1$ *on the support of f and $g = 0$ on the support of h.*

Note that $(x^*x)(xx^*) = 0$, and since f and g are uniform limits of polynomials on $[0, 1]$ without constant terms we have that $f(x^*x)$, $g(x^*x)$ and $h(x^*x)$ are each orthogonal to all the elements $f(xx^*)$, $g(xx^*)$ and $h(xx^*)$. Define $y = f(xx^*)xf(x^*x)$, note that $y^2 = 0$ and $y \neq 0$. For considering polynomials approximating f, we see that $y = xf(x^*x)^2$ and by definition of f, $x^*xf(x^*x)^2 \neq 0$.

Consider $p = g(x^*x) + g(xx^*)$, now $py = y = yp$. Since $(y^*y)(yy^*) = 0$, there *is a state σ of A such that $\sigma(y^*y) = 0$ but $\sigma(yy^*) > 0$. Further by Cauchy-Schwarz inequality, we have $\sigma(y) = 0$ and $\sigma(y^*) = 0$.*

Define the unital completely positive map ϕ on A by

$$\phi(a) = pap + \sigma(a)(I - p^2). \tag{5.1}$$

Clearly ϕ satisfies $\phi(I) = I$, $\phi(y) = y$, $\phi(y^*) = y^*$, $\phi(y^*y) = y^*y$, but $\phi(yy^*) \neq yy^*$.

Let S be the operator system generated by I, y and y^*y in A and let $C^*(S)$ be the C^*-algebra generated by S. We will show that S is not hyperrigid in $C^*(S)$. Consider the sequence of unital completely positive maps $\{\phi_n\}$ on $C^*(S)$ where $\phi_n = \phi$ for all n. Note that for all n, $\phi_n(s) = s \ \forall \ s \in S$, but $\phi_n(a) \neq a$ for $a = yy^* \in C^*(S)$, implying that S is not hyperrigid in $C^*(S)$. Therefore, S will be quasi hyperrigid but not hyperrigid.

Case 2: $sp(x^*x) = \{0, 1\}$. In this case x^*x and xx^* are orthogonal projections. If $x^*x + xx^* \neq 1$, define ϕ as in (5.1) with $p = x^*x + xx^*$, and let σ be a state of A satisfying $\sigma(x^*x) = 0,\ \sigma(xx^*) > 0$ then $\phi(I) = I,\ \phi(x) = x,\ \phi(x^*) = x^*$, $\phi(x^*x) = x^*x$ but $\phi(xx^*) \neq xx^*$.

Here too, let S_1 be the operator system generated by $I, x,\ x^*x$ in A and let $C^*(S_1)$ be the C^*-algebra by S_1, as the same argument above in case 1, S_1 is not hyperrigid in $C^*(S_1)$. Therefore, S_1 will be a quasi hyperrigid operator system but S_1 is not *hyperrigid.*

Suppose that $x^*x + xx^* = 1$, xx^* and x^*x being orthogonal equivalent projections in A. Then A can be expressed as a matrix algebra $M_2(B)$, where the C^*-algebra B is $*$-isomorphic to the relative commutant of $\{x, x^*\}$ in A. Since $A \neq M_2(\mathbf{C})$, we can find an element $b \in B_+$ of norm one which contains at least two non zero points in its spectrum then $a = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ satisfies $||a|| = 1$, $a^2 = 0$ and the spectrum of a^*a strictly contains $\{0, 1\}$. This returns us to the situation considered in case 1.

Remark 5.1.1. *In the infinite dimensional C^*-algebras considered in example 5.1.2 we can construct quasi hyperrigid operator systems which are not hyperrigid.*

Now we explore the notions of quasi hyperrigidity and weak Choquet boundary in the following results.

Proposition 5.1.2. *Let S be a separable operator system and $A = C^*(S)$. Then S is quasi hyperrigid if and only if for every non-degenerate representation $\pi : A \to B(H_\pi)$ on a separable Hilbert space, $\pi_{|S}$ has a unique UCP map extension of the form $V^*\pi V$, where $V : H_\pi \to H_\pi$ is an isometry.*

Proof. Assume that S is a quasi hyperrigid operator system in a C^*-algebra A. This means that for every non-degenerate representation π of A on a Hilbert space H_π and for every isometry $V : H_\pi \to H_\pi$ the condition $V^*\pi(s)V = \pi(s)$ for all s in S implies that $V^*\pi(a)V = \pi(a)$ for all a in A. So fix a non-degenerate representation

$\pi : A \to B(H_\pi)$. Every UCP map of the form $V^*\pi V$ agreeing with π on S will agree with π on A. Hence, $\pi_{|S}$ has a unique UCP map extension of the form $V^*\pi V$, where $V : H_\pi \to H_\pi$ is an isometry. Reversing the arguments we get the proof of the converse. $\square$

Proposition 5.1.3. *Let S be a separable operator system generating a C^*-algebra A. If S is quasi hyperrigid, then every irreducible representation of A is a weak boundary representation for S.*

Proof. We know that every irreducible representation is non-degenerate. Using above proposition the result is immediate. $\square$

Problem 5.1.1. *If every irreducible representation of A is a weak boundary representation for a separable operator system $S \subseteq A$, then is S quasi hyperrigid?*

We will settle the above problem for certain classes of C^*-algebras.

Proposition 5.1.4. *Let S be an operator system generating a C^*-algebra $A = C^*(S)$ and for each i in an index set I, let $\pi_i : A \to B(H_{\pi_i})$ be a representation such that ${\pi_i}_{|S}$ has unique UCP map extension of the form $V_{\pi_i}^* \pi_i V_{\pi_i}$, where $V_{\pi_i} : H_{\pi_i} \to H_{\pi_i}$ is an isometry. Then for the direct sum of representations $\pi = \oplus_{i\in I}\pi_i : A \to B(\oplus_{i\in I} H_{\pi_i})$, $\pi_{|S}$ has unique UCP map extension of the form $V_\pi^* \pi V_\pi$, where $V_\pi : \oplus_{i\in I} H_{\pi_i} \to \oplus_{i\in I} H_{\pi_i}$ is an isometry.*

Proof. Let $\Phi = V_\pi^* \pi V_\pi = V_\pi^* \oplus_{i\in I} \pi_i V_\pi : A \to B(\oplus_{i\in I} H_{\pi_i})$ be an extension of $\pi_{|S}$

for an isometry $V_\pi : \oplus_{i\in I} H_{\pi_i} \to \oplus_{i\in I} H_{\pi_i}$. For each $i \in I$, let $\Phi_i : A \to B(H_{\pi_i})$ be the UCP map

$$\Phi_i(a) = P_i \Phi(a)|_{H_{\pi_i}}, \qquad a \in A$$

where P_i is the projection onto H_{π_i}. Since Φ_i restricted to π_i on S has unique extension we have $\Phi_i(a) = \pi_i(a)$ for all $a \in A$. Equivalently $P_i\Phi(a)P_i = \pi(a)P_i$. Using Schwarz inequality,

$$\begin{aligned}
P_i\Phi(a)^*(1-P_i)\Phi(a)P_i &= P_i\Phi(a)^*\Phi(a)P_i - P_i\Phi(a)^*P_i\Phi(a)P_i \\
&\leq P_i\Phi(a^*a)P_i - \pi(a)^*P_i\Phi(a)P_i \\
&= \pi(a^*a)P_i - \pi(a)^*\pi(a)P_i \\
&= 0
\end{aligned}$$

Hence, $|(1-P_i)\Phi(a)P_i|^2 = 0$, and it follows that P_i commutes with the self adjoint family of operators $\Phi(A)$. Hence for every $a \in A$ we have

$$\Phi(a) = \sum_{i\in I} \Phi(a)P_i = \sum_{i\in I} P_i\Phi(a)P_i = \sum_{i\in I} \pi(a)P_i = \pi(a)$$

Hence, $\Phi(a) = V_\pi^*\pi(a)V_\pi = V_\pi^* \oplus_{i\in I} \pi_i(a)V_\pi = \pi(a)$ for all $a \in A$ and for an isometry $V_\pi : \oplus_{i\in I} H_{\pi_i} \to \oplus_{i\in I} H_{\pi_i}$.

□

Now we settle the problem 5.1.1 for C^*-algebras with countable spectrum.

Theorem 5.1.1. *Let $A = C^*(S)$ be the C^*-algebra generated by a separable operator system S such that A has countable spectrum. If every irreducible representation of A is a weak boundary representation for S then S is quasi hyperrigid.*

Proof. To prove S is quasi hyperrigid using proposition 5.1.2, it is enough to prove that for every representation $\pi : A \to B(H_\pi)$ of A on a separable Hilbert space, $\pi_{|_S}$ has the unique UCP map extension of the form $V_\pi^* \pi V_\pi$, where $V_\pi : H_\pi \to H_\pi$ is an isometry. Our assumption that spectrum of A is countable implies that A is a type I C^*-algebra, hence π decomposes uniquely into a direct integral of mutually disjoint type I factor representations. Using the fact that spectrum of A is countable again, the direct integral must be a countable direct sum. Therefore, π can be decomposed into a direct sum of subrepresentations $\pi_n : A \to B(H_{\pi_n})$

$$H_\pi = H_{\pi_1} \oplus H_{\pi_2} \oplus ..., \qquad \pi = \pi_1 \oplus \pi_2 \oplus ...$$

with the property that each π_n is unitarily equivalent to a finite or countable direct sum of copies of a single irreducible representation $\sigma_n : A \to B(H_{\sigma_n})$.

By our assumption, each map $\sigma_{n|_S}$ has the unique UCP map extension of the form $V_{\sigma_n}^* \sigma_n V_{\sigma_n}$, where $V_{\sigma_n} : H_{\sigma_n} \to H_{\sigma_n}$ is an isometry. Hence, the above decomposition expresses $\pi_{|_S}$ as a (double) direct sum. By proposition 5.1.4 it follows that $\pi_{|_S}$ has the unique UCP map extension of the form $V_\pi^* \pi V_\pi$, where $V : H_\pi \to H_\pi$ is an isometry. $\square$

5.2 Weak unique extension property

In this section, we introduce a weaker notion of unique extension property of representations of C^*-algebras by considering particular class of UCP maps. We solve the problem 5.1.1 for a Type I C^*-algebra.

Definition 5.2.1. *Let S be an operator system generating a C^*-algebra A. Let $\pi : A \to B(H_\pi)$ be a representation, then π is said to have weak unique extension property (WUEP) for S if π is the only UCP map extension of $\pi_{|S}$ of the form $V^*\pi(\cdot)V$, where V is an isometry on H_π.*

Kleski [27] proved the hyperrigidity conjecture of Arveson for a Type I C^*-algebra with an additional assumption on the co-domain. Since our problem 5.1.1 is similar to Arveson's hyperrigidity conjecture with weaker notions, Kleski's [27] results can be modified to our settings. The following results give partial answer to the problem 5.1.1.

Let A be a C^*-algebra and B be a C^*-subalgebra of A. A *conditional expectation* from a C^*-algebra A to a C^*-subalgebra B is a completely positive projection of norm 1. A C^*-algebra A is said to be *injective* if for every faithful representation $\pi : A \to B(K)$, there exists a conditional expectation $E : B(K) \to \pi(A)$. For example, if A is a nuclear C^*-subalgebra of $B(H)$, then A'' is injective. Consequence of the above fact is that if $\psi : S \to A''$ is UCP then there is a UCP map $\tilde{\psi} : A \to A''$ such that $\tilde{\psi}_{|S} = \psi$.

Let (X, μ) be a standard Borel measure space and let H_x be a separable Hilbert space for each $x \in X$. A *measurable field of Hilbert spaces* is a vector subspace $\mathcal{V}$ of $\prod_{x\in X} H_x$ closed under multiplication by $L^\infty(X,\mu)$ such that $x \mapsto \langle \xi(x), \eta(x)\rangle_x$ is measurable for all $\xi, \eta \in \mathcal{V}$ and $\int_X \langle \xi(x), \xi(x)\rangle \, d\mu(x) < \infty$ for all $\xi \in \mathcal{V}$. Define $\langle \xi, \eta\rangle = \int_X \langle \xi(x), \eta(x)\rangle_x \, d\mu(x)$ is a pre-inner product on $\mathcal{V}$. The completion of $\mathcal{V}$ is a separable Hilbert space H. We can identify H with a space of equivalence classes of measurable sections of the field H_x. We write $H = \int_X^{\oplus} H_x d\mu(x)$ and H is called as *direct integral* of Hilbert spaces H_x.

Let $T_x \in B(H_x)$, (T_x) is said to be *measurable field of bounded operators* if $(T_x\xi(x))$ is a measurable section for each measurable section ξ. If $||T_x||$ is uniformly bounded then (T_x) defines an operator $T \in B(H)$ such that $||T||$ is essential supremum of $||T_x||$. This operator T is said to be *decomposable* and is written as $T = \int_X^{\oplus} T_x d\mu(x)$.

Theorem 5.2.1. *Let S be a separable operator system in $B(H)$ generating a C^*-algebra A, and suppose A'' is injective. Suppose every factor representation $\pi : A \to B(H_\pi)$ has WUEP for S of A. Let ρ be a faithful representation of A on $B(K_\rho)$ and let $\gamma : \rho(A) \to B(K_\rho)$, $\gamma = V_1^* IdV_1$, where $V_1 : K_\rho \to K_\rho$ is an isometry such that $\gamma(\rho(s)) = \rho(s)$ for all $s \in S$. Then for every conditional expectation $E : B(K_\rho) \to \rho(A)''$, we have $E\gamma\rho(a) = \rho(a)$ for all $a \in A$.*

Proof. We first prove that the result for the case the Hilbert space K_ρ is separable. Let $E : B(K_\rho) \to \rho(A)''$ be a conditional expectation. Let $\gamma : B(K_\rho) \to B(K_\rho)$,

$\gamma = V_1^* I d V_1$, where $V_1 : K_\rho \to K_\rho$ is isometry such that $\gamma(\rho(s)) = \rho(s)$ for all $s \in S$. We will show that $E\gamma\rho = \rho$ for any conditional expectation E, under the assumption that every factor representation $\varphi : \rho(A) \to B(H_\varphi)$, $\varphi_{|_{\rho(S)}}$ has unique UCP map extension $V_2^*\varphi V_2$ for every isometry $V_2 : H_\varphi \to H_\varphi$.

Let $M := Z(\rho(A)'')$ be a commutative von Neumann algebra and acts on a separable Hilbert space, there is a weak* dense unital commutative C^*-subalgebra M_0 of M. Let X be the spectrum of M_0, we have $M_0 \cong C(X)$. There is a probability measure μ on X such that $M \cong L^\infty(X, \mu)$. The probability measure μ gives us a disintegration $K_\rho = \int_X^\oplus K_{\rho,x} d\mu$, and the identity representation of $\rho(A)''$ may be decomposed as

$$b = \int_X^\oplus \pi_x(b) d\mu(x),$$

for all $b \in \rho(A)''$ [39, 4.12]. After removing a set of measure zero from X, the resulting set (which we still cal X) has the property that each $\pi_{x|_{\rho(A)}}$ is a factor representation of $\rho(A)$. Since $E\gamma\rho(A)$ is contained in $\rho(A)''$, we write

$$E\gamma\rho(a) = \int_X^\oplus \pi_x(E\gamma\rho(a)) d\mu(x),$$

for all $a \in A$. Note that $\pi_x\rho$ is a factor representation of A for all $x \in X$. Now $\gamma\rho_{|S} = \rho_{|S}$ means that $\pi_x E\gamma\rho_{|S} = \pi_x\rho_{|S}$ for a.e $x \in X$. By our assumption, we conclude that $\pi_x E\gamma\rho = \pi_x\rho$ for a.e. $x \in X$; from this it follows that $E\gamma\rho = \rho$.

Now assume that K_ρ is not necessarily separable. Because A is separable, the representation ρ is unitarily equivalent to $\oplus\rho_i$, where each ρ_i is a representation acting on a separable Hilbert space K_{ρ_i}. So it is enough to show the claim for $\rho := \oplus\rho_i$. Fix a faithful separable representation $\sigma : A \to B(L_\sigma)$ with conditional expectation $F : B(L_\sigma) \to \sigma(A)''$. Then $\rho_i \oplus \sigma$ is a faithful separable representation of A for each i. Let P_i be the projection of K_ρ onto K_{ρ_i} and $P_i \in \rho(A)'$ for each i. Let Ad $P_i \circ E \oplus F$: $B(K_\rho \oplus L_\sigma) \to (\rho_i \oplus \sigma)(A)''$ be the conditional expectation. Using the results for separable representations above, we have $(\mathrm{Ad}P_i \circ E \oplus F)(\gamma\rho \oplus \sigma)(a) = (\rho_i \oplus \sigma)(a)$ for all $a \in A$. Thus, $(E \oplus F)(\gamma\rho \oplus \sigma) = \rho \oplus \sigma$, and hence $E\gamma\rho = \rho$. $\square$

Corollary 5.2.1. *Let S be an operator system generating a Type I C^*-algebra A. If every irreducible representation of A is a weak boundary representation for S, then for any representation $\pi : A \to B(K_\pi)$ and any UCP map $V^*IdV : \pi(A) \to B(K_\pi)$ for $V : K_\pi \to K_\pi$ is isometry such that $V^*Id(\pi(s))V = \pi(s)$ for all $s \in S$ and any conditional expectation $E : B(K_\pi) \to \pi(A)''$, $E(V^*IdV)\pi = \pi$.*

Proof. Fix a faithful representation ρ of A and a conditional expectation $F : B(K_\rho) \to \rho(A)''$. Applying above theorem to the faithful representation $\rho \oplus \pi$ using the conditional expectation $F \oplus E$; hence $(F \oplus E)(\rho \oplus (V^*IdV)\pi)(a) = (\rho \oplus \pi)(a)$ for all $a \in A$ and so $E(V^*IdV)\pi = \pi$. $\square$

Corollary 5.2.2. *Let S be a separable operator system generating a Type I C^*-algebra A. If every irreducible representation of A is a weak boundary representation for S, then for any UCP map $V^*\pi V : A \to A''$, where $\pi : A \to A''$ is a representation and $V \in A''$ is an isometry such that $V^*\pi(s)V = \pi(s)$ for all $s \in S$ implies that $V^*\pi(a)V = \pi(a)$ for all $a \in A$.*

Proof. When A is Type I, every factor representation is a multiple of an irreducible representation. If every irreducible representation is a weak boundary representation, direct sums of irreducible representations will have WUEP. So the hypothesis of the previous theorem are satisfied. Because $V^*\pi(A)V \subseteq A''$, $E(V^*\pi V) = V^*\pi V$ and so $V^*\pi(a)V = \pi(a)$ for all $a \in A$. □

5.3 Weak peak points

In this section, we will introduce the notion of weak peak point which is a non-commutative analogue of peak point but different from Arveson's peaking representation.

Definition 5.3.1. *Let A be an unital C^*-algebra and S be an operator system of A such that $A = C^*(S)$, the C^*-algebra generated by S. An element π of $\hat{A}$ is called a weak peak point for S if there exists $s \in S$ such that*

(i) $|\langle \pi(s)\xi_\pi, \xi_\pi \rangle| = \|s\|$ *for some* $\xi_\pi \in H_\pi$ *with* $\|\xi_\pi\| = 1$,

(ii) $|\langle \sigma(s)\xi_\sigma, \xi_\sigma \rangle| < \|s\|$ *for all* $\xi_\sigma \in H_\sigma$ *with* $\|\xi_\sigma\| = 1$,

where σ is any irreducible representation not equivalent to π. We will denote the set of all weak peak points for S by $P_w(S)$.

However the exact relation between weak peak points and peaking representations of an operator system calls for further study.

We observed that the Choquet boundary of an operator system is contained in weak Choquet boundary of it and this inclusion is strict. So it would be interesting to know which weak Choquet boundary points are Choquet boundary points of an operator system. The following theorem gives partial answer to this query.

Theorem 5.3.1. *Let S be an operator system in a C^*-algebra $A = C^*(S)$. If $\pi \in \hat{A}$ is a weak peak point for S, π is a weak boundary representation for S and $\pi_{|S}$ is pure, then π is a boundary representation for S.*

Proof. Let $\pi \in P_w(S)$. We want to show that π is a boundary representation for S. Let $K = \left\{\Psi \in CP\left(A, H_\pi\right) : \Psi_{|S} = \pi_{|S}\right\}$. Then K is a compact convex set with respect to the BW-topology. By Krein-Milman theorem, there exists an extreme element Φ of K.

We claim that Φ is pure on A. Choose non zero elements Φ_1 and Φ_2 of $CP\left(A, H_\pi\right)$ such that $\Phi(a) = \Phi_1(a) + \Phi_2(a),\ a \in A$. Since $\Phi_{|S}$ is pure and $\Phi_{|S} = \pi_{|S}$, there exist scalars $t_i \geq 0,\ i = 1, 2$ such that $\Phi_i(s) = t_i\pi(s)$ for every $s \in S$. If we take $t_i = 0$, $i = 1, 2$, and since $e \in S$, we get $\Phi_i(e) = 0,\ i = 1, 2$. Hence, $\Phi_i = 0,\ i = 1, 2$, which is not possible because of our selection of Φ_i. This gives that $t_i > 0,\ i = 1, 2$. Since

$e \in S, \pi(e) = 1 = t_1\pi_1(e)+t_2\pi_2(e)$ we get $t_1+t_2 = 1$. Now put $\Psi_i = t_i^{-1}\Phi_i,\ i = 1,2$ then $\Psi_i \in K,\ i = 1,2$. Therefore we get $\Phi = t_1\Psi_1 + t_2\Psi_2$. But Φ is an extreme point of K, hence $\Phi = \Psi_1 = \Psi_2$. Then $\Phi_i = t_i\Phi,\ i = 1,2$. This proves that Φ is pure.

Let $(V, H_{\pi'}, \pi')$ be the minimal Stinespring triple corresponding to Φ where π' is an irreducible representation. Then $\Phi(\cdot) = V^*\pi'(\cdot)V$. Since Φ is unital, $\Phi(1_A) = V^*\pi'(1_A)V = V^*V = I$, so V is isometric.

Now we show that $\pi' \sim \pi$. Let if possible, π is not equivalent to π'. Since $\pi \in P_w(S)$, there exists $s \in S$ such that

$$|\langle \pi(s)\xi_\pi, \xi_\pi \rangle| = \|s\| \text{ for some unit vector } \xi_\pi, \quad \text{and}$$

$$|\langle \pi'(s)\xi_{\pi'}, \xi_{\pi'} \rangle| < \|s\| \text{ for all unit vectors } \xi_{\pi'}.$$

Now,

$$\|s\| = |\langle \pi(s)\xi_\pi, \xi_\pi \rangle| = |\langle \Phi(s)\xi_\pi, \xi_\pi \rangle| = |\langle \pi'(s)V\xi_\pi, V\xi_\pi \rangle| < \|s\|.$$

This is a contradiction. Hence, $\pi' \sim \pi$. Therefore, $\pi' = U^*\pi U$ for some unitary $U : H_{\pi'} \to H_\pi$. Hence, $\Phi = V^*\pi' V = V^*U^*\pi UV = V_1^*\pi V_1$ where $V_1 = UV$ is an isometry. Thus, $\Phi(s) = V_1^*\pi(s)V_1$ for every $s \in S$. Since $\Phi_{|S} = \pi_{|S}$, we have $\pi(s) = V_1^*\pi(s)V_1$ for every $s \in S$. By our assumption π is weak boundary representation, hence $\pi(a) = V_1^*\pi(a)V_1$ for all $a \in A$ and therefore $\pi(a) = \Phi(a)$ for all $a \in A$. Thus, $\pi = \Phi$. $\square$

Arveson [4, page 179] introduced the notion of separating subalgebra $\mathcal{A}$ of a C^*-algebra A. He observed that, if π is an irreducible representation of A on a Hilbert space H_π then $\mathcal{A}$ separates π if and only if for every irreducible representation σ of A on H_σ and every isometry $V : H_\pi \to H_\sigma$ the condition $V^*\sigma(a)V = \pi(a)$ for all $a \in \mathcal{A}$ implies π and σ are unitarily equivalent on A. If $\mathcal{A}$ separates every irreducible representation of a C^*-algebra A then $\mathcal{A}$ is separating subalgebra of A. Arveson mentioned that using this result one can define the separating linear subspace (or operator system) of a C^*-algebra.

Remark 5.3.1. *Using Arveson [4, Theorem 2.4.5] we can observe that if S is an operator system in a C^*-algebra $A = C^*(S)$, $\pi \in \hat{A}$ is a weak peak point for S, π is a weak boundary representation for S (equivalently π is an irreducible finite representation of S) and $\pi_{|S}$ is pure then S separates π.*

Following examples illustrates the above theorem.

Example 5.3.1. *Let the Volterra integration operator V acting on the Hilbert space $H = L^2[0,1]$ is given by*

$$Vf(x) = \int_0^x f(t)dt, \qquad f \in L^2[0,1].$$

It is well known that V generates the C^*-algebra $K = K(H)$ of all compact operators.

Let $S = span\,(V, V^*, V^2, V^{2*})$ and S is hyperrigid [Theorem 2.6.2]. Let $\tilde{S} = S + \mathbb{C} \cdot \mathbf{1}$ be an operator system generating the C^*-algebra $\tilde{A} = K + \mathbb{C} \cdot \mathbf{1}$. The irreducible representations of $\tilde{A}$ are π and ρ given by

$$\pi(T + \lambda\mathbf{1}) = T, \quad \textit{for} \quad T \in K, \; \lambda \in \mathbb{C}$$

$$\rho(T + \lambda\mathbf{1}) = \lambda, \quad \textit{for} \quad T \in K, \; \lambda \in \mathbb{C}$$

In fact these are the only two irreducible representations upto unitary equivalence. $\tilde{S}$ is a hyperrigid operator system [Theorem 2.6.1] implying that π and ρ are boundary representations for $\tilde{S}$ of $\tilde{A}$. Also, $\tilde{S}$ is quasi hyperrigid and therefore π, ρ are weak boundary representations for $\tilde{S}$.

Let $V + V^* \in \tilde{S}$ be the projection on the space of constants and let the constant function $1 \in L^2[0, 1]$, $||1|| = 1$

$$|\,\langle \pi(V + V^*)1, 1\rangle\,| = 1 = ||V + V^*||.$$

For all $\xi_\rho \in \mathbb{C}$, $||\xi_\rho|| = 1$

$$|\,\langle \rho(V + V^*)\xi_\rho, \xi_\rho\rangle\,| = |\,\langle 0\xi_\rho, \xi_\rho\rangle\,| = 0 < ||V + V^*||.$$

Therefore π is a weak peak point.

Let $\mathbf{1} \in \tilde{S}$ *and* $1 \in \mathbb{C}$, $||1|| = 1$

$$|\langle \rho(\mathbf{1})1, 1\rangle| = 1 = ||\mathbf{1}||.$$

For all $\xi_\pi \in L^2[0,1]$, $||\xi_\pi|| = 1$

$$|\langle \pi(\mathbf{1})\xi_\pi, \xi_\pi\rangle| = |\langle 0\xi_\pi, \xi_\pi\rangle| = 0 < ||\mathbf{1}||.$$

Hence ρ *is a weak peak point. Also,* π *and* ρ *restricted to* $\tilde{S}$ *are pure.*

Example 5.3.2. *Let* $G = span\,(I, S, S^*, SS^*)$*, where* S *is the unilateral right shift in* $B(H)$ *and* I *the identity operator. Let* $A = C^*(G)$ *be the* C^**-algebra generated by* G*. We have,* $K(H) \subseteq A$*.* $A/K(H) \cong C(\mathbb{T})$ *is commutative, where* $\mathbb{T}$ *denotes the unit circle in* $\mathbb{C}$ *and the spectrum* $\hat{A}$ *of* A *can be identified with* $\{Id\} \cup \mathbb{T}$*. Since* S *is an isometry,* G *is hyperrigid [Theorem 2.6.3] and this will imply that all the irreducible representations of* A *are boundary representations for* S*. Clearly* G *is quasi hyperrigid, so all the irreducible representations are weak boundary representations for* S*.*

Now we prove that identity representation Id *of* A *is a weak peak point for* G*. Let* $e_1 = (1, 0, 0..., 0)$ *and let* $E = I - SS^* \in G$ *be the rank one projection. We have* $|\langle Id(E)e_1, e_1\rangle| = 1 = ||E||$ *and for any irreducible representation* π *which is not equivalent to identity,* $\pi(E) = 0$*. So we have* $|\langle \pi(E)\eta, \eta\rangle| = 0 < ||E||$ *for all unit* *vectors $\eta \in H_\pi$. This proves that Id is a weak peak point. Also, $Id_{|_G}$ is pure.*

Now we give a 'lighter' version of weak peak points where we don't insist on the condition (ii) being true for all unit vectors of the corresponding Hilbert space.

Definition 5.3.2. *Let A be an unital C^*-algebra and S be an operator system of A such that $A = C^*(S)$, the C^*-algebra generated by S. An element π of $\hat{A}$ is called a quasi weak peak point for S if there exists $s \in S$ such that*

(i) $|\langle \pi(s)\xi_\pi, \xi_\pi \rangle| = \|s\|$ for some $\xi_\pi \in H_\pi$ with $\|\xi_\pi\| = 1$,

(ii) $|\langle \sigma(s)\xi_\sigma, \xi_\sigma \rangle| < \|s\|$ for some $\xi_\sigma \in H_\sigma$ with $\|\xi_\sigma\| = 1$,

where σ is any irreducible representation not equivalent to π.

We now give a few examples.

Example 5.3.3. *For each $\lambda \in \mathbb{C}$, let $T_\lambda \in M_3(\mathbb{C})$ be given by $T_\lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$. Let $S_{T_\lambda} = span\{I, T_\lambda, {T_\lambda}^*\}$ denote the operator system generated by T_λ. Now, let $A = C^*(S_{T_\lambda}) = M_2(\mathbb{C}) \oplus \mathbb{C}$ be the C^*-algebra generated by S_{T_λ}. Consider the map $\pi : A \to \mathbb{C}$ which sends each $X \in A$ to its $(3,3)-$ entry. Thus, π is an irreducible representation of A onto $\mathbb{C}$. Define another irreducible representation $\rho : A \to M_2(\mathbb{C})$ by $\rho(X) = V^*XV$, where $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. It can be proved that ρ and π are the only* *irreducible representations (up to unitary equivalence) of A. We will prove that π is quasi weak peak point for $\lambda = \frac{1}{2}$. Let* $S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $\xi_\pi = 1$, $|\langle \pi(S)\xi_\pi, \xi_\pi\rangle| = 1 = \|S\|$. *Let* $\xi_\rho = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|\langle \rho(S)\xi_\rho, \xi_\rho\rangle| = \left|\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle\right| = 0 < \|S\|$. *Hence, π is a quasi weak peak point for $\lambda = \frac{1}{2}$.*

Example 5.3.4. *[23, Page 488] Let $X = [0,1]$ and $A = C(X)$. Consider $f : [0,1] \to \mathbb{R}$ which is a strictly positive and strictly increasing continuous function. Consider the C^*-algebra $A \otimes M_2$. Let G be operator system in $A \otimes M_2$ spanned by* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $F = \begin{bmatrix} 0 & 0 \\ f & 0 \end{bmatrix}$. *Here $C^*(G) = A \otimes M_2$, and the irreducible representations of $A \otimes M_2$ on $\mathbb{C}^2$ are given by $\rho_t, t \in [0,1]$ where* $\rho_t(F) = \begin{bmatrix} 0 & 0 \\ f(t) & 0 \end{bmatrix}$ *represents the point evaluation at t and by [23], the only boundary representation for G in $A \otimes M_2$ is ρ_1. We will show that ρ_1 is a weak peak point for G. Let* $S = \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix}$ *and* $\xi_{\rho_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ *for $t = 1$,* $\left|\left\langle \rho_1 \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle\right| = \left|\left\langle \begin{bmatrix} 0 & f(1) \\ f(1) & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle\right| = |f(1)| = ||S||$. *And for all $t \in [0,1)$, $|\langle \rho_t(S)\xi_t, \xi_t\rangle| < |f(1)|$ for all $\xi_t \in H_{\rho_t}$. Hence, ρ_1 is a weak peak point.*

Remark 5.3.2. *In the classical case, when $A = C(X)$, where X is a compact Hausdorff space, irreducible representations correspond to point evaluation functionals and thereby precisely to the points of X. Let π_x be the irreducible representation corresponding to $x \in X$. An $x_0 \in X$ is a weak peak point for $G \subseteq C(X)$ if there exists $g_0 \in G$ such that $\left|\left\langle \pi_{x_0}(g_0)\xi_{\pi_{x_0}}, \xi_{\pi_{x_0}} \right\rangle\right| = \|g_0\|$ for some $\xi_{\pi_{x_0}} \in H_{\pi_{x_0}}$ with $\|\xi_{\pi_{x_0}}\| = 1$ and $|\left\langle \pi_x(g_0)\xi_{\pi_x}, \xi_{\pi_x} \right\rangle| < \|g_0\|$ for all $\xi_{\pi_x} \in H_{\pi_x}$ with $\|\xi_{\pi_x}\| = 1$, where π_x is any irreducible representation not equivalent to π_{x_0}. i.e., $g_0(x_0) = \|g_0\|$ and $|g_0(x)| < \|g_0\|$ for every $x \neq x_0$ which implies that x_0 is a peak point for G. Hence, in the classical case both weak peak points and peak points coincide. In the classical case we can prove that quasi weak peak points and peak points also coincide using similar arguments. Hence, all the three notions viz. weak peak points, quasi weak peak points and peak points coincide in the classical case.*

Remark 5.3.3. *It is clear that the concepts and the corresponding analysis is more based on a modest setting than the much stronger notions employed by Arveson in his series of articles. However, it is revealed that there are non-trivial questions related to the structure of certain interesting operator spaces associated with isometries.*

Chapter 6

Conclusion

In this thesis, we established a characterization of unique extension property for representations in the context of a C^*-algebra generated by an operator system in terms of the orthogonal projectivity and orthogonal injectivity of Hilbert modules over the operator algebra underlying the operator system. Using this result we characterized hyperrigidity of operator systems in terms of orthogonality properties of Hilbert modules.

We also proved that unique extension property for unital completely positive maps on operator systems carry over to tensor product of such maps defined on the tensor product of operator systems. Using this we deduced that tensor product of two hyperrigid operator systems is hyperrigid in the spatial tensor product of C^*-algebras. Hopenwasser's result about tensor product of boundary representations follows as a special case.

We introduced the notions of weak boundary representations and quasi hyperrigidity in the non-commutative setting for operator systems in C^*-algebras. An analogue of Saskin's theorem relating quasi hyperrigidity and weak Choquet boundary for particular classes of C^*-algebras is proved. We introduced the notion of weak peak points in the non-commutative setting for operator systems in C^*-algebras. We proved that if an irreducible representation is a weak boundary representation and weak peak then it is a boundary representation.

Here we will mention some problems for further research.

Arveson's hyperrigid conjecture [10] states that, if every irreducible representation of a C^*-algebra A is a boundary representation for a separable operator system $S \subseteq A$, then S is hyperrigid. This conjecture is proved for particular classes of C^*-algebras, but the case of general C^*-algebra is still unexplored.

In Corollary 3.3.1, we characterized hyperrigidity of separable operator systems of the form $S = \mathcal{A} + \mathcal{A}^*$, where $\mathcal{A}$ is an operator algebra in terms of orthogonality properties of Hilbert modules over $\mathcal{A}$. This result has to be investigated for general hyperrigid operator system S with suitable operator algebra $\mathcal{A}$.

In Theorem 4.3.1, we proved that unique extension property of completely positive maps on operator systems in C^*-algebras carry over to tensor product of those maps defined on the tensor product of operator systems in the spatial tensor product of C^*-algebras. This result also needs to be investigated for maximal tensor product of C^*-algebras.

In Proposition 5.1.1, we proved that every irreducible finite representation of an operator system in a C^*-algebra is equivalent to a weak boundary representation for an operator system in the C^*-algebra. It will be interesting to examine the case where the irreducibility of representations is not assumed.

We solved the Problem 5.1.1 for C^*-algebras with countable spectrum and type I C^*-algebras. The Problem 5.1.1 is still open for general C^*-algebras.

In theorem 5.3.1, we showed that if an irreducible representation is a weak peak point and weak boundary then it is a boundary representation for an operator system. The converse of this result is yet to be studied.

In [11, Section 1.5] there are two more notions of classical peak points depending on the maps under consideration. The non-commutative analogue of these are yet to be studied.

Davidson and Kennedy [16] completely solved the problem of existence of boundary representations. Kleski [26] proved that "sup" can be replaced by "max" in the separable case. This implies that the Choquet boundary for a separable operator system is a boundary in the classical sense. The problem of replacing "sup" by "max" in the general case is to be studied.

Kleski [26] proved that every peaking representation for an operator system in a separable C^*-algebra is a boundary representation. The converse of this result is yet to be studied.

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