Ph.D. THESIS **MATHEMATICS**

A STUDY ON V_4-MAGIC AND BARYCENTRIC RING MAGIC GRAPHS

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28 February, 2018

CERTIFICATE

I hereby certify that the thesis entitled "A STUDY ON V_4-MAGIC AND BARYCENTRIC RING MAGIC GRAPHS" is a bonafide work carried out by **Ms.Vandana P.T.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

University of Calicut,
28 February, 2018.

Anil Kumar V.
(Research Supervisor)

DECLARATION

I hereby declare that the thesis, entitled "**A STUDY ON V_4-MAGIC AND BARYCENTRIC RING MAGIC GRAPHS**" is based on the original work done by me under the supervision of **Dr. Anil Kumar V.**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Contents

List of Symbols

G	Graph
$V(G)$	Vertex set of G
$E(G)$	Edge set of G
G^c or $\overline{G}$	Complement of G
$G_1 \vee G_2$	Join of G_1 and G_2
$G_1 \odot G_2$	Corona of two graphs
$G_1 \Box G_2$	Cartesian product of two graphs
$G_1[G_2]$	Composition of two graphs
$G_1 \diamond G_2$	Chain of two graphs
$\diamond G_n$	n-link chain with all links G
P_n	Path graph
C_n	Cycle graph
W_n	Wheel graph
H_n	Helm graph
K_n	Complete graph
$K_{m,n}$	Complete bipartite graph
$K_{n_1,n_2,\ldots,n_k}$	Complete k-partite graph
$K_{1,n}$	Star graph
$L(G)$	Line graph of G
$M(G)$	Middle graph of G
$T(G)$	Total graph of G
$S(G)$	Splitting graph of G
$\mu(G)$	Mycielski graph of G
V_4	Klein 4-group
ℓ	Edge labeling
ℓ^+	Vertex labeling

$\mathscr{V}_a$	a-sum V_4-magic graph
$\mathscr{V}_0$	zero-sum V_4-magic graph
$\mathscr{V}_{a,0}$	both a-sum and zero-sum V_4-magic graph
$\mathscr{BV}_a$	a-sum V_4-barycentric magic graph
$\mathscr{BV}_0$	zero-sum V_4-barycentric magic graph
$\mathscr{BV}_{a,0}$	both a-sum and zero-sum V_4-barycentric magic graph
Sun_n	Sun graph
$BSun(p,q)$	Broken Sun Graph
$CBSun(p,q)$	Consecutive Broken Sun
$C(n_1,n_2,\ldots,n_k)$	Chain of cycles
$W_{n,m}$	m-level wheel graph
$W(2,n)$	Web graph
$W(t,n)$	Generalized Web Graph
$W_0(t,n)$	Generalized Web Graph without center
$H(2,n)$	Closed helm
$H(t,n)$	Closed generalized helm
Fl_n	Flower graph
$D_n^{(m)}$	Windmill graph
F_m	Friendship graph
$T_n^{(m)}$	Snake graph
$BP(G)$	Bipyramid based on G
$BP(n)$	Bipyramid based on C_n
B_n	Book graph
$B(n,k)$	n-gon book of k pages
$C_n(t)$	One point union of t cycles
G_n	Gear graph
SF_n	Sunflower graph
$J_{n,m}$	Jahangir graph
$C_m@C_n$	Flower graph
$\Theta(a_1,a_2,\ldots,a_k)$	Generalized theta graph
L_n	Ladder graph
$H(n,n-3)$	Shell graph
$U(m,n)$	Umbrella graph
$U(m,n,k)$	Extended umbrella graph
$C_n(p,r)$	Cycle with r consecutive chords
$MS(n_1^{t_1},n_2^{t_2},\ldots,n_r^{t_r})$	Multiple shell
$MS(p^t)$	Balanced multiple shell
$\mathbb{Z}_p$	Set of all integers modulo p

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Chapter 1

Introduction

Graph theory is an important branch of Mathematics which has a wide range of applications in our day to day life. Generally graph theory have been motivated by the study of games and recreational mathematics. Graphs act as mathematical models for many real life problems. One of the interesting problems in the area of graph theory is that of labeling of graphs. Labeled graphs are becoming an increasingly useful family of mathematical model for a broad range of applications. They are useful in many coding theory problems, including the design of good radar location codes, synch-set codes, missile guidance codes and convolution codes with optimal nonstandard encoding of integers. Labeled graphs have also been applied in determining ambiguities in X-ray crystallographic analysis, design communication network addressing systems in determining optimal circuit layouts, database management etc. The efforts to find solutions to many practical problems in real life situations have also led to the development of several graph labeling methods-graceful, harmonious, prime, divisor, magic, antimagic, cordial, product cordial, prime cordial etc.

Most graph labeling methods trace their origin to one introduced by Rosa in 1967 or one given by Graham and Sloane in 1980. During the mid sixties J. Sedlacek introduced the magic labelings motivated by the notion of magic squares in number theory. Magic graphs are in fact generalizations of magic squares. A magic graph is a graph whose edges are labeled by positive integers so that the sum of the labels of the edges incident with a vertex is the same, independent of the choice of a vertex. The original concept of a group magic graph is due to Sedlacek. Afterwards, Kotzig and Rosa started the study of graph labeling especially edge-magic total labeling. Studies were made on $\mathbb{Z}-$magic and $\mathbb{Z}_k$-magic graphs. Stanley considered $\mathbb{Z}-$magic graphs and Doob considered A-magic graphs where A is an abelian group. Given a graph G, the problem of whether G admits a magic labeling is similar to the problem of deciding whether a set of linear homogeneous Diophantine equations has a solution [13]. At present, given an abelian group, no general algorithm is available for finding magic labeling of graphs. Recently there has been a great interest in the area of magic labeling due to so many of its applications.

For an abelian group A, written additively, any mapping $\ell : E(G) \longrightarrow A \setminus \{0\}$ is called a labeling, where 0 denote the identity element in A. For any abelian group A, a graph $G = (V, E)$ is said to be A-magic if there exists a labeling $\ell : E(G) \longrightarrow A \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \longrightarrow A$ defined by

$$\ell^+(u) = \sum\{\ell(uv) : uv \in E(G)\}$$

is a constant map [13]. Observe that A-magic labeling of a graph need not be unique. When $A = \mathbb{Z}_k$, then the graph is called k- magic [13]. The Klein 4 group $V_4 = \{0, a, b, c\} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a group of order 4 with $0 + 0 = 0,\ a + a = 0,\ b + b = 0,\ c + c = 0,\ a + b + c = 0,\ a + b = c,\ a + c = b,\ b + c = a$. When $A = V_4$, the graph is called V_4 magic graph. The concept of V_4 magic graphs was first introduced by S. M. Lee et al. in 2002 [13]. There has been an increasing interest in the study of V_4 magic graphs since the publication of [13]. We say that a graph G is a-sum V_4 magic if there exists a labeling $\ell : E(G) \rightarrow V_4 \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \rightarrow A$ satisfies $\ell^+(v) = a$ for all $v \in V(G)$ and for some nonzero element $a \in V_4$. If $\ell^+(v) = 0$, for all $v \in V(G)$, the graph is zero-sum V_4 magic.

A graph G is A- barycentric magic if there exists a labeling $\ell : E(G) \longrightarrow A \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \longrightarrow A$ defined by

$$\ell^+(v) = \sum\{\ell(uv) : uv \in E(G)\}$$

is a constant map and also satisfies $\ell^+(v) = \deg(v)\ell(u_v v)$ for all $v \in V$, and for some vertex u_v adjacent to v [34]. We say that a graph G is a-sum V_4 barycentric magic if there exists an $a \in V_4$ such that

$$\ell^+(v) = \deg(v)\ell(u_v v) = a$$

for all $v \in V$, and for some vertex u_v adjacent to v. When $a = 0$, G is said to be zero sum V_4 barycentric magic.

If a graph G and its line graph $L(G)$ are both a-sum V_4-magic, then G is called an a-sum V_4-bimagic graph. If a graph G and its line graph $L(G)$ are both zero-sum V_4-magic, then G is called a zero-sum V_4-bimagic graph. A graph G is called a V_4-bimagic graph if G and its line graph $L(G)$ are both a-sum V_4-magic or zero-sum V_4-magic.. If G or $L(G)$ is not V_4-magic, then G is called a non V_4-bimagic graph..

The concept of a ring magic graph is a very natural generalization of a group magic graph and it was introduced by W.C.Shiu and Richard M.Low in their paper [26]. Here we introduce the concept of an R-barycentric ring magic graph and characterize some $\mathbb{Z}_p$-barycentric ring magic graphs having vertices of degrees 2 and 3. Let R be a commutative ring with unity. A graph $G = (V, E)$ is said to be R-barycentric ring magic if there exists a labeling $\ell : E(G) \rightarrow R \setminus \{0\}$ of the edges of G by nonzero elements of R such that the induced vertex labelings $\ell^+ : V(G) \rightarrow R$ defined by $\ell^+(v) = \sum \ell(uv)$ where $(u, v) \in E$and $\ell^\times : V(G) \rightarrow R$ defined by $\ell^\times(v) = \prod \ell(uv)$ where $(u, v) \in E$ are constant maps and satisfies:

i) $\ell^+(v) = deg(v)\ell(u_v v)$, for all $v \in V(G)$, and for some vertex u_v adjacent to v.

ii) $\ell^\times(v) = \ell(u_v v)^{deg(v)}$, for all $v \in V(G)$, and for some vertex u_v adjacent to v.

Throughout this thesis we use the following notations:

(1) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs,

(2) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs,and

(3) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero-sum V_4 magic.

(4) $\mathscr{BV}_a$: the class of graphs that are a-sum V_4 barycentric magic.

(5) $\mathscr{BV}_0$: the class of graphs that are zero-sum V_4 barycentric magic.

(6) $\mathscr{BV}_{a,0}$: the class of graphs that are both a-sum and zero-sum V_4 barycentric magic.

(7) $\mathscr{A} := \{G : G, L(G) \in \mathscr{V}_a\}$.

(8) $\mathscr{B} = \{G : G \in \mathscr{V}_a$ and $L(G) \in \mathscr{V}_0\}$.

(9) $\mathscr{C} = \{G : G, L(G) \in \mathscr{V}_0\}$.

(10) $\mathscr{D} = \{G : G \in \mathscr{V}_0$ and $L(G) \in \mathscr{V}_a\}$.

(11) $\mathscr{E} = \{G : G \in \mathscr{V}_a$ and $L(G) \notin \mathscr{V}_a\}$.

(12) $\mathscr{F} = \{G : G \in \mathscr{V}_a$ and $L(G) \notin \mathscr{V}_0\}$.

(13) $\mathscr{G} = \{G : G \notin \mathscr{V}_a$ and $L(G) \in \mathscr{V}_a\}$.

(14) $\mathscr{H} = \{G : G \notin \mathscr{V}_a$ and $L(G) \in \mathscr{V}_0\}$.

(15) $\mathscr{I} = \{G : G \in \mathscr{V}_0$ and $L(G) \notin \mathscr{V}_a\}$.

(16) $\mathscr{J} = \{G : G \in \mathscr{V}_0$ and $L(G) \notin \mathscr{V}_0\}$.

(17) $\mathscr{K} = \{G : G \notin \mathscr{V}_0$ and $L(G) \in \mathscr{V}_a\}$.

(18) $\mathscr{L} = \{G : G \notin \mathscr{V}_0$ and $L(G) \in \mathscr{V}_0\}$.

(19) $\mathscr{M} = \{G : G \notin \mathscr{V}_a$ and $L(G) \notin \mathscr{V}_a\}$.

(20) $\mathscr{N} = \{G : G \notin \mathscr{V}_a$ and $L(G) \notin \mathscr{V}_0\}$.

(21) $\mathscr{O} = \{G : G \notin \mathscr{V}_0$ and $L(G) \notin \mathscr{V}_a\}$.

(22) $\mathscr{P} = \{G : G, L(G) \notin \mathscr{V}_0\}$.

The thesis mainly focus on a study of V_4-magic, V_4-barycentric magic, V_4 bimagic and $\mathbb{Z}_p$-barycentric ring magic graphs. The results are presented in the following chapters.

- Chapter 2: Preliminaries
- Chapter 3: V_4-Magic Labelings of Wheel Related Graphs
- Chapter 4: V_4-Magic Labelings of Shell Related Graphs
- Chapter 5: V_4-Magic Labelings of Some More Graphs
- Chapter 6: V_4-Barycentric Magic Graphs
- Chapter 7: Some Special V_4-Barycentric Magic Graphs
- Chapter 8: V_4-Bimagic Graphs
- Chapter 9: On $\mathbb{Z}_p$-Barycentric Ring Magic Graphs

In **Chapter 2**, we provide some basic definitions and results from graph theory, group theory and ring theory which are required for the subsequent chapters in the thesis.

In the first section of **Chapter 3**, we include definitions of some wheel related graphs. In section 2 we prove that if G is a (p,q) graph with vertex set $\{v_1, v_2, \ldots, v_p\}$, and $\ell : E(G) \to V_4 \setminus \{0\}$ is a labeling of G, then $\sum_{i=1}^{p} \ell^+(v_i) = 0$. Subsequently we prove following results:

- $C_n \in \mathscr{V}_a$ if and only if n is even and $C_n \in \mathscr{V}_0$ for all $n \geq 3$.
- If $C(n, k_1, k_2, \ldots, k_t) \subset \mathscr{V}_a$, then $n + k_1 + k_2 + \cdots + k_t$ is even. Then naturally a question arises whether the converse of this theorem is true. We conjecture that the converse of this result is true as well. Furthermore, we prove a special case of the conjecture which states that if $n + tk$ is even, then $C(n, \underbrace{k, k, \ldots, k}_{t}) \subset \mathscr{V}_a$.

Finally, we prove the following results for the graphs $Sun_n, BSun(p,q), CBSun(p,q), C_n \odot K_2, C_n \odot C_m, C_n \odot K_m, C_n \odot \overline{K_m}, C_m \diamond C_n$ and $\diamond[C_n]_m$.

- $Sun_n \in \mathscr{V}_a$ for all n and $Sun_n \notin \mathscr{V}_0$ for all n.
- $BSun(p,q)$ and $CBSun(p,q)$ is contained in $\mathscr{V}_a$ if and only if $p+q$ is even.
- $C_n \odot K_2 \in \mathscr{V}_a$ if and only if n is even and $C_n \odot K_2 \in \mathscr{V}_0$ for all $n \geq 3$.
- $C_n \odot C_m \in \mathscr{V}_a$ if and only if $n(m+1)$ is even and $C_n \odot C_m \in \mathscr{V}_0$ for all $m \geq 3$ and $n \geq 3$.
- $C_n \odot \overline{K_m} \in \mathscr{V}_a$ if and only if $n(m+1)$ is even and $C_n \odot \overline{K_m} \notin \mathscr{V}_0$ for all m and n, where $\overline{K_m}$ is the complement of the complete graph with m vertices.
- $C_m \diamond C_n \in \mathscr{V}_a$ if and only if $m+n$ is odd and $C_m \diamond C_n \in \mathscr{V}_0$ for all m and n.
- $\diamond[C_n]_m \in \mathscr{V}_a$ if and only if m is odd and n is even.

In the third section we consider some wheel related graphs and prove the following results.

- $W_n \in \mathscr{V}_a$ if and only if n is odd and $W_n \in \mathscr{V}_0$ for all n.
- If W_n is a- sum V_4-magic and if k is odd, then W_{nk} is a-sum V_4-magic.
- If W_n is zero-sum V_4 magic, so is W_{kn} for every $k \geq 2$.
- $W_{n,m} \in \mathscr{V}_a$ if and only if both m and n are odd and $W_{n,m} \in \mathscr{V}_0$ for all n and m.
- If $W_{n,m}$ is a- sum V_4-magic and if k is odd, then $W_{nk,m}$ is a-sum V_4-magic.
- $SW_n \notin \mathscr{V}_a$ for any $n \geq 3$ and $SW_n \in \mathscr{V}_0$ for any $n \geq 3$. Similar result holds for $SW_{n,m}$ also.
- Helm graph is neither a-sum V_4-magic nor zero-sum V_4-magic for any n.
- $W(2,n) \in \mathscr{V}_a$ if and only if n is odd but it does not belongs to $\mathscr{V}_0$ for any n.
- $W(t,n) \in \mathscr{V}_a$ if and only if n is odd and t is even but it does not belongs to $\mathscr{V}_0$ for any n and t.
- $W_0(t,n) \in \mathscr{V}_a$ if and only if $n(t+1)$ is even but it does not belongs to $\mathscr{V}_0$ for any n and t.
- $H(2,n) \notin \mathscr{V}_a$ for any n and $H(2,n) \in \mathscr{V}_0$ for all n.
- $H(t,n) \in \mathscr{V}_a$ if and only if both n and t are odd and $H(2,n) \in \mathscr{V}_0$ for all n.
- Flower graph is not a-sum V_4-magic for any n but it is zero-sum V_4-magic for all n.

In the first section of **Chapter 4** we provide the definition of a shell graph. Also include some well known shell related graphs. In the second section we prove the following results.

- Shell graph is a-sum V_4-magic if and only if n is even.
- $U(n,m) \notin \mathscr{V}_a$ if $m \geq 2$.
- $U(n,1) \in \mathscr{V}_a$ if n is odd.
- $U(m,n,k) \notin \mathscr{V}_a$ if $n \geq 2$.
- If $U(m,1,k) \in \mathscr{V}_a$ then $m+k$ is odd.
- If m is odd and k is even, then $U(m,1,k) \in \mathscr{V}_a$.
- If m is even and k is odd, then $U(m,1,k) \notin \mathscr{V}_a$.
- If $B(t,n_1,n_2,\ldots,n_t) \in \mathscr{V}_a$, then $n_1+n_2+\cdots+n_t$ is odd.
- If n and t are odd then $B(t,n,n,\ldots,n) \in \mathscr{V}_a$.
- $H(2n,n-2), H(2n,n-1) \in \mathscr{V}_a$ and $\mathscr{V}_0$ for all n.
- $H(4n+1,2n), H(4n+3,2n+2) \notin \mathscr{V}_a$ but belongs to $\mathscr{V}_0$ for all n.

- $U(4n+1, 2n, 1), U(4n+3, 2n+2, 1) \in \mathscr{V}_a$ for all n.
- $C_n(2, r) \in \mathscr{V}_a$ if and only if n is even and $2 \leq r \leq n-3$.
- $\mathcal{G}_r(n) \in \mathscr{V}_a$ if and only if $n + r$ is odd and $\mathcal{G}_r(n) \in \mathscr{V}_0$ if n+r is even.
- $G(n, n-3, k) \in \mathscr{V}_a$ if and only if nk is even and $G(n, n-3, k) \in \mathscr{V}_0$ for all n and k.

In the third section we prove that if the multiple shell graph $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}) \in \mathscr{V}_a$, then $\sum_{i=1}^{r}[(n_i-1)t_i]$ is odd. We conjecture that if $\sum_{i=1}^{r}[(n_i-1)t_i]$ is odd, then $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \in \mathscr{V}_a$. We prove some special cases of the conjecture which are as follows:

- $MS(n^t) \in \mathscr{V}_a$ if $(n-1)t$ is odd.
- $MS(n, n+1) \in \mathscr{V}_a$.
- $MS(n^t, (n+1)^t) \in \mathscr{V}_a$ for all odd t.
- $MS(n, m) \in \mathscr{V}_a$ if and only if $m + n$ is odd.

Now we have another question whether the multiple shell graph is zero-sum V_4-magic or not?. Again we state it as a conjecture. That is $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}) \in \mathscr{V}_0$ for all n_i and t_i. Moreover, we prove the following special cases.

- $MS(n^t) \in \mathscr{V}_0$ for n even, t odd and n even, t even.
- $MS(n^t) \in \mathscr{V}_0$ for n odd, t odd and n odd, t even.
- If $m + n$ is even, then $MS(n, m) \in \mathscr{V}_0$.

Now we prove that if the chain of multiple shell and star graph $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \diamond K_{1,m} \in \mathscr{V}_a$, then $\sum_{i=1}^{r}[(n_i-1)t_i] + m$ is odd. We conjecture that if $\sum_{i=1}^{r}[(n_i-1)t_i] + m$ is odd, then $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \diamond K_{1,m} \in \mathscr{V}_a$. We prove the following special cases of the conjecture.

- $MS(n^t) \diamond K_{1,m} \in \mathscr{V}_a$ if and only if $(n-1)t + m$ is odd.
- $MS(n, n+1) \diamond K_{1,m} \in \mathscr{V}_a$ if m is even.
- $MS(n^t, (n+1)^t) \diamond K_{1,m} \in \mathscr{V}_a$ if m is even and t is odd.

In **Chapter 5**, we include the definitions of some special graphs like Jahangir graph, windmill graph, friendship graph, one-point union of t-cycles, snake graph, the graph $C_m@C_n$, bipyramid graph, ladder graph, semiladder, planar grid, generalized Theta graph, n-gon book of k pages and book graph. We prove the following results.

- $J_{n,m} \in \mathscr{V}_a$ if and only if both n and m are odd and belongs to $\mathscr{V}_0$ for all n and m.

- The windmill graph $D_n^{(m)} \in \mathscr{V}_a$ if and only if m is odd and n is even and belongs to $\mathscr{V}_0$ for all n and m.
- $F_m \notin \mathscr{V}_a$ for any m but $F_m \in \mathscr{V}_0$ for all m.
- $C_n(t) \in \mathscr{V}_a$ if and only if n is even and t is odd.
- $T_n^{(m)} \in \mathscr{V}_a$ if and only if m is even and n is odd.
- For all $m, n \geq 3$, $C_m@C_n \in \mathscr{V}_a$ if and only if $n(m-1)$ is even.
- For any $n \geq 4$, the bipyramid graph $BP(n)$ is a-sum V_4-magic if and only if n is even. Then we prove the following results for $BP(G)$.
- If G is a-sum V_4-magic and number of vertices in G is odd, then $BP(G)$ is a-sum V_4-magic.
- If G is a-sum V_4-magic and number of vertices in G is even, then $BP(G)$ is 0-sum V_4-magic.
- If G is 0-sum V_4-magic and number of vertices in G is even, then $BP(G)$ is both a-sum V_4-magic and 0-sum V_4-magic.
- If G is 0-sum V_4-magic and number of vertices in G is odd, then $BP(G)$ is 0-sum V_4-magic.
- Ladders L_n and semiladders are a-sum V_4-magic for all n but $L_{n+2} \notin \mathscr{V}_0$ for any n.
- The planar grid $P_m \square P_n$ is a-sum V_4-magic if and only if mn is even.
- If the generalized Theta graph $\Theta(a_1, a_2, \cdots a_k)$ is a-sum V_4-magic then either odd number of a_i's are odd or even number of a_i's are even.
- Let $\Theta(a_1, a_2, \cdots a_k)$ be a generalized Theta graph. If k and even number of a_i's are even then $\Theta(a_1, a_2, \cdots a_k)$ is a-sum V_4 magic.
- For any $n \geq 3$ and $k \geq 1$, $B(n, k) \in \mathscr{V}_a$ if and only if $(n-2)k$ is even.
- The book B_n is a-sum V_4-magic and zero-sum V_4-magic for all n.
- For $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is a-sum V_4-magic if and only if $m + n$ is even.

Consider the complete graph K_n of order $n \geq 4$ with $V(K_n) = \{v_1, v_2, \cdots v_n\}$ and for each r such that $2 \leq r \leq n-2$, let $\mathcal{G}(n, r)$ be a spanning subgraph of K_n with $E(\mathcal{G}(n, r)) = E(K_n) - \{v_i v_j : 1 \leq i < j \leq r\}$. We prove that the graph $\mathcal{G}(n, r)$ is a-sum V_4-magic if and only if n is even. The graph $\mathcal{G}(n, r)$ is zero-sum V_4-magic for all n. This can be considered as a generalization of the result obtained by Sin Min Lee et.al. [13]. Consider the graph $\mathcal{G}(n, r)$ together with pendant edges at the vertices $v_1, v_2, \cdots, v_r$ and triangles at the vertices $v_{r+1}, v_{r+2}, \cdots, v_n$. We denote this graph by $\mathcal{G}_r^n(G)$. We prove that $\mathcal{G}_r^n(G)$ is a-sum V_4-magic if and only if both n and r are of the same parity and it does not belongs to $\mathscr{V}_0$.

In the first section of **Chapter 6**, we introduce the concept of V_4-barycentric magic graphs. In the second section we prove the following results:

- The star $K_{1,n} \in \mathscr{BV}_a$ if and only if n is odd and does not belongs to $\mathscr{BV}_0$ for any n.
- For $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is a-sum V_4-barycentric magic if and only if both m and n are odd and is zero-sum V_4-barycentric magic if and only if both m and n are even.

In the third section we prove that a tree t is a-sum V_4-barycentric magic if and only if the number of vertices of t is even and all its vertices have odd degrees. In the next section we prove the following.

- For $n \geq 2$, the complete graph $K_n \in \mathscr{BV}_a$ if and only if n is even and $K_n \in \mathscr{BV}_0$ if and only if n is odd.
- For any $n > 3$, $K_n \setminus e$, the complete graph with one edge removed, is neither a-sum V_4-barycentric magic nor zero-sum V_4-barycentric magic for any n.

In the fifth section we include the definitions of splitting graph and mycielski graph of a graph G. We investigate the splitting graph and mycielski graph of $K_{m,n}, K_{1,n}, C_n, P_n$ which belong to the classes $\mathscr{BV}_a$ and $\mathscr{BV}_0$. In the last section the following results are proved.

- The sun graph $C_n \odot K_1$ is a-sum V_4-barycentric magic for all n and is not zero-sum V_4-barycentric magic for any n.
- The wheel W_n is a-sum V_4-barycentric magic if and only if n is odd and is not zero-sum V_4-barycentric magic for any n.
- For any $n \geq 3$ and $k \geq 1$, the n-gon book of k pages is zero-sum V_4-barycentric magic if and only if k is odd but it is not a-sum V_4-barycentric magic.
- For any $n \geq 3$, the bipyramid graph $BP(n)$ is zero-sum V_4-barycentric magic if and only if n is even but it is not a-sum V_4-barycentric magic.
- The sunflower graph SF_n is neither zero-sum V_4-barycentric magic nor a-sum V_4-barycentric magic for any n.
- K_{n_1,n_2,n_3} is zero-sum V_4-barycentric magic if and only if n_1, n_2, n_3 are of same parity and is not a-sum V_4-barycentric magic for any $n_i, i = 1, 2, 3$.

We conclude this chapter by proving that the class $\mathscr{BV}_{a,0}$ is empty.

Chapter 7 is a continuation of chapter 6. Here we study some special V_4 barycentric magic graphs.

In the first section of **Chapter 8**, we define bimagic graphs and introduce the class of graphs $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}, \mathscr{K}, \mathscr{L}, \mathscr{M}, \mathscr{N}, \mathscr{O}$ and $\mathscr{P}$. In the second section we prove the following results.

- Let G be a V_4-bimagic (p,q) graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_p\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_q\}$. Then $\sum_{i=1}^{p} \ell^+(u_i) = \sum_{i=1}^{q} \ell^+(e_i) = 0$.

- Let G be an a-sum V_4-bimagic (p,q) graph. Then both p and q are even.

- The star graph does not belongs to the class $\mathscr{A}$ for all $n > 1$, belongs to the class $\mathscr{B}$ if and only if n is odd, belongs to the class $\mathscr{K}$ if and only if n is even and belongs to the class $\mathscr{L}$ for all n.

In the third section we prove the following results:

- Sun_n belongs to the classes $\mathscr{A}$, $\mathscr{B}$, $\mathscr{K}$ and $\mathscr{L}$ for all n.

- $BSun(n,k)$ and $CBSun(n,k)$ belongs to the classes $\mathscr{A}, \mathscr{B}$ and $\mathscr{K}$ if and only if $n+k$ is even and belongs to the $\mathscr{L}$-class for all n and k.

- W_n belongs to the classes $\mathscr{A}$ and $\mathscr{B}$ if and only if n is odd and belongs to the classes $\mathscr{C}$ and $\mathscr{D}$ for all n.

- $C_m@C_n$ belongs to the $\mathscr{A}$-class if and only if either m is odd and n is even or both m and n are even, belongs to the $\mathscr{B}$-class if and only if $n(m-1)$ is even, belongs to the $\mathscr{C}$-class for all $m, n \geq 3$ and belongs to the $\mathscr{D}$-class if and only if mn is even.

- $C_n(t)$ belongs to the classes $\mathscr{A}$ and $\mathscr{B}$ if and only if n is even and t is odd, belongs to the $\mathscr{C}$-class for all n and t and belongs to the $\mathscr{D}$-class if and only if nt is even.

- $J_{n,m}$ belongs to the classes $\mathscr{A}$ and $\mathscr{B}$ if and only if n and m are odd, belongs to the $\mathscr{C}$-class for all n and m, belongs to the $\mathscr{D}$-class if and only if $m(n+1)$ is even.

In the fourth section we prove the following results:

- L_n belongs to the $\mathscr{A}$-class and $\mathscr{D}$-class if and only if n is even and belongs to the classes $\mathscr{B}$ and $\mathscr{C}$ for all n.

- L_{n+2} belongs to the classes $\mathscr{A}$ and $\mathscr{K}$ if and only if n is even and belongs to the classes $\mathscr{B}$ and $\mathscr{L}$ for all n.

- The diamond graph belongs to the classes $\mathscr{B}, \mathscr{C}, \mathscr{E}$ and $\mathscr{I}$.

- The butterfly graph belongs to the classes $\mathscr{C}, \mathscr{D}, \mathscr{G}$ and $\mathscr{H}$.

- The kite graph belongs to the classes $\mathscr{G}, \mathscr{H}, \mathscr{K}$ and $\mathscr{L}$.

- The cricket graph belongs to the classes $\mathscr{H}, \mathscr{L}, \mathscr{M}$ and $\mathscr{O}$.

- The moth graph belongs to the classes $\mathscr{B}, \mathscr{E}, \mathscr{L}$ and $\mathscr{O}$.

In **Chapter 9**, we introduce the concept of R-barycentric ring magic graphs and characterize a few $\mathbb{Z}_p$-barycentric ring magic graphs having vertices of degrees 2 and 3. We introduce $k-$barycentric sequence in a commutative ring R with unity. In the second section we prove some lemmas which characterize barycentric sequences of lengths 2 and 3. In the third section we prove the following results.

- Any graph G is $\mathbb{Z}_2$- barycentric ring magic with same additive and multiplicative constant 1 if and only if all the vertices of G have odd degrees.
- A regular graph G is R-barycentric ring magic for any ring R.
- For every commutative ring R with unity, P_2 is R-barycentric ring magic and $P_n, n \geq 3$ is not R-barycentric ring magic.
- The cycle C_n is R-barycentric ring magic for every commutative ring R with unity.
- $C(n_1, n_2)$ and $C(n_1, n_2, \cdots, n_k)$ are $\mathbb{Z}_h$-barycentric magic if and only if h is even.
- If a graph is not $\mathbb{Z}_h$-barycentric magic, then it cannot be $\mathbb{Z}_h$-barycentric ring magic.

Finally we prove the following results.

- The sun graph $C_n \odot K_1$ is $\mathbb{Z}_p$-barycentric ring magic if and only if $p = 2$.
- Let p be an odd prime. Then the wheel graph W_n is $\mathbb{Z}_p$-barycentric ring magic if and only if there exists an $a \in \mathbb{Z}_p \setminus \{0\}$ such that $n \equiv 3(mod\ p)$ and $a^{n-3} \equiv 1(mod\ p)$. W_n is $\mathbb{Z}_2$ barycentric ring magic if and only if n is odd.
- The planar grid $P_m \times P_n$ is not $\mathbb{Z}_p$-barycentric ring magic for all m, n except $m = n = 2$ and for any p.
- The friendship graph F_m is $\mathbb{Z}_2$-barycentric ring magic for all m and $\mathbb{Z}_p$-barycentric ring magic for p odd if and only if there exists an $a \in \mathbb{Z}_p \setminus \{0\}$ such that $m \equiv 1(mod\ p)$ and $a^{2m-2} \equiv 1(mod\ p)$.
- The ladder graph L_n and semiladder are not $\mathbb{Z}_p$-barycentric ring magic for any n and any p and L_{n+2} is $\mathbb{Z}_p$-barycentric ring magic if and only if $p = 2$.
- Let p be an odd prime. Then the graph $B(n, k)$ is $\mathbb{Z}_p$-barycentric ring magic if and only if the following holds:
  - $(k-2)a + b \equiv 0(mod\ p)$
  - $a^{(k-2)}b \equiv 1(mod\ p)$
- $B(n, k)$ is $\mathbb{Z}_2$-barycentric ring magic if and only if k is odd. This result also holds for book graph B_n.
- $C_m @ C_n$ is $\mathbb{Z}_2$-barycentric ring magic for all m and n and if it is $\mathbb{Z}_p$-barycentric ring magic for an odd prime p, then $b_i \equiv b_j(mod\ p), i \neq j$ where both i and j are of same parity and b_i's are the edge labels of the cycle C_n.

Chapter 2

Preliminaries

In this chapter we present some basic definitions from graph theory, group theory and ring theory which are required for the subsequent chapters in the thesis. For notations and terminology not defined in this thesis the readers may refer to [4] and [20].

2.1 Basic Definitions from Graph Theory

Definition 2.1.1. *(see [20]) A (undirected) graph is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a nonempty finite set and $E(G)$ is a binary symmetric relation on $V(G)$. The elements of $V(G)$ are called vertices and elements of $E(G)$ are called edges.*

Denoting by $|S|$ the cardinality of a set S we define $p = |V|$ to be the order of G and $q = |E|$ to be the size of G. A (p, q) graph is a graph of order p and size q.

If an edge e corresponds to the vertex pair (u, v), we will write $e = uv$ and we say that the edge e joins the vertices u and v or u and v are adjacent vertices. If two distinct edges u and v are incident with a common point, then they are called adjacent edges.

Definition 2.1.2. *(see [4]) A graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n.*

Definition 2.1.3. *(see [4]) Let G be a graph. Then the complement G^c of G is defined by taking $V(G^c) = V(G)$ and making two vertices u and v adjacent in G^c if, and only if, they are nonadjacent in G.*

Definition 2.1.4. *(see [4]) A graph H is called a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. A subgraph H of G is a spanning subgraph of G, if $V(H) = V(G)$.*

Definition 2.1.5. *(see [4]) Let G be a graph and $v \in V$. The number of edges incident at $v \in G$ is called the degree of the vertex $v \in G$ and is denoted by $d_G(v)$ or simply $d(v)$. A* *graph G is called k-regular, if every vertex of G has degree k. A graph is said to be regular if it is k-regular for some nonnegative integer k. In particular a 3-regular graph is called a cubic graph.*

Definition 2.1.6. *(see [4]) A vertex of degree 1 is called a pendant vertex of G, whereas the unique edge of G incident to such a vertex of G is a pendant edge of G.*

Definition 2.1.7. *(see [4]) A walk in a graph G is an alternating sequence $W : v_0e_1v_1e_2v_2 \ldots e_nv_n$ of vertices and edges beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i; v_0 is the origin and v_n is the terminus of W. The walk is closed if $v_0 = v_n$ and is open otherwise. A walk is called a trail if all the edges appearing in the walk are distinct. It is called a path if all the vertices are distinct. A cycle is a closed trail in which the vertices are all distinct. Paths and cycles of n vertices are often denoted by P_n and C_n, respectively.*

Definition 2.1.8. *(see [7]) Two vertices u and v in a graph G are connected if there is a path in G from u to v. The graph is connected if every two vertices of G are connected.*

Definition 2.1.9. *(see [4]) A connected graph without cycles is defined as a tree.*

Definition 2.1.10. *(see [7]) For an integer $k \geq 1$, a graph G is a k-partite graph if $V(G)$ can be partitioned into k subsets $V_1, V_2, \ldots, V_k$(called partite sets) such that every edge joins vertices in two different partite sets. A complete k-partite graph G is a k-partite graph such that two vertices are adjacent in G if and only if the vertices belong to different partite sets. If $|V_i| = n_i$, for $1 \leq i \leq k$, then G is denoted by $K_{n_1,n_2,\ldots,n_k}$.*

A complete 2-partite graph is called a complete bipartite graph and it is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is called a star. A complete 3-partite graph is called complete tripartite graph and is denoted by K_{n_1,n_2,n_3}.

Definition 2.1.11. *(see [9]) Two graphs G and H are said to be disjoint if they have no vertex in common.*

2.2 Operations on Graphs

Let G_1 and G_2 be two graphs.

Definition 2.2.1. *(see [4]) The graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ is called the union of G_1 and G_2 and is denoted by $G_1 \cup G_2$. When G_1 and G_2 are disjoint, $G_1 \cup G_2$ is denoted by $G_1 + G_2$ and is called the sum of the graphs G_1 and G_2.*

Definition 2.2.2. *(see [4]) Let G_1 and G_2 be disjoint graphs. Then the join, $G_1 \vee G_2$ of G_1 and G_2 is the graph in which each vertex of G_1 is adjacent to every vertex of G_2.*

Definition 2.2.3. *(see [6]) The corona $G_1 \odot G_2$ of graphs G_1 and G_2 is the graph obtained by taking one copy of G_1, which has p_1 vertices, and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2.*

Definition 2.2.4. *(see [7]) The Cartesian Product of two graphs G_1 and G_2, commonly denoted by $G_1 \Box G_2$ or $G_1 \times G_2$, has vertex set $V(G) = V(G_1) \times V(G_2)$ and two distinct vertices (u, v) and (x, y) of $G_1 \Box G_2$ are adjacent if either $u = x$ and $vy \in E(G_2)$ or $v = y$ and $ux \in E(G_1)$.*

The cartesian product of graphs P_m and P_n denoted, $P_m \Box P_n$ is called a planar grid.

Definition 2.2.5. *(see [28]) Composition or lexicographic product of two graphs denoted by $G[H]$ has $V(G) \times V(H)$ as vertex set in which (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$.*

A graph G with a fixed vertex $u \in V(G)$ will be denoted by the ordered pair (G, u). Given two ordered pairs (G, u) and (H, v), one can construct another graph by linking these two graphs through identifying the vertices u and v. We will use the notation $(G, u) \diamond (H, v)$ for this construction or simply $G \diamond H$ if there is no ambiguity regarding the choices of u and v [18].

Definition 2.2.6. *(see [18]) Given n graphs $G_i (i = 1, 2, \ldots, n)$, the chain $G_1 \diamond G_2 \diamond \ldots G_n$ is the graph in which one of the vertices of G_i is identified with one of the vertices of G_{i+1}. If $G_i = G$, we use the notation $\diamond G_n$ for the n-link chain all of whose links are G.*

2.3 Line, Middle and Total Graphs

Definition 2.3.1. *(see [4]) The line graph of a graph G, denoted by $L(G)$, is a graph whose vertex set is in $1-1$ correspondence with the edge set of G and two vertices of $L(G)$ are joined by an edge if and only if the corresponding edges of G are adjacent in G.*

The concept of middle graph was introduced by J. Akiyama, T. Hamada and I. Yoshimura [1] in 1974.

Definition 2.3.2. *(see [1]) The middle graph of a graph G, denoted by $M(G)$, is the graph obtained from G by inserting a new vertex into every edge of G and by joining those pairs of these new vertices with edges which lie on adjacent edges of G.*

M. Behzad has introduced the notions of the total graph of a graph in [5].

Definition 2.3.3. *(see [5]) The total graph of a graph G, denoted by $T(G)$, is a new graph whose vertex set is the union of vertex and edge sets of G and two vertices of $T(G)$ are adjacent if they come from two adjacent vertices, two adjacent edges or an incident vertex with an edge.*

2.4 Basic Definitions from Group Theory and Ring Theory

Definition 2.4.1. *(see [11]) A binary operation $*$ on a set S is a rule that assigns to each ordered pair (a, b) of elements of S some element of S.*

Definition 2.4.2. *(see [11]) A binary operation on a set S is commutative if $a * b = b * a$ for all $a, b \in S$.*

Definition 2.4.3. *(see [11]) A binary operation on a set S is associative if $(a*b)*c = a*(b*c)$ for all $a, b, c \in S$.*

Definition 2.4.4. *(see [11]) A group $< G, * >$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:*

1 *The binary operation $*$ is associative.*

2 *There is an element e in G such that $e * x = x * e = x$ for all $x \in G$. (This element e is an identity element for $*$ on G.)*

3 *For each $a \in G$, there is an element a' in G with the property that $a' * a = a * a' = e$. (The element a' is an inverse of a with respect to the operation $*$.)*

Definition 2.4.5. *(see [11]) A group G is abelian if its binary operation $*$ is commutative.*

The group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$ is called the Klien 4-group. For simplicity we denote the elements by $0, a, b, c$ where $0+0 = 0,\ a+a = 0,\ b+b = 0,\ c+c = 0,\ a+b+c = 0,\ a+b = c,\ b+c = a,\ a+c = b$. The Klien 4-group is denoted by V_4.

Definition 2.4.6. *(see [11]) A ring $< R, +, . >$ is a set R together with two binary operations denoted by "+" and ".", which we call addition and multiplication, defined on R such that the following axioms are satisfied:*

1 *$< R, + >$ is an abelian group.*

2 *Multiplication is associative.*

3 *For all $a, b, c \in R$, the left distributive law, $a.(b+c) = a.b + a.c$ and the right distributive law $(a+b).c = a.c + b.c$ hold.*

Definition 2.4.7. *(see [11]) A ring in which the multiplication is commutative is a commutative ring. A ring with a multiplicative identity 1 such that $1x = x1 = x$ for all $x \in R$ is a ring with unity.*

Definition 2.4.8. *(see [11]) Let R be a ring with unity. An element $u \in R$ is a unit of R if it has a multiplicative inverse in R. If every nonzero element in R is a unit, then R is a division ring. A field is a commutative division ring.*

For a prime p, $\mathbb{Z}_p$ denotes the set of all integers modulo p. Observe that $\mathbb{Z}_p$ is a field.

Chapter 3

V_4-Magic Labelings of Wheel Related Graphs

In the first section of this chapter, we introduce a-sum and zero sum V_4 magic graphs. Some well known cycle related graphs and wheel related graphs are also included. In the second section of this chapter, we discuss some cycle related V_4 magic graphs. In the last section of this chapter we discuss some wheel related V_4 magic graphs.

3.1 Introduction

Let A be an abelian group. A mapping $\ell : E(G) \longrightarrow A \setminus \{0\}$ is called an edge labeling. We say that G is A-magic if there exists an edge labeling $\ell : E(G) \longrightarrow A \setminus \{0\}$ such that the induced vertex labeling $\ell^+ : V(G) \longrightarrow A$ defined by

$$\ell^+(v) = \sum \{\ell(uv) : uv \in E(G)\}$$

is a constant map [13]. If this constant is a where a is any nonzero element in A, then we say that ℓ is an a-sum A-magic labeling of G and G is said to be a-sum A-magic. We say that G is V_4-magic if there exists a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ such that the induced vertex labeling $\ell^+ : V(G) \longrightarrow V_4$ defined by

$$\ell^+(v) = \sum \{\ell(uv) : uv \in E(G)\}$$

is a constant map. If this constant is a where a is any nonzero element in V_4, then we say that ℓ is an a-sum V_4-magic labeling of G and G is said to be a-sum V_4-magic. If this constant is 0, then we say that ℓ is a zero-sum V_4-magic labeling of G and G is said to be zero-sum V_4-magic. In this chapter, we investigate a class of graphs in the following categories:

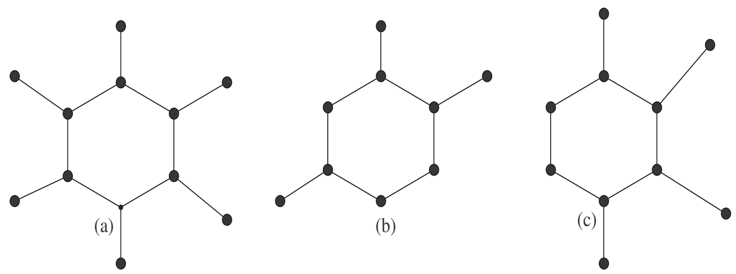


Figure 3.1: (a)Sun_6, (b)$BSun(6,3)$, (c) $CBSun(6,4)$

(i) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs,

(ii) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs,and

(iii) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero -sum V_4 magic.

Here we need the following.

Definition 3.1.1. *(see [6]) The sun on $2n$ vertices is a corona of the form $C_n \odot K_1$ where $n \geq 3$. The sun $C_n \odot K_1$ is denoted by Sun_n or S_n. A broken sun is a connected unicyclic subgraph of a sun. We denote by $BS(p,q)$ the set of broken suns with $n = p + q$ vertices and with a p-cycle, note that $BS(p;p) = C_p \odot K_1$. For $p > 2$ and $0 < q < p$, a consecutive broken sun, denoted by $CBSun(p,q)$ is the graph belonging to $BS(p,q)$ such that the subgraph induced by the vertices of degree 2 is a path on $p - q$ vertices. A broken sun (or a sun) is odd (resp. even) if p is odd (resp.even).*

Sun_6, $BSun(6,3)$ and $CBSun(6,4)$ are depicted in Figure 3.1.

Definition 3.1.2. *(see [18]) A wheel graph denoted by W_n is defined as $W_n = C_n + K_1$, where C_n for $n \geq 3$ is a cycle of length n.*

Definition 3.1.3. *(see [18]) A double-wheel graph $W_{n,2}$ can be obtained as join of $2C_n + K_1$, and inductively we can construct an m-level wheel graph denoted by $W_{n,m}$ as follows $W_{n,m} = mC_n + K_1$.*

Definition 3.1.4. *(see [12]) The helm H_n is the graph obtained from the wheel W_n by attaching a pendant edge at each vertex of the cycle C_n.*

Definition 3.1.5. *(see [12]) The web graph $W(2,n)$ is the graph obtained by joining the pendant points of a helm H_n to form a cycle and then adding a single pendant edge to each vertex of the outer cycle (see figure 3.2).*

Definition 3.1.6. *(see [12]) The generalized web graph $W(t,n)$ is the graph obtained by iterating the processes of constructing web graph $W(2,n)$ from the helm H_n, so that the web has t n-cycles (see figure 3.3).*

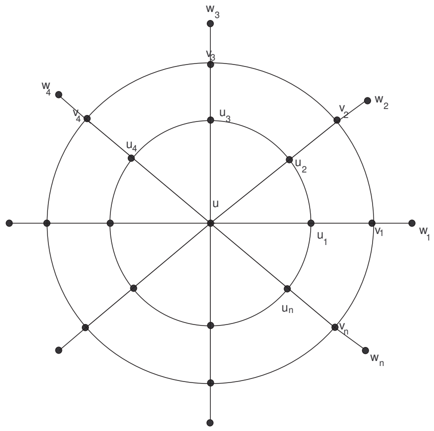


Figure 3.2: Web graph : $W(2, n)$

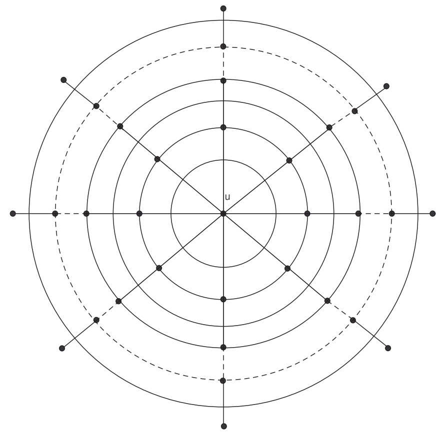


Figure 3.3: Generalised Web graph : $W(t, n)$,

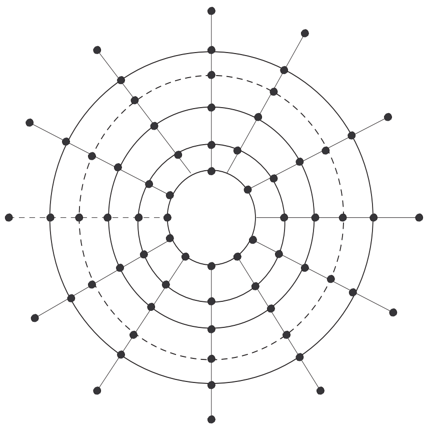


Figure 3.4: Generalised Web graph without centre: $W_0(t, n)$

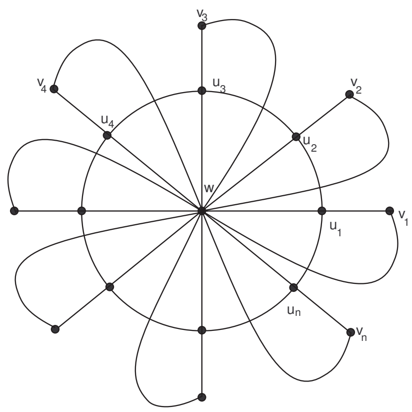


Figure 3.5: The flower graph: Fl_n

Definition 3.1.7. *(see [12]) The generalized web graph without center, $W_0(t, n)$ is the graph obtained by removing the central vertex of $W(t, n)$ (see figure 3.4).*

Definition 3.1.8. *(see [24]) A closed helm $H(2, n)$ is the graph obtained from a helm by joining each pendant vertex to form a cycle.*

Definition 3.1.9. *(see [24]) Closed generalized helms $H(t, n)$ are obtained by taking a generalized web and joining pendant vertices to form a cycle.*

Definition 3.1.10. *(see [24]) The flower graph Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to a central vertex of the helm (see figure 3.5).*

Definition 3.1.11. *(see [34]) The chain of cycles $C(n_1, n_2, \cdots, n_k)$ denotes the graph of k cycles $C_{n_1}, C_{n_2}, \cdots, C_{n_k}$ of sizes $n_1, n_2, \cdots, n_k$ such that C_{n_i} and $C_{n_{i+1}}$ have a common vertex, for $i = 1, 2, \cdots, k$.*

3.2 Cycle Related Graphs

Here we prove the following:

Lemma 3.2.1. *If G is a (p, q) graph with vertex set $\{v_1, v_2, \ldots, v_p\}$, and $\ell : E(G) \to V_4 \setminus \{0\}$ is a labeling of G, then*

$$\sum_{i=1}^{p} \ell^+(v_i) = 0.$$

Proof. Let v_i be any vertex of the graph G and $v_{i_1}, v_{i_2}, \cdots v_{i_k}$ be the k vertices adjacent to v_i. Then we have $\ell^+(v_i) = \sum_{j=1}^{k} \ell(v_i v_{i_j})$. In $\sum_{i=1}^{p} \ell^+(v_i)$, each $\ell(v_i v_j)$ appears twice. So $\sum_{i=1}^{p} \ell^+(v_i) = 0$. □

Theorem 3.2.2. *$C_n \in \mathscr{V}_a$ if and only if n is even.*

Proof. Assume that $C_n \in \mathscr{V}_a$. Then by lemma 3.2.1, we have $\sum_{i=1}^{n} \ell^+(u_i) = 0$. That is we get $na = 0$. This implies that n is even.
Conversely, assume that n is even. Define $\ell : E(C_n) \to V_4 \setminus \{0\}$ by

$$\ell(u_i u_{i+1}) = \begin{cases} b, & \text{for } i = 1, 3, \ldots, n-1, \\ c, & \text{for } i = 2, 4, \ldots, n. \end{cases}$$

Obviously, $\ell^+(u_i) = b + c = a$, for $i = 1, 2, \ldots, n$. Thus ℓ is an a-sum V_4-magic labeling of C_n. This completes the proof. □

Theorem 3.2.3. *$C_n \in \mathscr{V}_0$ for all $n \geq 3$.*

Proof. If we label all the edges of C_n by a, then we obtain $\ell^+(u_i) = 0$ for $i = 1, 2, \ldots, n$. □

Theorem 3.2.4. *If n is even, then $C_n \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from theorems 3.2.2 and 3.2.3. $\square$

Definition 3.2.5. *We denote by $C(n, \underbrace{k_1, k_2, \ldots, k_t}_{t})$ the class of all graphs obtained by identifying the apex vertices of t stars K_{1,k_i} $(i = 1, 2, \ldots, t)$ with t $(1 \leq t \leq n)$ vertices of C_n. Observe that $C(n, \underbrace{k, k, \ldots, k}_{n})$ is a unique graph.*

Theorem 3.2.6. *If $C(n, k_1, k_2, \ldots, k_t) \subset \mathscr{V}_a$, then $n + k_1 + k_2 + \cdots + k_t$ is even.*

Proof. Observe that each member of $C(n, k_1, k_2, \ldots, k_t)$ has $n + k_1 + k_2 + \cdots + k_t$ vertices. By lemma 3.2.1, $(n + k_1 + k_2 + \cdots + k_t)a = 0$. This implies that $n + k_1 + k_2 + \cdots + k_t$ is even. $\square$

Conjecture 3.2.7. *If $n + k_1 + k_2 + \cdots + k_t$ is even, then $C(n, k_1, k_2, \ldots, k_t) \subset \mathscr{V}_a$.*

We prove some special cases of conjecture 3.2.7.

Theorem 3.2.8. *If $n + tk$ is even, then $C(n, \underbrace{k, k, \ldots, k}_{t}) \subset \mathscr{V}_a$.*

Proof. Consider a graph G in the set $C(n, \underbrace{k, k, \ldots, k}_{t})$. We consider 4 cases:

Case 1: Suppose n, k and t are even. In this case, we label all edges of C_n as described in the proof of theorem 3.2.2 and label all the pendant edges by a. Then obviously this is an a-sum V_4-magic labeling of G.

Case 2: Suppose n, k are even and t is odd. In this case, the labeling is exactly similar to Case 1.

Case 3: Suppose n and t are even and k is odd. Without loss of generality assume that apex vertices of the t stars are at $u_1, u_{i_1}, u_{i_2}, \ldots, u_{i_t}$, $1 < i_1 < i_2 < \ldots < i_{t-1}$ of the cycle C_n. First, label all pendant edges by a. We label the edges of C_n as follows:
Consider the vertex u_{i_1}. If i_1 is even, we label the edges $u_n u_1, u_1 u_2, \ldots, u_{i_1} u_{i_1+1}$ as follows:

$$\begin{aligned}
\ell(u_n u_1) &= b, \\
\ell(u_i u_{i+1}) &= b, \text{ for } i = 1, 3, \ldots, i_1 - 1 \\
\ell(u_i u_{i+1}) &= c, \text{ for } i = 2, 4, \ldots, i_1 - 2 \\
\ell(u_{i_1} u_{i_1+1}) &= b.
\end{aligned}$$

If i_1 is odd, we label the edges $u_n u_1, u_1 u_2, \ldots, u_{i_1} u_{i_1+1}$ as follows

$$\begin{aligned}
\ell(u_n u_1) &= b, \\
\ell(u_i u_{i+1}) &= \begin{cases} b, & \text{for } i = 1, 3, \ldots, i_1 - 2 \\ c, & \text{for } i = 2, 4, \ldots, i_1 - 1. \end{cases}
\end{aligned}$$

$$\ell(u_{i_1}u_{i_1+1}) = c.$$

So, we have

$$\ell(u_{i_1-1}u_{i_1}) = \ell(u_{i_1}u_{i_1+1}) = \begin{cases} b & \text{if } i_1 \text{ even} \\ c & \text{if } i_1 \text{ odd} \end{cases}$$

Therefore for all i_1,

$$\ell^+(u_{i_1}) = a.$$

Furthermore,

$$\ell^+(u_i) = a \text{ for all } i.$$

Next, consider the vertex u_{i_2}. Here we consider the following cases:

i_1 and i_2 are even: In this case we label the edges $u_{i_1+1}u_{i_1+2}, u_{i_1+2}u_{i_1+3}, \ldots, u_{i_2}u_{i_2+1}$ consecutively by $c, b, c, b, \ldots, c, b, c, c$.

i_1 is even and i_2 is odd : In this case we label the edges $u_{i_1+1}u_{i_1+2}, u_{i_1+2}u_{i_1+3}, \ldots,$ $u_{i_2}u_{i_2+1}$ consecutively by $c, b, c, b, \ldots, c, b, b$.

i_1 is odd and i_2 is even : In this case we label the edges $u_{i_1+1}u_{i_1+2}, u_{i_1+2}u_{i_1+3}, \ldots,$ $u_{i_2}u_{i_2+1}$ consecutively by $b, c, b, c, \ldots, b, c, c$.

i_1 and i_2 are odd : In this case we label the edges $u_{i_1+1}u_{i_1+2}, u_{i_1+2}u_{i_1+3}, \ldots, u_{i_2}u_{i_2+1}$ consecutively by $b, c, b, c, \ldots, c, b, b$.

Proceeding like this, we eventually arrive at u_{i_t}. If i_t is even, then obviously $\ell(u_{i_t}u_{i_t-1}) = \ell(u_{i_t}u_{i_t+1}) = c$. Then label the edges $u_{i_t+1}u_{i_t+2}, u_{i_t+2}u_{i_t+3}, \ldots, u_{n-2}u_{n-1}$ consecutively by $b, c, b, c, \ldots, b, c$. If i_t is odd, then obviously $\ell(u_{i_t}u_{i_t-1}) = \ell(u_{i_t}u_{i_t+1}) = b$. Then label the edges $u_{i_t+1}u_{i_t+2}, u_{i_t+2}u_{i_t+3}, \ldots, u_{n-2}u_{n-1}$ consecutively by $c, b, c, b, \ldots, c$. Obviously this labeling is an a-sum V_4-magic labeling of G.

Case 4: n, k and t are odd. In this case, the labeling is similar to case 3.

This completes the proof. □

Theorem 3.2.9. *$Sun_n \in \mathscr{V}_a$ for all n.*

Proof. Observe that $\mathrm{Sun}_n = C(n, \underbrace{1, 1, \ldots, 1}_{n})$. So, the proof of the theorem follows from theorem 3.2.8. □

Theorem 3.2.10. *$Sun_n \notin \mathscr{V}_0$ for all n.*

Proof. Since Sun_n has pendant edges, $Sun_n \notin \mathscr{V}_0$. □

Theorem 3.2.11. *$BSun(p, q) \subset \mathscr{V}_a$ if and only if $p + q$ is even.*

Proof. Observe that any member in $BSun(p,q)$ has $p+q$ number of vertices. If $BSun(p,q) \in \mathscr{V}_a$, then by lemma 3.2.1, $(p+q)a = 0$. This implies that $p+q$ is even. Converse part follows from theorem 3.2.8. ☐

Theorem 3.2.12. *$CBSun(p,q) \subset \mathscr{V}_a$ if and only if $p+q$ is even.*

Proof. Proof follows from theorem 3.2.8. ☐

Theorem 3.2.13. *$C_n \odot K_2 \in \mathscr{V}_a$ if and only if n is even.*

Proof. Note that $C_n \odot K_2$ has $3n$ vertices. If $C_n \odot K_2 \in \mathscr{V}_a$, then by lemma 3.2.1, $3na = 0$. This implies that $na = 0$. Consequently, n is even. Conversely assume that n is even. Let $u_1, u_2, \dots u_n$ be the vertices of C_n and $v_{ij}, i = 1, 2, \dots, n,\ j = 1, 2$ be that of K_2. Now label the edges as follows:

$$\ell(u_i u_{i+1}) = \begin{cases} b, & i = 1, 3, \dots, n-1 \\ c, & i = 2, 4, \dots, n \end{cases}$$

$$\ell(u_i v_{ij}) = b$$

$$\ell(v_{ij} v_{i(j+1)}) = c$$

This gives an a-sum V_4-magic labeling of $C_n \odot K_2$. ☐

Theorem 3.2.14. *$C_n \odot K_2 \in \mathscr{V}_0$ for all $n \geq 3$.*

Proof. Label all the edges by a. The proof follows. ☐

Theorem 3.2.15. *If n is even, then $C_n \odot K_2 \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from theorems 3.2.13 and 3.2.14. ☐

Theorem 3.2.16. *$C_n \odot C_m \in \mathscr{V}_a$ if and only if $n(m+1)$ is even.*

Proof. Suppose $C_n \odot C_m \in \mathscr{V}_a$. Then by lemma 3.2.1, $n(m+1)a = 0$. This implies that $n(m+1)$ is even.

Conversely assume that $n(m+1)$ is even. Let the vertices of C_n be $(u_1, u_2, \dots, u_n, u_1)$ and let the vertices of C_m be $v_{ij}, i = 1, 2, \dots, n,\ j = 1, 2, \dots, m$ where $v_{i1} = u_i$. We consider the following cases.

Case 1: Assume that n is even and m is odd. Label all the edges by a. Obviously ℓ is an a-sum V_4-magic labeling of $C_n \odot C_m$.

Case 2: Suppose m and n are odd. In this case the labeling is exactly similar to Case 1.

Case 3: Suppose both m and n are even. Define $\ell : E(C_n \odot C_m) \to V_4 \setminus \{0\}$ by

$$\ell(u_i u_{i+1}) = \begin{cases} b & \text{for} \quad i = 1, 3, \ldots, n-1, \\ c & \text{for} \quad i = 2, 4, \ldots, n \end{cases}$$

For $\; i = 1, 2, \ldots, n :$

$$\ell(v_{ij} v_{i(j+1)}) = \ell(u_i v_{ij}) = a, \quad \text{for} \quad j = 1, 2, \ldots, m$$

end for

This completes the proof. □

Theorem 3.2.17. *$C_n \odot C_m \in \mathscr{V}_0$ for all $m \geq 3$ and $n \geq 3$.*

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(C_m) = \{v_1, v_2, \ldots, v_m\}$. We consider the following cases:

Case 1: Suppose n and m are even. Define $\ell : E(C_n \odot C_m) \to V_4 \setminus \{0\}$ by

$$\ell(u_i u_{i+1}) = a \quad \text{for} \quad i = 1, 2, \ldots, n,$$

$$\ell(v_i v_{i+1}) = \begin{cases} b & \text{for} \quad i = 1, 3, \ldots, m-1, \\ c & \text{for} \quad i = 2, 4, \ldots, m. \end{cases}$$

For $\; j = 1, 2, \ldots, n :$

$$\ell(u_j v_i) = a \quad \text{for} \quad i = 1, 2, \ldots, m.$$

end for

Obviously ℓ is a zero sum V_4-magic labeling of $C_n \odot C_m$.

Case 2: Suppose n is even and m is odd: Define $\ell : E(C_n \odot C_m) \to V_4 \setminus \{0\}$ by

$$\ell(u_j u_{j+1}) = a \quad \text{for} \quad j = 1, 2, \ldots, n,$$

$$\ell(v_i v_{i+1}) = \begin{cases} b & \text{for} \quad i = 1, 3, \ldots, m-2, \\ c & \text{for} \quad i = 2, 4, \ldots, m-1, \\ a & \text{for} \quad i = m. \end{cases}$$

For $\; j = 1, 2, \ldots, n :$

$$\ell(u_j v_i) = a \quad \text{for} \quad i = 2, \ldots, m-1,$$

$$\ell(u_j v_m) = b, \; \ell(u_j v_1) = c.$$

end for

Then we have ℓ is a zero sum V_4-magic labeling of $C_n \odot C_m$.

$$\ell^+(u_j) = a + a + (m-2)a + b + c = 0, \quad \text{for } j = 1, 2, \ldots, n,$$
$$\ell^+(v_i) = a + b + c = 0, \quad \text{for } i = 1, 2, \ldots, m.$$

Case 3: Suppose m and n are odd. In this case the labeling is exactly similar to Case 2.

Case 4: Suppose n is odd and m is even. In this case the labeling is exactly similar to Case 1.

This completes the proof. □

Theorem 3.2.18. *If $n(m+1)$ is even, then $C_n \odot C_m \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from theorems 3.2.16 and 3.2.17. □

Theorem 3.2.19. *$C_n \odot K_m \in \mathscr{V}_a$ if and only if $n(m+1)$ is even.*

Proof. Observe that $C_n \odot K_m$ has $n+mn$ vertices. If $C_n \odot K_m \in \mathscr{V}_a$, then we have $(m+1)na = 0$. This implies that $n(m+1)$ is even. We consider 3 cases:
Let the vertices of C_n be $u_1, u_2, \ldots, u_n$. We denote the j^{th} copy of K_m by K_m^j. Let the vertices of K_m^j be $\{v_{j,1}, v_{j,2}, \ldots, v_{j,m}\}$.

Case 1: Suppose n is even and m is odd. In this case label all the edges of K_m^j by a. Obviously, this is an a-sum V_4-magic labeling of $C_n \odot K_m$.

Case 2: Suppose n is even and m is even. In this case, first we label all edges of K_m^j by b, $j = 1, 2, \ldots, n$. Next, label all edges of C_n by $b, c, b, c, \ldots$ consecutively. Finally, label all edges $u_i v_{j,r}$ by c for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$; $r = 1, 2, \ldots, m$. Obviously, this is an a-sum V_4-magic labeling of $C_n \odot K_m$.

Case 3: Suppose n and m are odd. In this case, first we label all edges of K_m^j by b, $j = 1, 2, \ldots, n$. Next, label all edges of C_n by a. Finally, label all edges $u_i v_{j,r}$ by a for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$; $r = 1, 2, \ldots, m$. Obviously, this is an a-sum V_4-magic labeling of $C_n \odot K_m$.

This completes the proof. □

Theorem 3.2.20. *$C_n \odot \overline{K_m} \in \mathscr{V}_a$ if and only if $n(m+1)$ is even, where $\overline{K_m}$ is the complement of the complete graph with m vertices.*

Proof. Note that the graph $C_n \odot \overline{K_m}$ has $n(m+1)$ vertices. If $C_n \odot \overline{K_m} \in \mathscr{V}_a$, then we have $n(m+1)$ is even. Conversely, assume that $n(m+1)$ is even. Consider n copies of $\overline{K_m}$. Let $\overline{K_m}^j$ denotes the j^{th} copy of $\overline{K_m}$. Let

$$V(C_n) = \{u_1, u_2, \ldots, u_n\},$$
$$V(\overline{K_m}^j) = \{v_{j,1}, v_{j,2}, \ldots, v_{j,m}\},\ j = 1, 2, \ldots, n.$$

We consider 3 cases:

Case 1: Suppose n is even and m is odd. Define $\ell : V(C_n \odot \overline{K_m}) \to V_4 \setminus \{0\}$ by

$$\begin{aligned}
&\text{For } i = 1, 2, \ldots, n :\\
&\qquad \ell(u_i v_{j,r}) = a,\ j = 1, 2, \ldots, n; r = 1, 2, \ldots, m\\
&\qquad \ell(u_i u_{i+1}) = a\\
&\text{end for}
\end{aligned}$$

Then, we have ℓ is an a-sum V_4-magic labeling of $C_n \odot \overline{K_m}$.

Case 2: Suppose n and m are even. Define $\ell : V(C_n \odot \overline{K_m}) \to V_4 \setminus \{0\}$ by

$$\ell(u_i u_{i+1}) = \begin{cases} b, & \text{for } i = 1, 3, \ldots, n-1, \\ c, & \text{for } i = 2, 4, \ldots, n. \end{cases}$$

$$\begin{aligned}
&\text{For } i = 1, 2, \ldots, n :\\
&\qquad \ell(u_i v_{j,r}) = a,\ j = 1, 2, \ldots, n; r = 1, 2, \ldots, m\\
&\text{end for}
\end{aligned}$$

Then, we have ℓ is an a-sum V_4-magic labeling of $C_n \odot \overline{K_m}$.

Case 3: Suppose n and m are odd. In this case, the labeling is exactly similar to case 1.

This completes the proof. □

Theorem 3.2.21. *$C_n \odot \overline{K_m} \notin \mathscr{V}_0$ for all m and n.*

Proof. Since it has pendant edges, $C_n \odot \overline{K_m} \notin \mathscr{V}_0$ for all m and n. □

Theorem 3.2.22. *If $n(m+1)$ is even, then $C_n \odot \overline{K_m} \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from theorems 3.2.20 and 3.2.21. □

Theorem 3.2.23. *$C_m \diamond C_n \in \mathscr{V}_a$ if and only if $m + n$ is odd.*

Proof. Let the vertices of C_m and C_n be respectively, $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$. Assume that u_1 and v_1 are identified with a new vertex w. Then we have, $\sum_{i=2}^{m} \ell^+(u_i) + \sum_{i=2}^{n} \ell^+(v_i) + \ell^+(w) = 0$. This implies that $(m+n)$ is odd.

Conversely, assume that $m + n$ is odd. Then we consider two cases:

Case 1: Suppose m is even and n is odd. Define a mapping $\ell : E(C_m \diamond C_n) \to V_4 \setminus \{0\}$ by

$$\ell(u_i u_{i+1}) = \begin{cases} b & \text{for} \quad i = 1, 3, \ldots, m-1, \\ c & \text{for} \quad i = 2, 4, \ldots, m, \end{cases}$$

$$\ell(v_iv_{i+1}) = \begin{cases} c & \text{for} \quad i = 1, 3, \ldots, n, \\ b & \text{for} \quad i = 2, 4, \ldots, n-1. \end{cases}$$

Clearly ℓ is an a-sum magic labeling of $C_m \diamond C_n$.

Case 2: Suppose m is odd and n is even. The remaining part is exactly similar to Case 1.

This completes the proof. □

Theorem 3.2.24. *$\diamond[C_n]_m \in \mathscr{V}_a$ if and only if m is odd and n is even.*

Proof. Observe that $\diamond[C_n]_m$ has $mn - m + 1$ vertices. If $\diamond[C_n]_m \in \mathscr{V}_a$, then we have $[m(n-1)+1]a = 0$. This implies that $m(n-1)$ is odd. Consequently, m is odd and n is even.

Conversely, assume that n is even and m is odd. Consider m copies of C_n. Let the vertices of the i^{th} cycle C_n^i be $(u_1^i, u_2^i, \ldots, u_n^i, u_1^i)$. First, consider the pairs (C_n^1, u_1^1) and (C_n^2, u_1^2) and construct $G_1 = (C_n^1, u_1^1) \diamond (C_n^2, u_1^2)$. Next consider the pairs (G_1, u_2^2) and (C_n^3, u_1^3) and construct $G_2 = (G_1, u_2^2) \diamond (C_n^3, u_1^3)$. Proceeding like this, we finally arrive at $G_{m-1} = (G_{m-2}, u_2^{m-1}) \diamond (C_n^m, u_1^m)$. We need to show that $G = G_1 \diamond G_2 \diamond \cdots \diamond G_{m-1} \in \mathscr{V}_a$. We label the edges of G by the following table:

$i\backslash$edge	$u_1^iu_2^i$	$u_2^iu_3^i$	$u_3^iu_4^i$	$u_4^iu_5^i$	$u_5^iu_6^i$	$u_6^iu_7^i$	$\ldots$	$u_{n-1}^iu_n^i$	$u_n^iu_1^i$
1	b	c	b	c	b	c	$\ldots$	b	c
2	b	b	c	b	c	b	$\ldots$	c	b
3	b	c	b	c	b	c	$\ldots$	b	c
4	b	b	c	b	c	b	$\ldots$	c	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
m	b	c	b	c	b	c	$\ldots$	b	c

One can easily verify that this is an a-sum V_4-magic labeling of G. This completes the proof. □

Theorem 3.2.25. *$C_m \diamond C_n \in \mathscr{V}_0$ for all m and n.*

Proof. Label all the edges by a, we obtain $\ell^+ \equiv 0$. □

Theorem 3.2.26. *If $m + n$ is odd, then $C_m \diamond C_n \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from 3.2.23 and 3.2.25. □

Theorem 3.2.27. *$\diamond C_n \in \mathscr{V}_0$.*

Proof. If we label all edges of $\diamond C_n$ by a, we obtain a zero sum V_4-magic labeling of $\diamond C_n$. □

3.3 Wheel Related Graphs

Theorem 3.3.1. *$W_n \in \mathscr{V}_a$ if and only if n is odd.*

Proof. Suppose W_n admits an a- sum V_4-magic labeling. Then by lemma 3.2.1, we have $\sum_{i=1}^{n} \ell^+(u_i) = \ell^+(u)$ which shows that $na = a,\ a \neq 0$. This implies that n is odd.

Conversely, assume that n is odd. We will prove that W_n admits an a-sum V_4-magic labeling. Let $\ell : E(W_n) \to V_4 \setminus \{0\}$ be a labeling of W_n such that $\ell(uu_i) = a$ for all i . Since n is odd, $\sum_{i=1}^{n} \ell(uu_i) = a$. Thus $\ell^+(u) = a$. Note that $\ell(u_iu_{i+1}) \in V_4$ for $i = 1, 2, \ldots, n$, where $u_{n+1} = u_1$. Therefore, $2\sum_{i=1}^{n} \ell(u_iu_{i+1}) = 0$. This implies that $\sum_{i=1}^{n} \ell(u_iu_{i+1}) = 0, a, b$ or c. Without loss of generality assume that $\sum_{i=1}^{n} \ell(u_iu_{i+1}) = 0$. The other cases are similar. Note that $\sum_{i=1}^{n} \ell(u_iu_{i+1}) = 0$ can be written as:

$$\ell(u_1u_2) + \sum_{i=2}^{n} \ell(u_iu_{i+1}) = 0. \tag{3.1}$$

Let us take $\ell(u_1u_2) = a$. One can assign b or c to $\ell(u_1u_2)$ instead of a. If $\ell(u_1u_2) = a$, the second term in equation (3.1) can be taken as a. That is,

$$\sum_{i=2}^{n} \ell(u_iu_{i+1}) = a. \tag{3.2}$$

Note that equation (3.2) can be written as:

$$\ell(u_2u_3) + \sum_{i=3}^{n} \ell(u_iu_{i+1}) = a. \tag{3.3}$$

For an a-sum V_4-magic graph, we need

$$\ell(u_1u_2) + \ell(u_2u_3) + \ell(uu_2) = a.$$

This equation implies that $\ell(u_2u_3) = a$. Hence $\ell^+(u_2) = a$. From equation (3.3), we have $\sum_{i=3}^{n} \ell(u_iu_{i+1}) = 0$. That is,

$$\sum_{i=3}^{n} \ell(u_iu_{i+1}) = 0 \tag{3.4}$$

Again, equation (3.4) can be written as:

$$\ell(u_3u_4) + \sum_{i=4}^{n} \ell(u_iu_{i+1}) = 0. \tag{3.5}$$

For an a-sum V_4-magic graph, we need

$$\ell(u_3u_4) + \ell(uu_3) + \ell(u_2u_3) = a$$

This implies that $\ell(u_3u_4) = a$. Hence $\ell^+(u_3) = a$, If we continue this process we finally arrive at $\ell(u_nu_1) = a$ and $\ell^+(u_1) = a$. Thus ℓ is an a-sum V_4-magic labeling of W_n. □

A step by step procedure for finding an a-sum magic map for W_n, when n is odd is given below:

1. For $i = 1, 2, \ldots$, set $\ell(uu_i) = a$ or b or c.
2. Consider the equation $\sum_{i=1}^{n} \ell(u_iu_{i+1}) = 0$. Assume that $\ell(uu_i) = a$.
3. Split 0 into two parts. We have the following possibilities:
   $a + a = 0, \quad b + b = 0, \quad c + c = 0$
4. Consider the first sum $a + a = 0$ and take $\ell(u_1u_2)$ as a. Then $\sum_{i=2}^{n} \ell(u_iu_{i+1}) = a$. One can consider the other two cases also.
5. Split the summation $\sum_{i=2}^{n} \ell(u_iu_{i+1}) = a$ in the following form
   $$\ell(u_2u_3) + \sum_{i=3}^{n} \ell(u_iu_{i+1}) = a.$$
   Find the value of $\ell(u_2u_3)$ from the following equation:
   $$\ell(u_1u_2) + \ell(uu_2) + \ell(u_2u_3) = a.$$
6. Continue this processes up to the $(n-1)^{\text{th}}$ step. Finally the value of $\ell(u_nu_1)$ is determined by the equation:
   $$\ell(u_{n-1}u_n) + \ell(uu_n) + \ell(u_1u_n) = a$$

Observe that a-sum V_4-magic labeling of W_n is not unique. The following is another procedure for obtaining an a-sum V_4-magic labeling of W_n.

(1) Consider the equation
$$\sum_{i=1}^{n} \ell(u_iu_{i+1}) = 0, a, b \text{ or } c. \tag{3.6}$$
Without loss of generality assume that
$$\sum_{i=1}^{n} \ell(u_iu_{i+1}) = 0. \tag{3.7}$$
The equation (3.7) can be written as:
$$\ell(u_1u_2) + \sum_{i=2}^{n} \ell(u_iu_{i+1}) = 0. \tag{3.8}$$

Assign a or b or c to $\ell(u_1u_2)$. Let us assign a to $\ell(u_1u_2)$. Then from equation (3.8) one obtain,

$$\sum_{i=2}^{n} \ell(u_iu_{i+1}) = a. \tag{3.9}$$

Equation (3.9) can be written as:

$$\ell(u_2u_3) + \sum_{i=3}^{n} \ell(u_iu_{i+1}) = a. \tag{3.10}$$

Assign any value to $\ell(u_2u_3)$ from the set $\{a, b, c\}$. Let us assume that $\ell(u_2u_3) = b$. Choose $\ell(uu_2)$ such that

$$\ell(u_1u_2) + \ell(uu_2) + \ell(u_2u_3) = a. \tag{3.11}$$

Hence we have $\ell^+(u_2) = a$.

(2) From equation (3.9), we have

$$\sum_{i=3}^{n} \ell(u_iu_{i+1}) = c \tag{3.12}$$

Applying the same procedure as explained above, one obtain:

$$\ell^+(u_3) = a.$$

(3) Continue the above processes. Finally, we obtain $\ell^+(u_1) = a$.

(4) Since ℓ is a labeling of W_n, by lemma 3.2.1, we have

$$\ell^+(u) = \sum_{i=1}^{n} \ell^+(u_i) = na$$

Since n is odd, we have $na = a$. Therefore, we have $\ell^+(u) = a$.

Theorem 3.3.2. *$W_n \in \mathscr{V}_0$, if n is odd.*

Proof. Suppose n is odd. Define $\ell : E(W_n) \to V_4 \setminus \{0\}$ by

$$\ell(uu_i) = \begin{cases} a, & \text{for } i = 1, 2, \ldots, n-2, \\ c, & \text{for } i = n-1, \\ b, & \text{for } i = n. \end{cases}$$

$$\ell(u_iu_{i+1}) = \begin{cases} b, & \text{for } i = 1, 3, \ldots, n-2 \\ c, & \text{for } i = 2, 4, \ldots, n-3 \\ a, & \text{for } i = n-1 \\ c, & \text{for } i = n. \end{cases}$$

Obviously ℓ is a zero sum V_4-magic labeling of W_n. $\square$

Theorem 3.3.3. *$W_n \in \mathscr{V}_0$, if n is even*

Proof. Let $\ell : E(W_n) \to V_4 \setminus \{0\}$ be a labeling of W_n. Then $2\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0$. This implies that $\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0, a, b$ or c. Without loss of generality assume that

$$\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0. \tag{3.13}$$

Rest of the proof is exactly similar to the algorithm for finding the a-sum V_4-magic labeling of W_n explained above subject to the condition that no element will repeat consecutively on the outer circle of W_n. $\square$

Theorem 3.3.4. *If $n \equiv 0(\text{mod } 3)$, then W_n admits a zero sum V_4-magic labeling.*

Proof. Define $\ell : E(W_n) \to V_4 \setminus \{0\}$ as follows:

$$\ell(u_i u_{i+1}) = \begin{cases} b, & \text{for } i = 1, 4, 7, \ldots, n-2, \\ c, & \text{for } i = 2, 5, 8, \ldots, n-1, \\ a, & \text{for } i = 3, 6, 9, \ldots, n. \end{cases}$$

$$\ell(uu_i) = \begin{cases} c, & \text{for } i = 1, 4, 7, \ldots, n-2, \\ a, & \text{for } i = 2, 5, 8, \ldots, n-1, \\ b, & \text{for } i = 3, 6, 9, \ldots, n. \end{cases}$$

Obviously ℓ is a zero sum magic labeling of W_n. $\square$

Theorem 3.3.5. *If W_n is a-sum V_4-magic and if k is odd, then W_{nk} is a-sum V_4-magic.*

Proof. Assume that W_n is a- sum V_4 magic. Then by theorem 3.3.1, we have n is odd. Since k is odd this implies that nk is odd. Hence theorem 3.3.1 tells us that W_{nk} is a-sum V_4 magic. $\square$

Next, we will explain a procedure for obtaining an a-sum V_4-magic labeling W_{nk} if an a-sum V_4-magic labeling of W_n is known.
Let $C_{n,1} : v_1, v_2, v_3, \ldots, v_n, v_1$ and v be the center vertex of W_n. Let $C_{nk,1} : u_1, u_2, \ldots, u_{nk}, u_1$ and u be the center vertex of $W_{nk,1}$. Let $\ell : E(W_n) \to V_4 \setminus \{0\}$ be an a-sum V_4-magic labeling of W_n. Whenever $m \equiv i(\text{mod } n)$, define a function $\ell' : E(W_{nk}) \to V_4 \setminus \{0\}$ by

$$\ell'(uu_m) = \ell(vv_i), \text{ for } m \equiv i(\text{mod } n)$$
$$\ell'(u_m u_{m+1}) = \ell(v_i v_{i+1}), \text{ for } m \equiv i(\text{mod } n).$$

If ℓ'^{+} is the induced vertex labeling of W_{nk}, then $\ell'^{+}(u) = k\ell^{+}(v) = ka = a$ and

$$\begin{aligned}\ell'^{+}(u_i) &= \ell'(u_{m-1}u_m) + \ell'(uu_m) + \ell'(u_m u_{m+1}) \\ &= \ell(u_{(m-1)(\text{mod } n)} u_{m(\text{mod } n)}) + \ell(uu_{m(\text{mod } n)}) + \ell(u_{m(\text{mod } n)} u_{(m+1)(\text{mod } n)}) = a.\end{aligned}$$

Hence ℓ' is an a-sum V_4-magic labeling of W_{nk}.

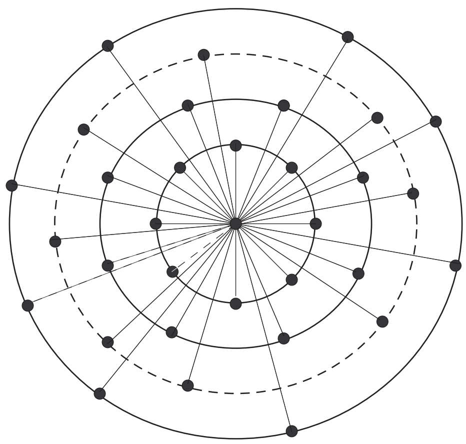


Figure 3.6: An m-level Wheel Graph: $W_{n,m}$

Theorem 3.3.6. *If W_n is zero-sum V_4 magic, so is W_{kn} for every $k \geq 2$.*

Let $C_{n,1}, \ldots, C_{n,m}$ represent the cycles of $W_{n,m}$ at levels $1, \ldots, m$, respectively, as shown in figure 3.6. Let $u_{1,j}, u_{2,j}, \ldots, u_{n,j}, u_{1,j}$ are the vertices of the cycle $C_{n,j}$ and u is the central vertex of $W_{n,m}$.

Theorem 3.3.7. *$W_{n,m} \in \mathscr{V}_a$ if and only if both m and n are odd.*

Proof. If $W_{n,m} \in \mathscr{V}_a$, then by lemma 3.2.1, we have $(mn)a = a$. This implies that mn is odd or equivalently m and n are both odd.

Conversely, assume that both m and n are odd. If we label all the edges of $W_{n,m}$ by a, then obviously $\ell^+(u) = a$ and $\ell^+(u_{ij}) = a$. □

Theorem 3.3.8. *$W_{n,m} \in \mathscr{V}_0$ for all m and n.*

Proof. We consider the following cases.

Case 1. m is odd and n is even. Define a labeling $\ell : W_{n,m} \to V_4 \setminus \{0\}$ by:

$$\ell(uu_{ij}) = a$$
$$\ell(u_{ij}u_{(i+1)j}) = \begin{cases} b, & i = 1, 3, \ldots, n-1 \\ c, & i = 2, 4, \ldots, n \end{cases}$$

Case 2. m is even and n is odd.

Define a labeling $\ell : W_{n,m} \to V_4 \setminus \{0\}$ by:

$$\ell(uu_{ij}) = a, \ i = 1, 4, \ldots, n$$

$$\ell(uu_{2j}) = b,\ \ell(uu_{3j}) = c$$
$$\ell(u_{2j}u_{3j}) = a$$
$$\ell(u_{ij}u_{(i+1)j}) = \begin{cases} c, & i = 1, 4, 6, \ldots, n-1 \\ b, & i = 3, 5, 7, \ldots, n. \end{cases}$$

Case 3. m and n are even: The proof is similar to Case 1.

Case 4. m and n are odd: The proof is similar to Case 2.

This completes the proof. □

Theorem 3.3.9. *If $W_{n,m} \in \mathscr{V}_a$, then $W_{kn,m} \in \mathscr{V}_a$ if k is odd.*

Proof. $W_{n,m} \in \mathscr{V}_a$ implies that mn is odd. This implies that both m and n are odd. Now, $W_{kn,m} \in \mathscr{V}_a$ if mnk is odd. This implies that k is odd. □

Definition 3.3.10. *A subdivided wheel graph denoted by SW_n is obtained by dividing each spoke uu_i. Similarly, we can define the subdivided m-level graph $SW_{n,m}$.*

Theorem 3.3.11. *$SW_n \notin \mathscr{V}_a$ for any $n \geq 3$.*

Proof. Assume that SW_n admits an a-sum V_4-magic labeling. Then by lemma 3.2.1, we have $na + na = a$. This implies that $a = 0$. The proof follows. □

Theorem 3.3.12. *$SW_n \in \mathscr{V}_0$ for any $n \geq 3$.*

Proof. We consider two cases:

Case 1: If n is even, define $\ell : E(SW_n) \to V_4 \setminus \{0\}$ as follows:

$$\ell(uv_i) = a,\ \text{for } i = 1, 2, 3, \ldots, n,$$
$$\ell(v_iu_i) = a,\ \text{for } i = 1, 2, 3, \ldots, n,$$
$$\ell(u_iu_{(i+1)}) = \begin{cases} b, & \text{for } i = 1, 3, \ldots, n-1, \\ c, & \text{for } i = 2, 4, \ldots, n. \end{cases}$$

Obviously $\ell^+(u) = \ell^+(u_i) = \ell^+(v_i) = 0$. Hence $SW_n \in \mathscr{V}_0$ if n is even.

Case 2: If n is odd, define $\ell : E(SW_n) \to V_4 \setminus \{0\}$ as follows:

$$\ell(uv_i) = \begin{cases} a, & \text{for } i = 1, 2, 3, \ldots, n-2, \\ b, & \text{for } i = n-1, \\ c, & \text{for } i = n. \end{cases}$$

$$\ell(v_iu_i) = \begin{cases} a, & \text{for } i = 1, 2, \ldots, n-2, \\ b, & \text{for } i = n-1, \\ c, & \text{for } i = n. \end{cases}$$

$$\ell(u_i u_{(i+1)}) = \begin{cases} b, & \text{for } i = 2, 4, \ldots, n-3, \\ c, & \text{for } i = 1, 3, \ldots, n-2, \\ a, & \text{for } i = n-1 \\ b, & \text{for } i = n. \end{cases}$$

Observe that $\ell^+(u) = \ell^+(u_i) = \ell^+(v_i) = 0$. Hence $SW_n \in \mathscr{V}_0$ if n is odd.

This completes the proof. $\square$

Let $u_{1,j}, u_{2,j}, \ldots, u_{n,j}, u_{1,j}$ are the vertices of the cycle $C_{n,j}$, $v_{1,j}, v_{2,j}, \ldots, v_{n,j}$ are the subdivisions corresponding to the edges $uu_{i,j}$ and u is the central vertex of $SW_{n,m}$.

Theorem 3.3.13. *$SW_{n,m} \notin \mathscr{V}_a$ for any n and m.*

Proof. By lemma 3.2.1, we have $(2mn)a = a$ which is impossible. Hence the proof follows. $\square$

Theorem 3.3.14. *$SW_{n,m} \in \mathscr{V}_0$ for any n and m.*

Proof. We consider two cases:

Case 1: If n is even, for $j = 1, 2, \ldots, m$, define $\ell : E(SW_{n,m}) \to V_4 \setminus \{0\}$ as follows:

$$\ell(uv_{i,j}) = a \text{ for } i = 1, 2, 3, \ldots, n,$$
$$\ell(v_{i,j}u_{i,j}) = a \text{ for } i = 1, 2, 3, \ldots, n,$$
$$\ell(u_{i,j}u_{i+1,j}) = \begin{cases} b, & \text{for } i = 1, 3, \ldots, n-1, \\ c, & \text{for } i = 2, 4, \ldots, n. \end{cases}$$

Obviously ℓ is a zero-sum magic labeling of $SW_{n,m}$.

Case 2: If n is odd, for $j = 1, 2, \ldots, m$, define $\ell : E(SW_{n,m}) \to V_4 \setminus \{0\}$ as follows:

$$\ell(uv_{i,j}) = \begin{cases} a, & \text{for } i = 1, 2, 3, \ldots, n-2, \\ b, & \text{for } i = n-1 \\ c, & \text{for } i = n. \end{cases}$$

$$\ell(v_{i,j}u_{i,j}) = \begin{cases} a, & \text{for } i = 1, 2, 3, \ldots, n-2, \\ b, & \text{for } i = n-1, \\ c, & \text{for } i = n. \end{cases}$$

$$\ell(u_{i,j}u_{i+1,j}) = \begin{cases} c, & \text{for } i = 1, 3, \ldots, n-2 \\ b, & \text{for } i = 2, 4, \ldots, n-3 \\ a, & \text{for } i = n-1 \\ b, & \text{for } i = n. \end{cases}$$

Obviously ℓ is a zero-sum magic labeling of $SW_{n,m}$.

This completes the proof. $\square$

Theorem 3.3.15. *$H_n \notin \mathscr{V}_a$ for any n.*

Proof. By lemma 3.2.1 we have, $\sum_{i=1}^{n} \ell^+(u_i) + \sum_{i=1}^{n} \ell^+(v_i) = \ell^+(u)$ where $u_1, u_2, \ldots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \ldots, v_n$ are the pendant vertices corresponding to the spokes uu_i and u is the central vertex of W_n. Suppose $H_n \in \mathscr{V}_a$ for some n. This implies $na + na = a$ which holds if and only if $a = 0$. Hence $H_n \notin \mathscr{V}_a$ for any n. $\square$

Theorem 3.3.16. *$H_n \notin \mathscr{V}_0$ for any n.*

Proof. Since it has pendant edges, $H_n \notin \mathscr{V}_0$ for any n. $\square$

Theorem 3.3.17. *$W(2, n) \in \mathscr{V}_a$ if and only if n is odd.*

Proof. Assume that $W(2, n) \in \mathscr{V}_a$. Then from lemma 3.2.1, we have $\sum_{i=1}^{n} \ell^+(u_i) + \sum_{i=1}^{n} \ell^+(v_i) + \sum_{i=1}^{n} \ell^+(w_i) = \ell^+(u)$ where $u_1, u_2, \ldots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \ldots, v_n$ are the vertices of $C_{n,2}$, $w_1, w_2, \ldots, w_n$ are the pendant vertices and u is the centre of $W(2, n)$. Thus we get $na + na + na = a$. This implies that $na = a$. This equation holds if and only if n is odd.

Conversely, assume that n is odd. Define a mapping $\ell : E(W(2, n)) \to V_4 \setminus \{0\}$ by

$$\ell(uu_i) = a \quad \text{for} \quad i = 1, 2, \ldots, n,$$

$$\ell(u_iu_{i+1}) = \begin{cases} b, & \text{for} \quad i = 1, 3, \ldots, n-2, \\ c, & \text{for} \quad i = 2, 4, \ldots, n-1, \\ a, & \text{for} \quad i = n. \end{cases}$$

$$\ell(v_iv_{i+1}) = \begin{cases} b, & \text{for} \quad i = 1, 3, \ldots, n-2, \\ c, & \text{for} \quad i = 2, 4, \ldots, n-1, \\ a, & \text{for} \quad i = n. \end{cases}$$

$$\ell(u_iv_i) = \begin{cases} c, & \text{for} \quad i = 1, \\ a, & \text{for} \quad i = 2, 3, \ldots, n-1, \\ b, & \text{for} \quad i = n. \end{cases}$$

$$\ell(v_iw_i) = a.$$

Obviously, $\ell^+(u) = a$ and $\ell^+(u_i) = \ell^+(v_i) = \ell(w_i) = a$ for $i = 1, 2, \ldots, n$. $\square$

An a-sum V_4-magic labeling is shown in figure 3.7.

Theorem 3.3.18. *$W(2, n) \notin \mathscr{V}_0$ for any n.*

Proof. Since it has pendant edges, $W(2, n) \notin \mathscr{V}_0$ for any n. $\square$

Theorem 3.3.19. *$W(t, n) \in \mathscr{V}_a$ if and only if n is odd and t is even.*

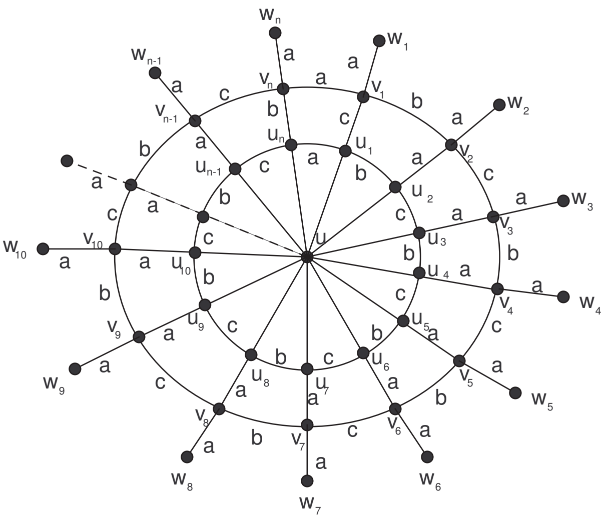


Figure 3.7: An a-sum V_4-magic labeling of $W(2,n)$

Proof. Assume that $W(t,n) \in \mathscr{V}_a$. Then by lemma 3.2.1, we have $\sum_{i=1}^{t}\sum_{i=1}^{n}\ell^+(u_{i,j}) + \sum_{i=1}^{n}\ell^+(v_i) = \ell^+(u)$ where $u_{1,j}, u_{2,j}, \ldots, u_{n,j}$ are the vertices of the cycle $C_{n,j}$, $j = 1, 2, \ldots, t$ and $v_1, v_2, \ldots, v_n$ are the pendant vertices and u is the hub of $W(2,n)$. That is $n(t+1)a = a$. This implies that n is odd and t is even. Conversely, assume that n is odd and t is even. Define a labeling $\ell : E(W(t,n)) \to V_4 \setminus \{0\}$ by:

$$\text{For } i = 1, 2, \ldots, n \text{ do:}$$
$$\ell(uu_{i,1}) = \ell(u_{i,t}v_i) = a$$
$$\ell(u_{i,j}u_{i,(j+1)}) = a, \text{ for } j = 1, 2, \ldots, t$$
$$\text{end for}$$
$$\text{For } j = 1, 2, \ldots, t \text{ do:}$$
$$\ell(u_{ij}u_{(i+1)j}) = \begin{cases} b, & i = 1, 3, \ldots, n-2 \\ c, & i = 2, 4, \ldots, n-1 \\ a, & i = n. \end{cases}$$
$$\text{end for}$$

With this labeling $W(t,n) \in \mathscr{V}_a$ for all n and t. ☐

Theorem 3.3.20. *$W(t,n) \notin \mathscr{V}_0$ for any n and any t.*

Proof. Since it has pendant edges, $W(t,n) \notin \mathscr{V}_0$ for any n and any t. ☐

Theorem 3.3.21. *$W_0(t,n) \in \mathscr{V}_a$ if and only if $n(t+1)$ is even.*

Proof. First, assume that $W_0(t,n) \in \mathscr{V}_a$. Then by lemma 3.2.1, we have $nta + na = 0$. This implies that $n(t+1)$ is even.

Conversely, assume that $n(t+1)$ is even. We consider the following cases:

Case 1: If n and t are even, define $\ell : E(W_0(t,n)) \to V_4 \setminus \{0\}$ as follows:

$$\ell(u_{i,1}u_{(i+1),1}) = a \text{ for } i = 1,2,3,\ldots,n$$

For $\quad j = 2,3,\ldots,t:$

$$\ell(u_{i,j}u_{i+1,j}) = \begin{cases} c, & \text{for } i = 1,3,\ldots,n-1 \\ b, & \text{for } i = 2,4,\ldots,n \end{cases}$$

end for

For $\quad j = 1,2,\ldots,n:$

$$\ell(u_{i,j}u_{i,j+1}) = a \text{ for } i = 1,2,\ldots,t-1$$

end for

$$\ell(u_{i,t}v_i) = a \text{ for } i = 1,2,3,4,\ldots,n.$$

Case 2: Assume that n is even and t is odd. In this case the labeling is exactly similar to Case 1.

Case 3: If n is odd and t is odd, define $\ell : E(W_0(t,n)) \to V_4 \setminus \{0\}$ as follows:

$$\ell(u_{i,1}u_{(i+1),1}) = a \text{ for } i = 1,2,3,\ldots,n$$

For $\quad j = 2,3,\ldots,t:$

$$\ell(u_{i,j}u_{i+1,j}) = \begin{cases} b, & \text{for } i = 1,3,\ldots,n-2 \\ c, & \text{for } i = 2,4,\ldots,n-1. \end{cases}$$

end for

$$\ell(u_{n,j}u_{1,j}) = a \text{ for } j = 1,2,\ldots,t$$

$$\ell(u_{i,t}v_i) = a, \text{ for } i = 1,2,\ldots,n$$

For $\quad k = 2,3,\ldots,n-1:$

$$\ell(u_{k,j}u_{k,j+1}) = a, \text{ for } j = 1,2,3,\ldots,t-1,$$

end for

$$\ell(u_{1,j}u_{1,j+1}) = \begin{cases} a, & \text{for } j = 1,3,\ldots,t-2, \\ c, & \text{for } j = 2,4,\ldots,t-1, \end{cases}$$

$$\ell(u_{n,j}u_{n,j+1}) = \begin{cases} a, & \text{for } j = 1,3,\ldots,t-2, \\ b, & \text{for } j = 2,4,\ldots,t-1. \end{cases}$$

Obviously $\ell^+(u_{i,j}) = a$ and $\ell^+(v_i) = a$.

□

Theorem 3.3.22. *$W_0(t,n) \notin \mathscr{V}_0$ for any n and t.*

Proof. Since it has pendant edges, $W_0(t,n) \notin \mathscr{V}_0$ for any n and t. □

Theorem 3.3.23. *$H(2,n) \notin \mathscr{V}_a$ for any n.*

Proof. Assume that $H(2,n) \in \mathscr{V}_a$. Then by lemma 3.2.1, we have $\sum_{i=1}^{n} \ell^+(u_i) + \sum_{i=1}^{n} \ell^+(v_i) = \ell^+(w)$ where $u_1, u_2, \ldots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \ldots, v_n$ are the vertices of the cycle $C_{n,2}$ and w is the central vertex. From this we get $na + na = a$. This implies that $a = 0$. This is a contradiction. □

Theorem 3.3.24. *$H(2,n) \in \mathscr{V}_0$ for all n.*

Proof. **Case 1:** Assume that n is even. Define a labeling $\ell : E(H(2,n)) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned} &\text{For} \quad i = 1, 2, \ldots, n, \\ &\ell(u_i w) = \ell(u_i u_{i+1}) = \ell(u_i v_i) = a \\ &\text{end for} \\ &\ell(v_i v_{i+1}) = \begin{cases} b, & \text{for} \quad i = 1, 3, \ldots, n-1, \\ c, & \text{for} \quad i = 2, 4, \ldots, n. \end{cases} \end{aligned}$$

Obviously, ℓ is a zero- sum V_4-magic labeling of $H(2,n)$.

Case 2: Assume that n is odd. Define a labeling $\ell : E(H(2,n)) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned} \ell(u_1 w) &= a,\ \ell(u_2 w) = b,\ \ell(u_3 w) = c, \\ \ell(u_i w) &= a, \quad \text{for} \quad i = 4, 5, \ldots, n, \\ \ell(u_i u_{i+1}) &= a \quad \text{for} \quad i = 1, 2, \ldots, n, \\ \ell(u_1 v_1) &= a,\ \ell(u_2 v_2) = b,\ \ell(u_3 v_3) = c, \\ \ell(u_i v_i) &= a, \quad \text{for} \quad i = 4, 5, \ldots, n, \\ \ell(v_1 v_2) &= c,\ \ell(v_2 v_3) = a,\ \ell(v_3 v_4) = b, \\ \ell(v_i v_{i+1}) &= c \quad \text{for} \quad i = 4, 6, \ldots, n-1, \\ \ell(v_i v_{i+1}) &= b \quad \text{for} \quad i = 5, 7, \ldots, n. \end{aligned}$$

One can easily verify that ℓ is a zero sum magic labeling of $H(2,n)$.

□

Theorem 3.3.25. *$H(t,n) \in \mathscr{V}_a$ if and only if both n and t are odd.*

Proof. First, assume that $H(t,n) \in \mathscr{V}_a$. Then by lemma 3.2.1, we have $\sum_{j=1}^{t} \sum_{i=1}^{n} \ell^+(u_{i,j}) = \ell^+(w)$ where $u_{1,j}, u_{2,j}, \ldots, u_{n,j}$ are the vertices of the cycle $C_{n,j}$, and w is the central vertex.

Hence we get $(nt+1)a = 0$. This implies that both n and t are odd. Conversely, assume that both n and t are odd. Define $\ell : E(H(t,n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
&\ell(u_{1,1}w) = a,\\
&\ell(u_{i,1}w) = b, \quad \text{for} \quad i = 2,3,\ldots,n,\\
&\text{For } j = 1,2,\ldots,t-1:\\
&\qquad \ell(u_{i,j}u_{i+1,j}) = \begin{cases} c, & \text{for} \quad i = 1,3,\ldots,n-2,\\ b, & \text{for} \quad i = 2,4,\ldots,n-1,n \end{cases}\\
&\text{end for}\\
&\ell(u_{i,t}u_{i+1,t}) = \begin{cases} b, & \text{for} \quad i = 1,3,\ldots,n,\\ a, & \text{for} \quad i = 2,4,\ldots,n-1 \end{cases}\\
&\text{For } j = 1,2,\ldots,t-1:\\
&\qquad \ell(u_{1,j}u_{1,j+1}) = a,\\
&\qquad \ell(u_{i,j}u_{i,j+1}) = b, \quad \text{for} \quad i = 2,3,\ldots,n-1,\\
&\text{end for}\\
&\ell(u_{n,j}u_{n,j+1}) = \begin{cases} c, & \text{for} \quad j = 1,3,\ldots,t-2,\\ b, & \text{for} \quad j = 2,4,\ldots,t-1. \end{cases}
\end{aligned}$$

Obviously ℓ is an a-sum magic labeling of $H(t,n)$. □

Theorem 3.3.26. *$H(t,n) \in \mathscr{V}_0$ for all n and t.*

Proof. **Case 1:** Assume that n is even. Define $\ell : E(H(t,n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
&\text{For} \quad i = 1,2,\ldots,n:\\
&\qquad \ell(u_{i,1}w) = a,\\
&\qquad \ell(u_{i,j}u_{i,j+1}) = \ell(u_{i,j}u_{i+1,j}) = a, \quad \text{for} \quad j = 1,2,\ldots,t-1,\\
&\text{end for}\\
&\ell(u_{i,t}u_{i+1,t}) = \begin{cases} b, & \text{for} \quad i = 1,3,\ldots,n-1,\\ c, & \text{for} \quad i = 2,4,\ldots,n. \end{cases}
\end{aligned}$$

Obviously, ℓ is a zero sum V_4-magic labeling of $E(H(t,n)$.

Case 2: Assume that n is odd. Define $\ell : E(H(t,n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
&\ell(u_{1,1}w) = a, \quad \ell(u_{2,1}w) = b \quad \ell(u_{3,1}w) = c\\
&\ell(u_{i,1}w) = a, \quad \text{for} \quad i = 4,5,\ldots,n,\\
&\text{For} \quad j = 1,2,\ldots,t-1:\\
&\qquad \ell(u_{i,j}u_{i+1,j}) = a, \quad \text{for} \quad i = 1,2,\ldots,n,\\
&\qquad \ell(u_{1,j}u_{1,j+1}) = a,\ \ell(u_{2,j}u_{2,j+1}) = b,\ \ell(u_{3,j}u_{3,j+1}) = c,
\end{aligned}$$

$$\ell(u_{i,j}u_{i,j+1}) = a, \quad \text{for} \quad i = 4, 5, , \ldots, n,$$

end for

$$\ell(u_{1,t}u_{2,t}) = c,\ \ell(u_{2,t}u_{3,t}) = a,\ \ell(u_{3,t}u_{4,t}) = b,$$

$$\ell(u_{i,t}u_{i+1,t}) = c, \quad \text{for} \quad i = 4, 6, , \ldots, n-1,$$

$$\ell(u_{i,t}u_{i+1,t}) = b, \quad \text{for} \quad i = 5, 7, , \ldots, n.$$

Obviously ℓ is a zero sum magic labeling of $E(H(t, n)$.

□

Theorem 3.3.27. *$H(t, n) \in \mathscr{V}_{a,0}$ if and only if both n and t are odd.*

Proof. Proof follows from theorems 3.3.25 amd 3.3.26. □

Theorem 3.3.28. *$Fl_n \notin \mathscr{V}_a$ for any n.*

Proof. Suppose $Fl_n \in \mathscr{V}_a$. Then by lemma 3.2.1, we have $\sum_{i=1}^{n} \ell^+(u_i) + \sum_{i=1}^{n} \ell^+(v_i) = \ell^+(w)$ where u_i are the vertices on the cycle, v_i are the corresponding vertices outside the cycle and w is the central vertex. Thus we get $na + na = a$. This implies that $a = 0$ which is a contradiction. □

Theorem 3.3.29. *$Fl_n \in \mathscr{V}_0$ for all n.*

Proof. If we label all the edges by a, we obtain that, $\ell^+(u_i) = \ell^+(v_i) = \ell^+(w) = 0$. □

Corollary 3.3.30. *$Fl_n \notin \mathscr{V}_{a,0}$ for any n.*

Proof. The proof follows from theorems 3.3.28 and 3.3.29. □

Chapter 4

V_4-Magic Labelings of Shell Related Graphs

In the first section of this chapter, definition of some shell related graphs are provided. In the second section of this chapter, we discuss some shell related V_4 magic graphs. In the last section of this chapter we discuss some multiple shell related V_4 magic graphs.

4.1 Introduction

For positive integers n, k, $1 \leq k \leq n-3$, $H(n,k)$ is used to denote the cycle C_n with k chords sharing a common endpoint called the apex. In general $H(n,k)$ represents a family of graphs. For certain choices of n and k, the family $H(n,k)$ may be singleton. For example, when $k = n-3$, the family $H(n, n-3)$ is singleton, called a shell (see figure 4.1) [10]. Observe that the shell $H(n, n-3)$ is the same as the fan $F_{n-1} = P_{n-1} + K_1$. For, $2 \leq p \leq n-r$, let $C_n(p,r)$ denote cycle $C_n : (v_0, v_1, \ldots, v_{n-1}, v_0)$ with consecutive r chords $v_0v_p, v_0v_{p+1}, \ldots, v_0v_{p+r-1}$.

Definition 4.1.1. *(see [14]) An umbrella graph $U(m,n)$ is defined to be a graph obtained by joining a path P_n with the apex of a shell $H(m, m-3)$.*

Definition 4.1.2. *(see [14]) An extended umbrella graph $U(m,n,k)$ is a graph obtained by identifying the pendant vertex of the umbrella $U(m,n)$ with the center(apex) of the star $K_{1,k}$.*

Definition 4.1.3. *(see [25]) A multiple shell $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r})$ is a graph formed by t_i shells of width n_i each, $1 \leq i \leq r$, which have a common apex.*

Thus a multiple shell is a one point union of many shells. Observe that the multiple shell $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r})$ has $\sum_{i=1}^{n}(n_i - 1)t_i + 1$ vertices. If there are k shells with a common apex, then it is called a k- tuple shell.

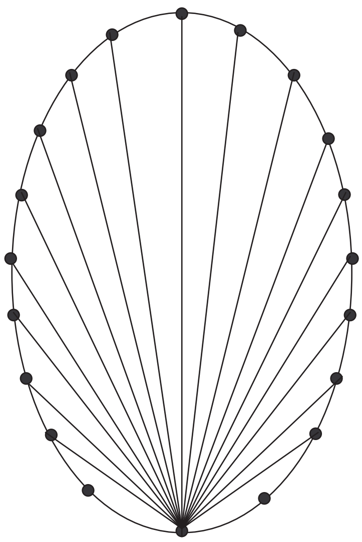


Figure 4.1: The Shell graph $H(n, n-3)$

Definition 4.1.4. *(see [25]) A multiple shell is said to be balanced if it is of the form $MS(p^t)$ or of the form $MS(p^t, (p+1)^s)$.*

4.2 Shell Related Graphs

Theorem 4.2.1. *$H(n, n-3) \in \mathscr{V}_a$ if and only if n is even.*

Proof. Assume that $H(n, n-3) \in \mathscr{V}_a$. Then $\ell^+(u_i) = a$ for $i = 0, 1, \ldots, n-1$. Then by lemma 3.2.1, we have

$$\sum_{i=0}^{n-1} \ell^+(u_i) = 0. \tag{4.1}$$

Thus we get $na = 0$. This implies that n is even.

Conversely, assume that n is even. We need to show that $H(n, n-3) \in \mathscr{V}_a$. Let the vertices of $H(n, n-3)$ be $v_0, v_1, \ldots, v_{n-1}$. Assume that v_0 be the apex of $H(n, n-3)$. Define $\ell : E(H(n, n-3)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_0 v_i) = \begin{cases} c, & \text{for } i = 1, n-1, \\ a, & \text{for } i = 2, 3, \ldots, n-2, \end{cases}$$

$$\ell(v_i v_{i+1}) = b, \quad \text{for } i = 1, 2, 3, \ldots, n-2.$$

Then we have

$$\ell^+(v_i) = \begin{cases} c + c + (n-3)a = a, & \text{for } \ i = 0, \\ b + c = a, & \text{for } \ i = 1, n-1, \\ b + b + a = a, & \text{for } \ i = 2, 3, \ldots, n-2. \end{cases}$$

This completes the proof. □

Theorem 4.2.2. *(see [28]) $H(n, n-3) \in \mathscr{V}_0$ if n is even.*

Theorem 4.2.3. *If n is even, $H(n, n-3) \in \mathscr{V}_{a,0}$.*

Proof. From theorem 4.2.1 we have $H(n, n-3) \in \mathscr{V}_a$ if and only if n is even. From theorem 4.2.2 it follows that $H(n, n-3) \in \mathscr{V}_0$ if n is even. Combining this the proof follows. □

Theorem 4.2.4. *$U(n, m) \notin \mathscr{V}_a$ if $m \geq 2$.*

Proof. Since any graph with a path pendant of length at least two is non-magic, $U(n, m) \notin \mathscr{V}_a$ if $m \geq 2$. □

Theorem 4.2.5. *$U(n, 1) \in \mathscr{V}_a$ if n is odd.*

Proof. Let the vertices of $U(n, 1)$ be $\{v_0, v_1, v_2, \ldots, v_{n-1}, u_n\}$, where v_0 is the apex of $H(n, n-3)$ and u_n is the pendant vertex. Define $\ell : E(U(n, 1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_0 v_i) = \begin{cases} c, & \text{for } i = 1, n-1, n, \\ a, & \text{for } i = 2, 3, \ldots, n-2, \end{cases}$$

$$\ell(v_i v_{i+1}) = b, \text{ for } i = 1, 2, 3, \ldots, n-2.$$

Then we have,

$$\ell^+(v_i) = \begin{cases} c + c + (n-3)a + a = a, & \text{for } i = 0, \\ b + c = a, & \text{for } i = 1, n-1, n, \\ b + b + a = a, & \text{for } i = 2, 3, \ldots, n-2. \end{cases}$$

This completes the proof. □

Theorem 4.2.6. *$U(m, n, k) \notin \mathscr{V}_a$ if $n \geq 2$.*

Proof. Assume that $n \geq 2$ and $U(m, n, k) \in \mathscr{V}_a$. Let v_0 be the apex of $H(m, m-3)$ and u_{n-1} be the apex of $K_{1,k}$. Let $V(H(m, m-3)) = \{v_0, v_1, \ldots, v_{m-1}\}$, $V(P_n) = \{v_0, u_1, u_2, \ldots, u_{n-1}\}$ and $V(K_{1,n}) = \{u_{n-1}, w_1, w_2, \ldots, w_k\}$. Since $\ell^+(v) = a$ for all $v \in V(U(m, n, k))$, we can label all pendant vertices of $U(m, n, k)$ by a. Assume that $\ell(u_{n-2} u_{n-1}) = x$, $x \in V_4 \setminus \{0\}$. Since $\ell^+(u_{n-1}) = a$, $ka + x = a$. This implies that $x = (k-1)a$. Hence, $x = 0$, if k is odd and $x = a$, if k is even. Observe that $x = 0$ is not admissible. Moreover, $x = a$ implies that $\ell(u_{n-3}u_{n-2}) = 0$. This is also not admissible. This completes the proof. □

Theorem 4.2.7. *If $U(m,1,k) \in \mathscr{V}_a$ then $m + k$ is odd.*

Proof. Observe that $U(m,1,k)$ has $m+k+1$ vertices. If $U(m,1,k) \in \mathscr{V}_a$, then by lemma 3.2.1 we have $(m+k+1)a = 0$. This implies that $m+k$ is odd. □

Theorem 4.2.8. *If m is odd and k is even, then $U(m,1,k) \in \mathscr{V}_a$.*

Proof. Let $V(H(m,m-3)) = \{v_0, v_1, \ldots, v_{m-1}\}$ and $V(K_{1,k}) = \{u_0, u_1, \ldots, u_k\}$. Define $\ell : U(m,1,k) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
\ell(v_0v_i) &= \begin{cases} c & \text{for } i = 1, m-1, \\ a, & \text{for } i = 2, 3, \ldots, m-2, \end{cases} \\
\ell(v_iv_{i+1}) &= b \;\text{ for }\; i = 1, 2, 3, \ldots, m-2, \\
\ell(v_0u_0) &= a. \\
\ell(u_0u_j) &= a \;\text{ for }\; j = 1, 2, \ldots, k.
\end{aligned}$$

Then we have,

$$\ell^+(v_i) = \begin{cases} c + c + (m-3)a + a = a, & \text{for } \; i = 0, \\ b + c = a, & \text{for } \; i = 1, n-1, \\ b + b + a = a, & \text{for } i = 2, 3, \ldots, m-2, \end{cases}$$

$$\ell^+(u_i) = \begin{cases} ka + a = a & \text{for } i = 0, \\ a, & \text{for } i = 1, 2, 3, \ldots, k. \end{cases}$$

This completes the proof. □

Theorem 4.2.9. *If m is even and k is odd, then $U(m,1,k) \notin \mathscr{V}_a$.*

Proof. Label all the pendant edges of the star by a and label the edge v_0u_0 by x. If $U(m,1,k) \in \mathscr{V}_a$, then $ka + x = a$. This implies that $x = 0$. This is a contradiction. The result now follows. □

Let $B(t, n_1, n_2, \ldots, n_t)$ be the graph obtained by identifying each pendant vertex v_i of the star $K_{1,t}$ with apex of shells $H(n_i, n_i - 3)$, $i = 1, 2, \ldots, t$. Then we have the following:

Theorem 4.2.10. *If $B(t, n_1, n_2, \ldots, n_t) \in \mathscr{V}_a$, then $n_1 + n_2 + \cdots + n_t$ is odd.*

Proof. Observe that $B(t, n_1, n_2, \ldots, n_t)$ has $n_1 + n_2 + \cdots + n_t + 1$ vertices. So, we have by lemma 3.2.1, $(n_1 + n_2 + \cdots + n_t + 1)a = 0$. This implies that $n_1 + n_2 + \cdots + n_t$ is odd. □

Theorem 4.2.11. *If n and t are odd then $B(t,n,n,\ldots,n) \in \mathscr{V}_a$.*

Proof. Let the vertex set of $K_{1,t}$ be $\{v_0, v_1, v_2, \ldots, v_t\}$, where v_0 is the apex. Consider t copies of the shell $H(n, n-3)$. Let $H^i(n, n-3)$ be the i^{th} copy of $H(n, n-3)$. Let the vertex set of $H^i(n, n-3)$ be $\{v_i, v_1^i, v_2^i, \ldots, v_{n-1}^i\}$, where v_i is the apex. Define a labeling $\ell : E(B(t,n,n,\ldots,n)) \to V_4 \setminus \{0\}$ by

$$\ell(v_0 v_i) = a, \text{ for } i = 1, 2, \ldots, t,$$

$$\text{For } i = 1, 2, \ldots, t:$$

$$\begin{cases} \ell(v_i v_1^i) = c, \\ \ell(v_i v_{n-1}^i) = c, \\ \ell(v_j^i v_{j+1}^i) = b, \text{ for } j = 1, 2, \ldots, n-2, \\ \ell(v_i v_j^i) = a, \text{ for } j = 2, 3, \ldots, n-2. \end{cases}$$

$$\text{end for}$$

Obviously ℓ is an a-sum magic labeling of $B(t,n,n,\ldots,n)$. ◻

Let $H(2n, n-2)$ be the graph obtained by taking the cycle $C_{2n} : (v_0, v_1, \ldots, v_{2n-1}, v_0)$ and its chords $v_0v_3, v_0v_5, \ldots, v_0v_{2n-3}$. Observe that $H(2n, n-2)$ has $n-2$ chords. We have the following theorem:

Theorem 4.2.12. *$H(2n, n-2) \in \mathscr{V}_a$ for all n.*

Proof. We consider two cases.

Case 1: Assume that n is even. Let $n = 2t$. Observe that in this case, the graph $H(2n, n-2)$ has $4t$ vertices. Let the vertex set of $H(2n, n-2)$ be $\{v_0, v_1, \ldots, v_{4t-1}\}$, where v_0 is the apex. For convenience, we denote the vertex v_0 by v_{4t}. Define $\ell : E(H(2n, n-2)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \begin{cases} c & \text{for } \ i = 1, 4t-1, \\ b & \text{for } \ i = 2, 4t, \end{cases}$$

$$\ell(v_{i-1}v_i) = \ell(v_i v_{i+1}) = \begin{cases} c & \text{for } \ i = 3, 7, 11, \ldots, 4t-5, \\ b & \text{for } \ i = 5, 9, 13, \ldots, 4t-3, \end{cases}$$

$$\ell(v_0 v_i) = a \ \text{ for } \ i = 3, 5, 7, \ldots, 4t-3.$$

Obviously,

$$\ell^+(v_i) = \begin{cases} b + c + (2t-2)a = a & \text{for } \ i = 0, \\ b + c = a, & \text{for } \ i = 1, 2, 4, 6, \ldots, 2t+4, 4t-4, 4t-2, 4t-1, \\ c + c + a = a, & \text{for } \ i = 3, 7, \ldots, 4t-5, \\ b + b + a = a, & \text{for } \ i = 5, 9, \ldots, 4t-3. \end{cases}$$

Case 2: Assume that n is odd. Let $n = 2t+1$. In this case, the graph has $4t+2$ vertices. Let the vertex set of $H(2n, n-2)$ be $\{v_0, v_1, v_2, \ldots, v_{4t+2}\}$. For convenience, we denote the vertex v_0 by v_{4t+2}. Define $\ell : E(H(2n, n-2)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \begin{cases} c & \text{for } \ i = 1, 4t+2, \\ b & \text{for } \ i = 2, 4t+1, \end{cases}$$

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } \ i = 3, 7, 11, \ldots, 4t-5, 4t-1, \\ b & \text{for } \ i = 5, 9, 13, \ldots, 4t-3, \end{cases}$$

$$\ell(v_0v_i) = a, \quad \text{for } \ i = 3, 5, 9, \ldots, 4t-1.$$

Obviously,

$$\ell^+(v_i) = \begin{cases} c + c + (2t-1)a = a, & \text{for } \ i = 0, \\ b + c = a, & \text{for } \ i = 1, 2, 4, 6, \ldots, 4t, 4t+1, \\ c + c + a = a, & \text{for } \ i = 3, 7, \ldots, 4t-1, \\ b + b + a = a, & \text{for } \ i = 5, 9, \ldots, 4t-3. \end{cases}$$

This completes the proof.

□

Theorem 4.2.13. *$H(2n, n-2) \in \mathscr{V}_0$ for all n.*

Proof. We consider two cases.

Case 1: Suppose n is even. Let $n = 2t$. Let the vertex set of $H(2n, n-2)$ be $\{v_0, v_1, v_2, \ldots, v_{4t-1}\}$. Define $\ell : E(H(2n, n-2)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } \ i = 2, 6, 10, 14, \ldots, 4t-6, 4t-2, 4t, \\ b & \text{for } \ i = 4, 8, 12, \ldots, 4t-4, \end{cases}$$

$$\ell(v_0v_i) = a \text{ for } \ i = 3, 5, 7, \ldots, 4t-3.$$

where $v_{4t} = v_0$. Obviously,

$$\ell^+(v_i) = \begin{cases} c + c + (2t-2)a = 0 & \text{for } \ i = 0, \\ c + c = 0, & \text{for } \ i = 1, 2, 6, \dots, 4t-2, 4t-1, \\ a + b + c = 0, & \text{for } \ i = 3, 5, 7, \dots, 4t-3, \\ b + b = 0, & \text{for } \ i = 4, 8, 12, \dots, 4t-4. \end{cases}$$

Case 2: Assume that n is odd. Let $n = 2t + 1$. In this case, the graph has $4t + 2$ vertices. Define $\ell : E(H(2n, n-2)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \ell(v_i v_{i+1}) = \begin{cases} c & \text{for } \ i = 1, 2, 6, 10, 14, \dots, 4t-2, \\ b & \text{for } \ i = 4, 8, 12, \dots, 4t-4, 4t, 4t+1, \end{cases}$$

$$\ell(v_0 v_i) = a \text{ for } \ i = 3, 5, 7, \dots, 4t-1,$$

where $v_{4t+2} = v_0$. Obviously,

$$\ell^+(v_i) = \begin{cases} b + c + (2t-1)a = 0, & \text{for } \ i = 0, \\ c + c = 0, & \text{for } \ i = 1, 2, 6, \dots, 4t-2, \\ a + b + c = 0, & \text{for } \ i = 3, 5, \dots, 4t-1, \\ b + b = 0, & \text{for } \ i = 4, 8, \dots, 4t, 4t+1. \end{cases}$$

This completes the proof. ☐

Theorem 4.2.14. $H(2n, n-2) \in \mathscr{V}_{a,0}$.

Proof. From theorem 4.2.12 we have, $H(2n, n-2) \in \mathscr{V}_a$ for all n and from theorem 4.2.13 we get, $H(2n, n-2) \in \mathscr{V}_0$ for all n. Combining these results the proof follows. ☐

Let $H(2n, n-1)$ be the graph obtained by taking the cycle $C_{2n} : (v_0, v_1, \dots, v_{2n-1}, v_0)$ and its alternate chords $v_0v_2, v_0v_4, \dots, v_0v_{2n-2}$. Observe that $H(2n, n-1)$ has $n-1$ chords. We have the following theorem:

Theorem 4.2.15. $H(2n, n-1) \in \mathscr{V}_a$ *for all* n.

Proof. **Case 1:** Assume that n is even. Let $n = 2t$. Observe that in this case, the graph $H(2n, n-1)$ has $4t$ vertices. Define $\ell : E(H(2n, n-1)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \ell(v_i v_{i+1}) = \begin{cases} c & \text{for } \ i = 4, 8, 12, \dots, 4t-4, 4t, \\ b & \text{for } \ i = 2, 6, 10, \dots, 4t-2, \end{cases}$$

$$\ell(v_0 v_i) = a \text{ for } \ i = 2, 4, 6, \dots, 4t-2.$$

Obviously,

$$\ell^+(v_i) = \begin{cases} c+c+(2t-1)a = a, & \text{for } i=0, \\ b+c = a, & \text{for } i = 1,3,5,7,\ldots,4t-3,4t-1, \\ a+b+b = a, & \text{for } i = 2,6,\ldots,4t-2, \\ c+c+a = a, & \text{for } i = 4,8,\ldots,4t-4. \end{cases}$$

Case 2: Assume that n is odd. Let $n = 2t+1$. In this case, the graph has $4t+2$ vertices. Define $\ell : E(H(2n, n-1)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \begin{cases} c & \text{for } i = 1, \\ b & \text{for } i = 4t+2, \end{cases}$$

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 4,8,12,\ldots,4t-4,4t, \\ b & \text{for } i = 2,6,10,\ldots,4t-2, \end{cases}$$

$$\ell(v_0v_i) = \quad a \qquad \text{for } i = 2,4,6,\ldots,4t,$$

where $v_{4t} = v_0$. Obviously,

$$\ell^+(v_i) = \begin{cases} b+c+(2t)a = a, & \text{for } i=0, \\ b+c = a, & \text{for } i = 1,3,5,7,\ldots,4t-1,4t+1, \\ c+c+a = a, & \text{for } i = 4,8,\ldots,4t, \\ b+b+a = a, & \text{for } i = 2,6,\ldots,4t-2. \end{cases}$$

This completes the proof. □

Theorem 4.2.16. *(see [28]) $H(2n, n-1) \in \mathscr{V}_0$ for all n.*

Theorem 4.2.17. *$H(2n, n-1) \in \mathscr{V}_{a,0}$ for all n.*

Proof. From theorem 4.2.15 we have $H(2n, n-1) \in \mathscr{V}_a$ for all n and from theorem 4.2.16 it follows that $H(2n, n-1) \in \mathscr{V}_0$ for all n. Combining this the result follows. □

Let $H(4n+1, 2n)$ be the graph obtained by taking the cycle $C_{4n+1} := (v_0, v_1, \ldots, v_{4n}, v_0)$, the consecutive middle chords v_0v_{2n} and v_0v_{2n+1} and all alternate chords symmetrically placed between the apex, that is, the chords:$v_0v_2, v_0v_4, \ldots, v_0v_{2n-2}$; $v_0v_{2n+3}, v_0v_{2n+5}, \ldots, v_0v_{4n-1}$.

Theorem 4.2.18. *$H(4n+1, 2n) \notin \mathscr{V}_a$.*

Proof. Since the order of the graph $H(4n+1, 2n)$ is odd, $H(4n+1, 2n) \notin \mathscr{V}_a$. □

Theorem 4.2.19. *(see [28]) $H(4n+1, 2n) \in \mathscr{V}_0$.*

Let $U(4n+1, 2n, 1)$ be the graph obtained by identifying the apex of $H(4n+1, 2n)$ with a vertex of K_2. Then we have the following:

Theorem 4.2.20. *$U(4n+1, 2n, 1) \in \mathscr{V}_a$ for all n.*

Proof. We consider two cases:

Case 1: Suppose $n = 2t$. Then $U(4n+1, 2n, 1)$ has $8t+2$ vertices. Let the vertex set of $H(4n+1, 2n)$ be $\{v_0, v_1, v_2, \ldots, v_{8t}\}$ and let u be the pendant vertex. Define $\ell : E(U(4n+1, 2n, 1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_{i-1}v_i) = \ell(v_i v_{i+1}) = \begin{cases} c \text{ for } i = 2, 6, 10, \ldots, 4t-2, 4t+3, 4t+7, \ldots, 8t-1, \\ b \text{ for } i = 4, 8, 10, \ldots, 4t-4, 4t, 4t+1, 4t+4, 4t+8, \ldots, 8t+1 \end{cases}$$

$$\begin{aligned} \ell(v_0 v_i) &= a \text{ for } i = 2, 4, \ldots, 4t, 4t+1, 4t+3, \ldots, 8t-1, \\ \ell(v_0 u) &= a. \end{aligned}$$

where $v_{8t+1} = v_0$ and $v_{8t+2} = v_1$. We have,

$$\ell^+(v_i) = \begin{cases} a + b + b + 4ta = a \text{ for } i = 0, \\ b + c = a \text{ for } i = 1, 3, 5, \ldots, 4t-1, 4t+2, 4t+4, \ldots, 8t, \\ c + c + a = a \text{ for } i = 2, 6, 10, \ldots, 4t-2, 4t+3, 4t+7, \ldots, 8t-1, \\ b + b + a = a \text{ for } i = 4, 8, 10, \ldots, 4t-4, 4t, 4t+1, 4t+4, 4t+8, \ldots, 8t-3 \end{cases}$$

Case 2: Suppose $n = 2t+1$. In this case $U(4n+1, 2n, 1)$ has $8t+6$ vertices. Let the vertex set of $H(4n+1, 2n)$ be $\{v_0, v_1, \ldots, v_{8t+4}\}$ and let the pendant vertex be u. Define $\ell : E(U(4n+1, 2n, 1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_{i-1}v_i) = \ell(v_i v_{i+1}) = \begin{cases} b, \text{ for } i = 2, 6, 10, \ldots, 4t+2, 4t+3, 4t+7, \ldots, 8t+3, \\ c, \text{ for } i = 4, 8, 10, \ldots, 4t-4, 4t, 4t+5, 4t+9, \ldots, 8t+5 \end{cases}$$

where $v_{8t+5} = v_0$ and $v_{8t+6} = v_1$ and

$$\begin{aligned} \ell(v_0 v_i) &= a \text{ for } i = 2, 4, \ldots, 4t+2, 4t+3, 4t+5, \ldots, 8t+3 \\ \ell(v_0 u) &= a. \end{aligned}$$

We have,

$$\ell^+(v_i) = \begin{cases} a + c + c + (4t+2)a = a \text{ for } i = 0, \\ b + c = a \text{ for } i = 1, 3, \ldots, 4t+1, 4t+6, 4t+8, \ldots, 8t+4, \\ c + c + a = a \text{ for } i = 4, 8, 10, \ldots, 4t-4, 4t, 4t+5, 4t+9, \ldots, 8t+1, \\ b + b + a = a \text{ for } i = 2, 6, 10, \ldots, 4t+2, 4t+3, 4t+7, \ldots, 8t+3. \end{cases}$$

This completes the proof. $\square$

Let $H(4n+3, 2n+2)$ denotes cycle $C_{4n+3} := (v_0, v_1, \dots, v_{4n+2}, v_0)$, the four consecutive middle chords $v_0v_{2n}, v_0v_{2n+1}, v_0v_{2n+2}, v_0v_{2n+3}$ and all alternate chords symmetrically placed between the apex, that is, the chords : $v_0v_2, v_0v_4, \dots, v_0v_{2n-2}; v_0v_{2n+5}, v_0v_{2n+7}, \dots, v_0v_{4n+1}$. Then we have the following:

Theorem 4.2.21. *$H(4n+3, 2n+2) \notin \mathscr{V}_a$.*

Proof. Since the graph has odd number of vertices, $H(4n+3, 2n+2) \notin \mathscr{V}_a$ for any n. $\square$

Theorem 4.2.22. *(see [28]) $H(4n+3, 2n+2) \in \mathscr{V}_0$.*

Let $U(4n+3, 2n+2, 1)$ be the graph obtained by identifying the apex of $H(4n+1, 2n+2)$ with a vertex of K_2. Then we have the following:

Theorem 4.2.23. *$U(4n+3, 2n+2, 1) \in \mathscr{V}_a$.*

Proof. We consider two cases.

Case 1: Suppose $n = 2t$. Then $U(4n+3, 2n+2, 1)$ has $8t+4$ vertices. Let the vertex set of $U(4n+3, 2n+2)$ be $\{v_0, v_1, \dots, v_{8t+2}\}$. Let u be the pendant vertex of $U(4n+3, 2n+2, 1)$. Define $\ell : E(U(4n+3, 2n+2, 1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c \text{ for } i = 2, 6, 10, \dots, 4t-2, 4t+5, 4t+9, \dots, 8t+1, \\ b \text{ for } i = 4, 8, \dots, 4t, 4t+1, 4t+2, 4t+3, 4t+7, 4t+11, \\ \hfill \dots, 8t+3. \end{cases}$$

where $v_{8t+3} = v_0$ and $v_{8t+4} = v_1$ and

$$\ell(v_0v_i) = a \text{ for } i = 2, 4, 6, \dots, 4t-2, 4t, 4t+1, 4t+2, 4t+3, 4t+5, \dots 8t+1,$$
$$\ell(v_0u) = a.$$

We have, $\ell^+(u) = a$ and

$$\ell^+(u_i) = \begin{cases} a + (4t+2)a + b + b = a \text{ for } i = 0, \\ b + c = a \text{ for } i = 1, 3, 5, \dots, 4t-1, 4t+4, 4t+6, 4t+8, \dots 8t+2, \\ c + c + a = a \text{ for } i = 2, 6, 10, \dots, 4t-2, 4t+5, 4t+9, \dots, 8t+1, \\ b + b + a = a \text{ for } i = 4, 8, \dots, 4t, 4t+1, 4t+2, 4t+3, 4t+7, 4t+11, \\ \hfill \dots, 8t-1. \end{cases}$$

Case 2: Suppose $n = 2t+1$. In this case, the graph has $8t+7$ vertices. Define $\ell : E(U(4n+3, 2n+2, 1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} b \text{ for } i = 2, 6, \dots, 4t+2, 4t+3, 4t+4, 4t+5, 4t+9, \dots, 8t+5, \\ c \text{ for } i = 4, 8, \dots, 4t, 4t+7, 4t+11, \dots, 8t+7. \end{cases}$$

where $v_{8t+7} = v_0$ and $v_{8t+8} = v_1$ and

$$\ell(v_0v_i) = a \text{ for } i = 2, 4, 6, \dots, 4t+2, 4t+3, 4t+4, 4t+5, 4t+7, \dots, 8t+5,$$
$$\ell(v_0u) = a.$$

We have,

$$\ell(v_i) = \begin{cases} c + c + (4t+2)a + a = a \text{ for } i = 0, \\ b + b + a = a \text{ for } i = 2, 6, \dots, 4t+2, 4t+3, 4t+4, 4t+5, 4t+9, \dots, 8t+5, \\ c + c + a = a \text{ for } i = 4, 8, \dots, 4t, 4t+7, 4t+11, \dots, 8t+3, \\ b + c = a \text{ for } i = 1, 3, 5, \dots, 4t-3, 4t+6, 4t+8, \dots, 8t+6. \end{cases}$$

This completes the proof. □

Theorem 4.2.24. *$C_n(2, r) \in \mathscr{V}_a$ if n is even and $2 \le r \le n-3$.*

Proof. Let the vertex set of $C_n(2, r)$ be $\{u_0, u_1, u_2, \dots, u_{n-1}\}$, where u_0 is the apex. Here we consider two cases:

Case 1: Suppose r is odd. Now, we give the labeling to the edges of G as follows:

$$\ell(u_0u_1) = b,$$

$$\left\{ \begin{array}{l} \text{for } i = 1, 2, \dots, r+1 : \\ \qquad \ell(u_iu_{i+1}) = c. \\ \text{end for} \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{for } i = r+2, r+4, \dots, n-1 : \\ \qquad \ell(u_iu_{i+1}) = b. \\ \text{end for} \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{for } i = r+3, r+5, \dots, n-2 : \\ \qquad \ell(u_iu_{i+1}) = c. \\ \text{end for} \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{for } i = 2, 3, \dots, r+1 : \\ \qquad \ell(u_0u_i) = a. \\ \text{end for} \end{array} \right.$$

Observe that,

$$\ell^+(u_i) = \begin{cases} b+b+ra=a, & \text{for } \ i=0, \\ c+c+a=a, & \text{for } \ i=2,3,\ldots,r+1, \\ b+c=a, & \text{for } \ i=1,r+2,r+3,\ldots,n-1. \end{cases}$$

Case 2: Suppose r is even. In this case, labeling is similar to case 1.

This completes the proof. □

Theorem 4.2.25. *If $2 \le p \le n-r$ and n is even, then $C_n(p,r) \in \mathscr{V}_a$.*

Proof. Label the edges as follows:

$$\ell(u_0u_1) = b,$$

$$\left\{\begin{array}{l} \text{for } i = 1,2,\ldots,r+1: \\ \qquad \ell(u_iu_{i+1}) = c. \\ \text{end for} \end{array}\right.$$

$$\left\{\begin{array}{l} \text{for } i = r+2,r+4,\ldots,n-1: \\ \qquad \ell(u_iu_{i+1}) = b. \\ \text{end for} \end{array}\right.$$

$$\left\{\begin{array}{l} \text{for } i = r+3,r+5,\ldots,n-2: \\ \qquad \ell(u_iu_{i+1}) = c. \\ \text{end for} \end{array}\right.$$

$$\left\{\begin{array}{l} \text{for } i = p,p+1,\ldots,p+r-1: \\ \qquad \ell(u_0u_i) = a. \\ \text{end for} \end{array}\right.$$

Observe that,

$$\ell^+(u_i) = \begin{cases} b+b+ra=a, & \text{for } \ i=0, \\ c+c+a=a, & \text{for } \ i=2,3,\ldots,r+1, \\ b+c=a, & \text{for } \ i=1,r+2,r+3,\ldots,n-1. \end{cases}$$

□

Let $\mathcal{G}_r(n)$ denote graph $P_{n-1} + \overline{K_r}$ (see figure 4.2). Let the vertex sets of P_{n-1} and K_r be $\{v_1, v_2, \ldots, v_{n-1}\}$ and $\{u_1, u_2, \ldots, u_r\}$, respectively.

Theorem 4.2.26. *$\mathcal{G}_r(n) \in \mathscr{V}_a$ if and only if $n+r$ is odd.*

Proof. Assume that $\mathcal{G}_r(n) \in \mathscr{V}_a$. Then by lemma 3.2.1, we have $\sum_{i=1}^{n-1} \ell^+(v_i) + \sum_{j=1}^{r} \ell^+(u_j) = 0$ which implies that $(n-1)a + ra = 0$. This shows that $n+r$ is odd.
Conversely, assume that $n+r$ is odd. We consider two cases.

Case 1: Suppose n is odd and r is even. Let $n = 2s+1$. Now, we give the labeling to the edges of $\mathcal{G}_r(n)$ as follows:

$$\begin{cases} \text{For } j = 1, 2, \ldots, r: \\ \ell(u_j v_1) = b, \\ \ell(u_j v_{2s}) = c. \\ \text{end for} \end{cases}$$

$$\ell(u_1 v_i) = a \quad \text{for } i = 2, 3, \ldots, 2s-1,$$

$$\ell(v_i v_{i+1}) = \begin{cases} a, & \text{for } i = 1, 3, \ldots, 2s-1, \\ c, & \text{for } i = 2, 4, \ldots, 2s-2, \end{cases}$$

$$\begin{cases} \text{For } j = 2, 3, \ldots, r: \\ \ell(u_j v_i) = b \quad \text{for } i = 2, 3, \ldots, 2s-1 \\ \text{end for} \end{cases}$$

We have

$$\ell^+(v_i) = \begin{cases} rb + a = a & \text{for } i = 1, \\ rc + a = a & \text{for } i = 2s, \\ a + c + a + (r-1)b = a, & \text{for } i = 2, 3, \ldots, 2s-1 \end{cases}$$

$$\ell^+(u_i) = \begin{cases} b + c + (2s-2)a = a & \text{for } i = 1 \\ b + c + (2s-2)b = a, & \text{for } i = 2, 3, \ldots, r. \end{cases}$$

Case 2: Suppose n is even and r is odd. Let $n = 2s$. Now, we give the labeling to the edges of $\mathcal{G}_r(n)$ as follows:

$$\ell(v_i v_{i+1}) = c \text{ for } \; i = 1, 2, \ldots, 2s-2,$$

For $\; j = 1, 2, \ldots, r:$

$$\begin{cases} \ell(u_j v_1) = b, \\ \ell(u_j v_{2s-1}) = b, \\ \ell(u_j v_i) = a \text{ for } \; i = 2, 3, \ldots, 2s-1. \end{cases}$$

end for

We have

$$\ell^+(v_i) = \begin{cases} rb + c = a \text{ for } i = 1, 2s-1 \\ c + c + ra = a \text{ for } i = 2, 3, \ldots, 2s-2 \end{cases}$$

$$\ell^+(u_i) = b + b + (2s-3)a = a \text{ for } \; i = 1, 2, \ldots, r.$$

This completes the proof. □

Theorem 4.2.27. *$\mathcal{G}_r(n) \in \mathscr{V}_0$ if $n{+}r$ is even.*

Proof. Assume that $n + r$ is even. We consider two cases.

Case 1: Suppose n and r are both odd. Let $n = 2s + 1$. Now, we give the labeling to the edges of $\mathcal{G}_r(n)$ as follows:

$$\ell(v_i v_{i+1}) = \begin{cases} b & \text{for } i = 1, 3, \ldots, 2s-1, \\ c & \text{for } i = 2, 4, \ldots, 2s-2, \end{cases}$$

$$\ell(u_1 v_i) = b \text{ for } \; i = 1, 2s$$

$$\ell(u_1 v_i) = a \text{ for } \; i = 2, 3, \ldots, 2s-1$$

$$\begin{cases} \text{for } \; j = 2, 3, \ldots, r: \\ \ell(u_j v_i) = a \text{ for } \; i = 1, 2, 3, \ldots, 2s \\ \text{end for} \end{cases}$$

We have,

$$\ell^+(v_i) = \begin{cases} b + b + (r-1)a = 0 \;\text{ for } i = 1, 2s \\ b + c + ra = 0 \text{ for } \; i = 2, 3, \ldots, 2s-1, \end{cases}$$

$$\ell^+(u_j) = \begin{cases} b + b + (2s-2)a = 0 \text{ for } j = 1, \\ 2sa = 0 \text{ for } \; j = 2, 3, \ldots, r. \end{cases}$$

Case 2: Suppose both n and r are both even.

$$\ell(u_1 v_{2s-1}) = c,$$
$$\ell(u_1 v_s) = a,$$
$$\ell(v_i v_{i+1}) = a, \quad \text{for } i = 1, 2, \ldots, 2s-2,$$
$$\ell(u_1 v_i) = b, \quad \text{For } i = 1, 2, 3, \ldots, s-1, s+1, s+2, \ldots 2s, 2s-2$$

$$\begin{cases} \text{for } \ j = 2, 3, \ldots, r : \\ \ell(u_j v_1) = c \\ \ell(u_j v_{2s-1}) = b \\ \ell(u_j v_s) = a \\ \ell(u_j v_i) = b, \quad \text{for } i = 2, 3, \ldots, s-1, s+1, s+2, \ldots, 2s-2 \\ \text{end for} \end{cases}$$

We have,

$$\ell^+(v_i) = \begin{cases} a + b + (r-1)c = 0 \ \text{ for } \ i = 1 \\ a + a + b + (r-1)b = 0 \ \text{ for } \ i = 2, 3, \ldots, s-1, s+1, s+2, \ldots, 2s-2 \\ a + a + a + (r-1)a = 0 \ \text{ for } \ i = s \\ c + a + (r-1)b = 0 \ \text{ for } \ i = 2s-1 \end{cases}$$

$$\ell^+(u_i) = (2s-3)b + a + c = 0 \ \text{ for } \ i = 1, 2, \ldots, r,$$

This completes the proof. □

Let $G(n, n-3, k)$ denote the graph obtained by taking the union of k copies of $H(n, n-3)$ having the edges v_0v_1's identified (see figure 4.3).
Let $\{v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,n-1}\}$ be the vertex set of the i^{th} copy of $H(n, n-3)$. Then

Theorem 4.2.28. *$G(n, n-3, k) \in \mathscr{V}_a$ if and only if nk is even.*

Proof. Assume that $G(n, n-3, k) \in \mathscr{V}_a$. Then by lemma 3.2.1, we have $na + (k-1)(n-2)a = 0$. This implies that nk is even.
Conversely, assume that nk is even. We consider the following cases:

Case 1: Assume that both n and k are even. In this case, we label the edges of $G(n, n-3, k)$ as follows:

$$\ell(v_{1,0}v_{1,1}) = a,$$

$$\begin{cases} \text{For } \ i = 1, 2, \ldots, k \\ \qquad \ell(v_{i,j}v_{i,j+1}) = c, \quad \text{for} \quad j = 1, 2, 3, \ldots, n-2, \\ \text{end for} \end{cases}$$

$$\begin{cases} \text{For } \ i = 1, 2, \ldots, k \\ \qquad \ell(v_{1,0}v_{i,j+1}) = a, \quad \text{for} \quad j = 1, 2, 3, \ldots, n-3, \\ \text{end for} \end{cases}$$

$$\ell(v_{1,0}v_{i,n-1}) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, k.$$

So, we have $\ell^+(v_{i,j}) = a$ for all i, j.

Case 2: Assume that n is even and k is odd. In this case, the labeling is exactly similar to case 1 with only difference is that $\ell(v_{1,0}v_{1,1}) = a$ is to be replaced by $\ell(v_{1,0}v_{1,1}) = b$.

Case 3: Assume that n is odd and k is even. In this case, the labeling is similar to case 1.

This completes the proof. $\square$

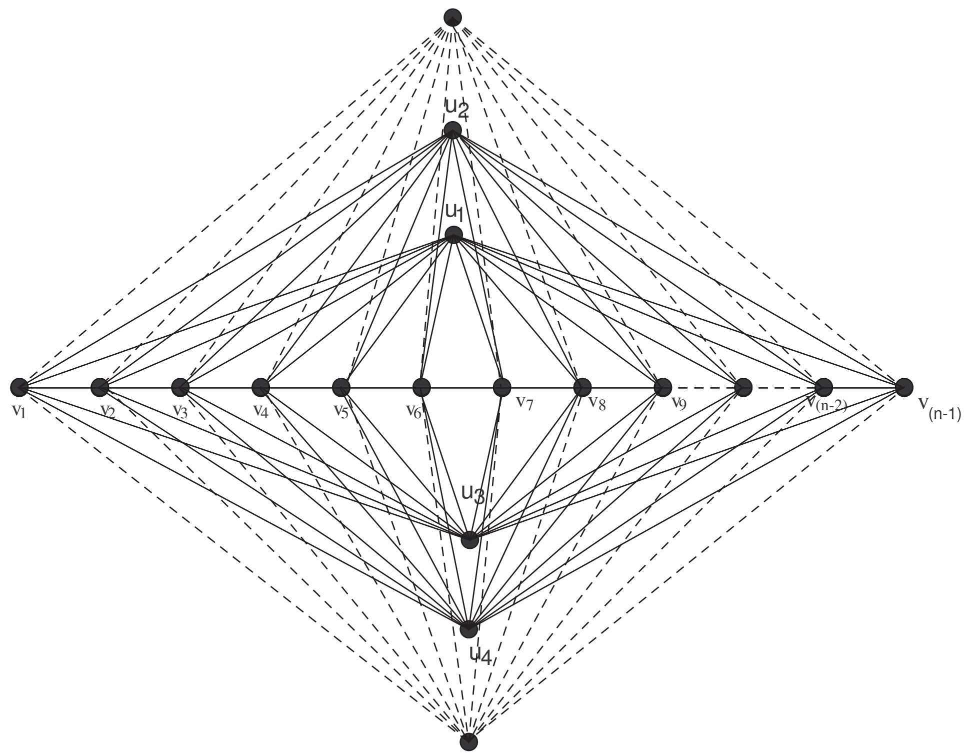


Figure 4.2: The graph $\mathcal{G}_r(n)$

Theorem 4.2.29. *$G(n, n-3, k) \notin \mathscr{V}_a$ if n and k are both odd.*

Proof. Assume that $G(n, n-3, k) \in \mathscr{V}_a$. Since n and k are both odd, nk is odd. Then by lemma 3.2.1, we have $\sum_{j=0}^{n-1} \ell^+(v_{1,j}) + \sum_{i=2}^{k} \sum_{j=2}^{n-1} \ell^+(v_{i,j}) = 0$ which implies that $nka = 0$. In turn we get $a = 0$. This is a contradiction. The result now follows. $\square$

Theorem 4.2.30. *$G(n, n-3, k) \in \mathscr{V}_0$ if n and k are odd.*

Proof. Now, we give the labeling to the edges of G as follows:

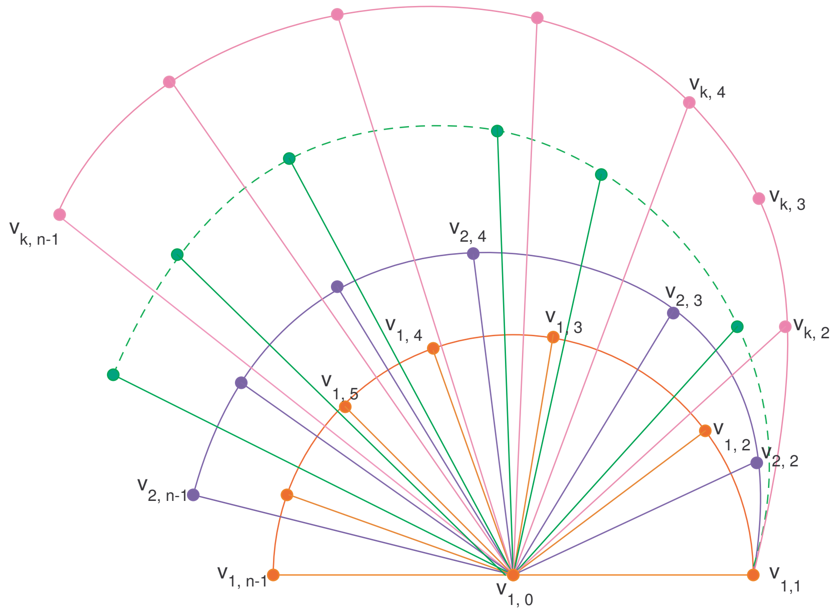


Figure 4.3: The graph $G(n, n-3, k)$

$\ell(v_{1,0}v_{1,1}) = b,$

$\ell(v_{i,n-1}v_{1,0}) = b, \quad \text{for } i = 1, 2, \ldots, k,$

for $\quad i = 1, 2, \ldots, k:$

$$\begin{cases} \ell(v_{i,j}v_{i,j+1}) = b, \;\; j = 1, 3, \ldots, n-2, \\ \ell(v_{i,j}v_{i,j+1}) = c, j = 2, 4, \ldots, n-3, \\ \ell(v_{1,0}v_{i,j}) = a, j = 2, 3, \ldots, n-2. \end{cases}$$

end for

Obviously,

$$\ell^+(v_{i,j}) = \begin{cases} k(n-3)a + 2b = 0, & \text{for } i = 1, j = 0, \\ b + b = 0, & \text{for } i = 1, 2, 3, \ldots, k; j = n-1 \\ b + kb = 0, & \text{for } i = 1, j = 1, \\ b + c + a = 0, & \text{for } i = 1, 2, \ldots, k; j = 2, 3, \ldots, n-2. \end{cases}$$

This completes the proof. $\square$

Theorem 4.2.31. *$G(n, n-3, k) \in \mathscr{V}_0$ if n is even and k odd.*

Proof. Now, we give the labeling to the edges of G as follows:

$$
\begin{aligned}
&\ell(v_{1,0}v_{1,1}) = c,\\
&\ell(v_{i,n-1}v_{1,0}) = b, \quad i = 1, 2, 3, \ldots, k,\\
&\text{for} \quad i = 1, 2, \ldots, k:\\
&\begin{cases}
\ell(v_{i,j}v_{i,j+1}) = c, \ j = 1, 3, \ldots, n-3,\\
\ell(v_{i,j}v_{i,j+1}) = b, \ j = 2, 4, \ldots, n-2,\\
\ell(v_{1,0}v_{i,j}) = a, \ j = 2, 3, \ldots, n-2.
\end{cases}\\
&\text{end for}
\end{aligned}
$$

Obviously,

$$
\ell^+(v_{i,j}) = \begin{cases}
k(n-3)a + kb + c = a + b + c = 0, \quad \text{for } i = 1, j = 0,\\
b + b = 0, \quad \text{for } i = 1, 2, 3, \ldots, k; j = n-1,\\
c + kc = 0, \quad \text{for } i = 1, j = 1,\\
b + c + a = 0, \quad \text{for } i = 1, 2, \ldots, k; j = 2, 3, \ldots, n-2.
\end{cases}
$$

This completes the proof. □

Theorem 4.2.32. *$G(n, n-3, k) \in \mathscr{V}_0$ if n and k are even.*

Proof. We give the labeling to the edges of G as follows:

$$
\begin{aligned}
&\ell(v_{1,0}v_{1,1}) = a,\\
&\ell(v_{1,n-1}v_{1,0}) = b,\\
&\ell(v_{1,j}v_{1,j+1}) = \begin{cases}
c, & \text{for} \quad j = 1, 3, \ldots, n-3,\\
b, & \text{for} \quad j = 2, 4, \ldots, n-2,
\end{cases}\\
&\text{For} \quad i = 2, 3, \ldots, k:\\
&\begin{cases}
\ell(v_{i,j}v_{i,j+1}) = b, \ j = 1, 3, \ldots, n-3,\\
\ell(v_{i,j}v_{i,j+1}) = c, \ j = 2, 4, \ldots, n-2,
\end{cases}\\
&\text{end for}\\
&\ell(v_{1,0}v_{i,n-1}) = c, \quad \text{for} \quad i = 2, 3, \ldots, k,\\
&\text{For} \quad i = 1, 2, 3, \ldots, k:\\
&\qquad \ell(v_{1,0}v_{i,j}) = a, \ j = 2, 3, \ldots, n-2.\\
&\text{end for}
\end{aligned}
$$

Then we have,

$$\ell^+(v_{i,j}) = \begin{cases} +a + (n-3)ka + (k-1)c = a+b+c = 0, & \text{for } i = 1, j = 0, \\ a + c + (k-1)b = a+b+c = 0, & \text{for } i = 1, j = 1, \\ c + c = 0, & \text{for } i = 2, 3, \ldots, k; j = n-1 \\ b + b = 0, & \text{for } i = 1, j = n-1, \\ b + c + a = 0 & \text{for } i = 1, 2, \ldots, k; j = 2, 3, \ldots, n-2. \end{cases}$$

□

Theorem 4.2.33. *$G(n, n-3, k) \in \mathscr{V}_0$ if n is odd and k is even.*

Proof. We give the labeling to the edges of G as follows:

$$\begin{aligned}
&\ell(v_{1,n-1}v_{1,0}) = c \\
&\ell(v_{1,0}v_{1,1}) = a \\
&\text{For } i = 1, 2, \ldots, k: \\
&\qquad \ell(v_{1,0}v_{i,j}) = a, \quad \text{for } j = 1, 2, \ldots, n-2, \\
&\text{end for} \\
&\text{For} \quad i = 2, 3, \ldots, k: \\
&\qquad \ell(v_{1,0}v_{i,n-1}) = b \\
&\text{end for} \\
&\qquad \ell(v_{1,j}v_{1,j+1}) = c, \quad \text{for} \quad j = 1, 3, \ldots, n-2, \\
&\qquad \ell(v_{1,j}v_{1,j+1}) = b, \quad \text{for} \quad j = 2, 4, \ldots, n-3, \\
&\qquad \ell(v_{1,1}v_{i,2}) = b, \quad \text{for} \quad i = 2, \ldots, k, \\
&\text{for} \quad i = 2, 3, \ldots, k: \\
&\qquad \ell(v_{i,j}v_{i,j+1}) = c, \; j = 2, 4, \ldots, n-3, \\
&\qquad \ell(v_{i,j}v_{i,j+1}) = b \; j = 3, 5, \ldots, n-2, \\
&\text{end for}
\end{aligned}$$

Obviously,

$$\ell^+(v_{i,j}) = \begin{cases} c + (k-1)b + (n-3)ka + a = 0, & \text{for } i = 1, j = 0, \\ a + c + (k-1)b = 0, & \text{for } i = 1, j = 1, \\ b + b = 0, & \text{for } i = 2, \ldots, k, j = n-1, \\ c + c = 0, & \text{for } i = 1, j = n-1, \\ a + b + c = 0, & \text{for } i = 1, 2, \ldots, k; j = 2, \ldots, n-2. \end{cases}$$

This completes the proof. □

4.3 Multiple Shell Graphs

Let G denotes the multiple shell $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r})$. Let $\{u, v_{k,j}^{t_i}\}$, $j = 1, 2, \ldots, n_i$, $1 \le i \le r$ be the vertices of G with apex u.

Theorem 4.3.1. *If $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}) \in \mathscr{V}_a$, then $\sum_{i=1}^{r}[(n_i-1)t_i]$ is odd.*

Proof. Let $\ell : E(G) \to V_4 \backslash \{0\}$ is a labeling of G, then by lemma 3.2.1, $\sum_{i=1}^{r}\sum_{k=1}^{t_i}\sum_{j=1}^{n_i-1}\ell^+(v_{k,j}^{t_i})+$ $\ell^+(u) = 0$. Hence if $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}) \in \mathscr{V}_a$, it follows that $\sum_{i=1}^{r}[(n_i-1)t_i]a + a = 0$ which implies that $\sum_{i=1}^{r}[(n_i-1)t_i]$ is odd. □

Conjecture 4.3.2. *If $\sum_{i=1}^{r}[(n_i-1)t_i]$ is odd, then $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \in \mathscr{V}_a$.*

We prove that the conjecture is true for $r = 1$.

Corollary 4.3.3. *$MS(n^t) \in \mathscr{V}_a$ if $(n-1)t$ is odd.*

Proof. Assume that $(n-1)t$ is odd. This implies that n is even and t is odd. Observe that $MS(n^t)$ is the one point union of t shells $H^i(n, n-3)$, $i = 1, 2, \ldots, t$. Let the vertex set of $H^i(n, n-3)$ be $\{u_{0,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,n-1}\}$, where $u_{0,0}$ is the apex of all shells. Now, we give the labeling to the edges of G as follows:

$$\begin{aligned}
&\text{For} \quad i = 1, 2, 3, \ldots, t: \\
&\begin{cases}
\ell(u_{0,0}u_{i,1}) = \ell(u_{0,0}u_{i,n-1}) = b, \\
\ell(u_{i,j}u_{i,j+1}) = c, \ j = 1, 2, \ldots, n-2, \\
\ell(uu_{i,j}) = a, \ j = 2, 3, \ldots, n-2.
\end{cases} \\
&\text{end for}
\end{aligned}$$

Then we have,

$$\ell^+(u_{i,j}) = \begin{cases}
(n-3)ta + tb + tb = a + b + b = a, & \text{for } i = 0, j = 0, \\
b + c = a, & \text{for } i = 1, 2, 3, \ldots, t; j = 1, n-1, \\
c + c + a = a, & \text{for } i = 1, 2, 3, \ldots, t; 2, 3, \ldots, n-2.
\end{cases}$$

This completes the proof. □

Next we prove that the conjecture is true for $r = 2$, $n_1 = n$, $n_2 = n+1$ and $t_1 = t_2 = 1$.

Corollary 4.3.4. *$MS(n, n+1) \in \mathscr{V}_a$.*

Proof. Observe that $MS(n, n+1)$ is the one point union of $H(n, n-3)$ and $H(n+1, n-2)$. Let $V(H(n, n-3)) = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$ and $V(H(n+1, n-2)) = \{v_0, v_1, v_2, \ldots, v_n\}$. Assume that $u_o = v_0$ be the apex of both the shells $H(n, n-3)$ and $H(n+1, n-2)$. Now, we give the labeling to the edges of $MS(n, n+1)$ as follows:

$$\ell(u_iu_{i+1}) = c, \quad \text{for} \quad i = 1, 2, 3, \dots, n-2,$$
$$\ell(u_0u_i) = \begin{cases} a, & \text{for} \quad i = 2, 3, \dots, n-2, \\ b, & \text{for} \quad i = 1, n-1. \end{cases}$$
$$\ell(v_iv_{i+1}) = c, \quad \text{for} \quad j = 1, 2, 3, \dots, n-1,$$
$$\ell(v_0v_i) = \begin{cases} a, & \text{for} \quad i = 2, 3, \dots, n-1, \\ b, & \text{for} \quad i = 1, n. \end{cases}$$

We have,

$$\ell^+(u_i) = \begin{cases} (n-3)a + (n-2)a + 4b = (2n-5)a = a, & \text{for } i = 0, \\ b + c = a, & \text{for } i = 1, n-1 \\ a + c + c = a, & \text{for } 2, 3, \dots, n-2, \end{cases}$$
$$\ell^+(v_i) = \begin{cases} b + c = a, & \text{for } i = 1, n \\ a + c + c = a, & \text{for } 2, 3, \dots, n-1, \end{cases}$$

This completes the proof. □

Corollary 4.3.5. *$MS(n^t, (n+1)^t) \in \mathscr{V}_a$ for all odd t.*

Proof. Label the edges of all the t copies of $H(n, n-3), H(n+1, n-2)$ as in corollary 4.3.4. The proof follows. □

Corollary 4.3.6. *$MS(n, m) \in \mathscr{V}_a$ if and only if $m + n$ is odd.*

Proof. Assume that $MS(n, m) \in \mathscr{V}_a$. Observe that $MS(n, m)$ has $(m + n - 1)$ vertices. Then by theorem 4.3.1, $m + n$ is odd.

Conversely, assume that $m+n$ is odd. We need to show that $MS(n, m) \in \mathscr{V}_a$. We consider two cases.

Case 1: Suppose n is even and m is odd. Assume that v_0 is the apex of both the shells and, let

$$V(H(n, n-3)) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$$
$$V(H(m, m-3)) = \{v_0, u_1, u_2, \dots, u_{m-1}\}.$$

Now, we give the labeling to the edges of $H(n, m, n-3, m-3)$ as follows:

$$\ell(v_0v_i) = \begin{cases} a, & \text{for } i = 2, \dots, n-2, \\ b, & \text{for } i = 1, n-1. \end{cases}$$

$$\ell(v_iv_{i+1}) = c, \quad \text{for } i = 1, 2, \ldots, n-2,$$
$$\ell(u_ju_{j+1}) = c, \quad \text{for } j = 1, 2, \ldots, m-2,$$
$$\ell(v_0u_j) = \begin{cases} b, & \text{for } j = 1, m-1, \\ a, & \text{for } j = 2, \ldots, m-2. \end{cases}$$

We have

$$\ell^+(v_i) = \begin{cases} b + b + (n-3)a + b + b + (m-3)a = a & \text{for } \ i = 0, \\ b + c = a & \text{for } \ i = 1, n-1, \\ c + c + a = a & \text{for } \ i = 2, 3, 4, \ldots, n-2. \end{cases}$$
$$\ell^+(u_i) = \begin{cases} b + c = a & \text{for } \ i = 1, m-1, \\ c + c + a = a & \text{for } \ i = 2, 3, 4, \ldots, m-2. \end{cases}$$

Case 2: Suppose n is odd and m is even. In this case the labeling is exactly similar to case 1.

This completes the proof.

Conjecture 4.3.7. *$MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}) \in \mathscr{V}_0$ for all n_i and t_i.*

We prove some special cases of conjecture 4.3.7.

Corollary 4.3.8. *$MS(n^t) \in \mathscr{V}_0$ for n even, t odd and n even, t even.*

Now, we give the labeling to the edges of $MS(n^t)$ as follows:

$$\ell(u_{0,0}u_{i,1}) = b, \quad \text{for} \quad i = 1, 2, \ldots, t,$$
$$\ell(u_{0,0}u_{i,n-1}) = c, \quad \text{for} \quad i = 1, 2, \ldots, t,$$
For $\quad i = 1, 2, 3, \ldots, t :$
$$\begin{cases} \ell(u_{i,j}u_{i,j+1}) = b, & \text{for} \quad j = 1, 3, \ldots, n-3, \\ \ell(u_{i,j}u_{i,j+1}) = c, & \text{for} \quad j = 2, 4, \ldots, n-2, \\ \ell(u_{0,0}u_{i,j}) = a, & \text{for} \quad j = 2, 3, \ldots, n-2, \end{cases}$$
end for

Then we have,

$$\ell^+(u_{i,j}) = \begin{cases} (n-3)ta + tb + tc = a + b + c = 0, & \text{for } i = 0, j = 0, \\ b + b = 0, & \text{for } i = 1, 2, 3, \ldots, t; j = 1, \\ c + c = 0, & \text{for } i = 1, 2, 3, \ldots, t; j = n-1, \\ b + c + a = 0, & \text{for } i = 1, 2, 3, \ldots, t; j = 2, 3, \ldots, n-3. \end{cases}$$

This completes the proof.

Corollary 4.3.9. *$MS(n^t) \in \mathscr{V}_0$ for n odd, t odd and n odd, t even.*

Proof. We give the labeling to the edges of $MS(n^t)$ as follows:

$$\begin{aligned}
&\ell(u_{0,0}u_{i,1}) = b, \quad \text{for} \quad i = 1, 2, \ldots, t,\\
&\ell(u_{0,0}u_{i,n-1}) = b, \quad \text{for} \quad i = 1, 2, \ldots, t,\\
&\text{For} \quad i = 1, 2, 3, \ldots, t:\\
&\begin{cases}
\ell(u_{i,j}u_{i,j+1}) = b, \quad \text{for} \quad j = 1, 3, \ldots, n-2,\\
\ell(u_{i,j}u_{i,j+1}) = c, \quad \text{for} \quad j = 2, 4, \ldots, n-3,\\
\ell(uu_{i,j}) = a, \quad \text{for} \quad j = 2, 3, \ldots, n-3,
\end{cases}\\
&\text{end for}
\end{aligned}$$

Then we have,

$$\ell(u_{i,j}) = \begin{cases}
(n-3)ta + tb + tb = 0, \quad \text{for } i = 0, j = 0\\
b + b = 0, \quad \text{for } i = 1, 2, 3, \ldots, t, j = 1, n-1\\
b + c + a = 0, \quad \text{for } i = 1, 2, 3, \ldots, t, j = 2, 3, \ldots, n-2.
\end{cases}$$

This completes the proof. □

Corollary 4.3.10. *If $m + n$ is even, then $MS(n, m) \in \mathscr{V}_0$.*

Let the vertex set of the graph be as in corollary 4.3.6. We consider two cases.

Case 1: Suppose both n and m are even. Now, we give the labeling to the edges of $MS(n, m)$ as follows:

$$\begin{aligned}
\ell(v_0v_i) &= \begin{cases} b & \text{for } i = 1,\\ c & \text{for } i = n-1, \end{cases}\\
\ell(v_iv_{i+1}) &= \begin{cases} c & \text{for } i = 2, 4, 6, \ldots, n-2,\\ b & \text{for } i = 3, 5, 8, \ldots, n-3, \end{cases}\\
\ell(v_0v_i) &= a \;\; \text{for } i = 2, \ldots, n-2,\\
\ell(v_1v_2) &= \ell(u_1u_2) = b,\\
\ell(v_0u_j) &= \begin{cases} b & \text{for } j = 1,\\ c & \text{for } j = m-1, \end{cases}\\
\ell(u_ju_{j+1}) &= \begin{cases} c & \text{for } j = 2, 4, 6, \ldots, m-2,\\ b & \text{for } j = 3, 5, 8, \ldots, m-3, \end{cases}\\
\ell(u_0u_j) &= a \;\; \text{for } j = 2, \ldots, m-2.
\end{aligned}$$

We have,

$$\ell^+(v_i) = \begin{cases} b + c + (n-3)a + b + c + (m-3)a = 0 & \text{for } i = 0, \\ b + b = 0 & \text{for } i = 1, \\ c + C = 0 & \text{for } i = n-1, \\ b + c + a = 0 & \text{for } i = 2, 3, 4, \ldots, n-2. \end{cases}$$

$$\ell^+(u_j) = \begin{cases} b + b = 0 & \text{for } j = 1, \\ c + c = 0 & \text{for } j = m-1, \\ a + b + c = 0 & \text{for } j = 2, 3, 4, \ldots, m-2. \end{cases}$$

Case 2: Suppose both m and n are odd. In this case, the labeling is similar to case 1.

This completes the proof. □

Consider the multiple shell $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\}$ with vertex set $\{u, v_{k,j}^{t_i}\}$, $j = 1, 2, \ldots, n_i$, $1 \le i \le r$. Let $K_{1,m}$ denotes the star graph with vertex set $\{v, v_1, v_2, \ldots, v_m\}$. Here v denotes the apex of $K_{1,n}$. Let $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \diamond K_{1,m}$ denotes the graph obtained by identifying the vertices u and v.

Theorem 4.3.11. *If $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \diamond K_{1,m} \in \mathscr{V}_a$, then $\sum_{i=1}^{r}[(n_i - 1)t_i] + m$ is odd.*

Proof. Suppose that $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \diamond K_{1,m} \in \mathscr{V}_a$. Let G denotes the graph $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \diamond K_{1,m}$. Then if $\ell : E(G) \to V_4 \backslash \{0\}$ is a labeling of G, then by lemma 3.2.1, $\sum_{i=1}^{r} \sum_{k=1}^{t_i} \sum_{j=1}^{n_i-1} \ell^+(v_{k,j}^{t_i}) + \sum_{i=1}^{m} \ell^+(v_i) + \ell^+(u) = 0$. Thus we have $[\sum_{i=1}^{r}[(n_i - 1)t_i] + m + 1]a = 0$. That is $\sum_{i=1}^{r}[(n_i - 1)t_i] + m$ is odd. □

Conjecture 4.3.12. *If $\sum_{i=1}^{r}[(n_i - 1)t_i] + m$ is odd, then $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \diamond K_{1,m} \in \mathscr{V}_a$.*

We prove some special cases of the conjecture 4.3.12.

Corollary 4.3.13. *$MS(n^t) \diamond K_{1,m} \in \mathscr{V}_a$ if and only if $(n-1)t + m$ is odd.*

Proof. Assume that $MS(n^t) \diamond K_{1,m} \in \mathscr{V}_a$. Then by lemma 3.2.1, we have $[(n-1)t + m + 1]a = 0$. This implies that $(n-1)t + m$ is odd.

Conversely, assume that $(n-1)t + m$ is odd. Then we have the following cases:

Case 1: Suppose n is even, t is odd and m is even. Let $u_{0,0}$ be the apex of both $MS(n^t)$ and $K_{1,m}$. Let $\{u_{0,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,n-1}\}$ be the vertex set of the i^{th} copy of $H^i(n, n-3)$ and let $\{u_{0,0}, v_{1,1}, v_{1,2}, \ldots, v_{1,m}\}$ be the vertex set of $K_{1,m}$. Now, we give the labeling to the edges of G as follows:

$$\ell(u_{0,0}u_{i,1}) = b, \quad \text{for} \quad i = 1, 2, \ldots, t,$$

$$\begin{aligned}
&\ell(u_{0,0}u_{i,n-1}) = b, \quad \text{for} \quad i = 1, 2, \ldots, t,\\
&\text{For} \quad i = 1, 2, 3, \ldots, t:\\
&\begin{cases} \ell(u_{i,j}u_{i,j+1}) = c, \quad \text{for} \quad j = 1, 2, \ldots, n-2,\\ \ell(u_{0,0}u_{i,j}) = a, \quad \text{for} \quad j = 2, 3, \ldots, n-2, \end{cases}\\
&\text{end for}\\
&\ell(u_{0,0}v_{1,k}) = a, \quad \text{for} \quad k = 1, 2, 3, \ldots, m.
\end{aligned}$$

Then we have,

$$\ell^+(u_{i,j}) = \begin{cases} (n-3)ta + tb + tb + ma = a + b + b = a, & \text{for } i = 0, j = 0\\ b + c = a, & \text{for } i = 1, 2, \ldots, t, j = 1, n-1\\ c + c + a = a, & \text{for } i = 1, 2, \ldots, t, j = 1, 2, 3, \ldots, n-2 \end{cases}$$
$$\ell^+(v_{i,k}) = a \quad \text{for } i = 1; k = 1, 2, 3, \ldots, m$$

Case 2: n is even, t is even and m is odd. In this case, the labeling is similar to case 1.

Case 3: n is odd, t is even and m is odd. In this case, the labeling is similar to case 1.

Case 4; m, n and t are odd. In this case, the labeling is similar to case 1.

This completes the proof. □

Corollary 4.3.14. *$MS(n, n+1) \diamond K_{1,m} \in \mathscr{V}_a$ if m is even.*

Proof. First, label all the edges of $K_{1,m}$ by a. Next, label all edges of $MS(n, n+1)$ as described in Corollary 4.3.4. Then we can easily verify that this labeling is an a-sum V_4-magic labeling of $MS(n, n+1) \diamond K_{1,m}$. □

Corollary 4.3.15. *$MS(n^t, (n+1)^t) \diamond K_{1,m} \in \mathscr{V}_a$ if m is even and t is odd.*

Proof. Labeling is similar to Corollary 4.3.14. □

Chapter 5

V_4-Magic Labelings of Some More Graphs

In the first section of this chapter, definition of some more cycle related graphs and path related graphs are provided. Some wellknown book graphs, ladder graphs and K_n-related graphs are also included. In the second section of this chapter, we discuss some cycle related V_4-magic graphs. In the third section of this chapter we discuss ladder related V_4-magic graphs. In the fourth section path related V_4-magic and in the fifth section some book related V_4-magic graphs are discussed. In the last section of this chapter we discuss $K_{m,n}$ and some K_n-related V_4-magic graphs.

5.1 Introduction

Here we need the following:

Definition 5.1.1. *(see [16]) The windmill graph $D_n^{(m)}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common.*

The graph $D_3^{(m)}$ is called the Dutch windmill graph or the friendship graph, F_m [22].

Definition 5.1.2. *(see [28]) A snake graph is formed by taking n-copies of a cycle C_m and identifying exactly one edge of each copy to a distinct edge of the path $P_{(n+1)}$, which is called as the backbone of the snake. It is denoted by $T_n^{(m)}$ (see figure 5.1).*

Definition 5.1.3. *(see [28]) The book B_n is the graph $S_n \Box P_2$ where S_n is the star with $n+1$ vertices.*

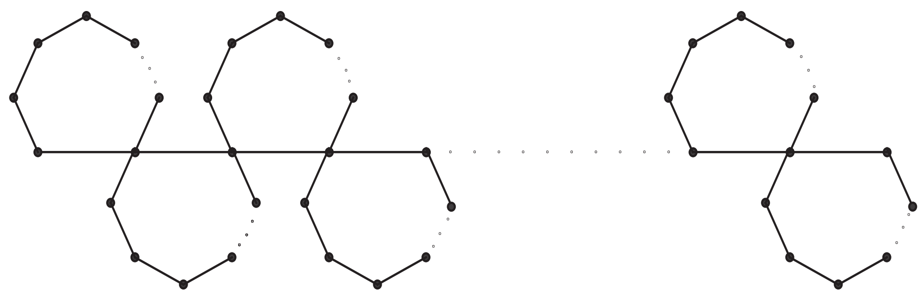


Figure 5.1: Snake Graph $T_n^{(m)}$

Definition 5.1.4. *(see [13]) When k copies of C_n share a common edge it will form the n-gon book of k pages and is denoted by $B(n,k)$.*

Definition 5.1.5. *(see [28]) One point union of any number of connected graphs is obtained by identifying one vertex from each graph. One point union of t cycles each of length n is denoted by $C_n(t)$.*

Definition 5.1.6. *(see [21]) The sunflower graph SF_n is obtained from a wheel with the central vertex v_0 and the cycle $C_n : v_1v_2\dots v_nv_1$ and additional vertices $w_1, w_2, \dots, w_n$ where w_i is joined by edges to v_i, v_{i+1} where v_{i+1} is taken modulo n.*

Definition 5.1.7. *(see [17]) The Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph consisting of a cycle C_{nm} with one additional vertex called the central vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.*

Definition 5.1.8. *(see [13]) Given a cycle C_n construct a cycle C_m on each edge of this cycle. The resulting graph is denoted by $C_m@C_n$.*

Let $N_2 = \{v_1, v_2\}$ be the disconnected graph of order two.

Definition 5.1.9. *(see [13]) Given a graph G, we can define the bipyramid based on G to be $G \vee N_2$. This graph will be denoted by $BP(G)$. The graph $C_n \vee N_2$ is called the bipyramid based on C_n and is denoted by $BP(n)$.*

Definition 5.1.10. *(see [13]) Given k natural numbers $a_1, a_2, \cdots a_k$, if we connect the two vertices of $N_2 = \{u, v\}$ by k parallel paths of length $a_1, a_2, \cdots a_k$, the resulting graph is called the generalized Theta graph and is denoted by $\Theta(a_1, a_2, \cdots a_k)$.*

Note that in this graph, deg(u)= deg(v)= k and all the other vertices are of degree two.

Definition 5.1.11. *(see [28]) The graph $P_2 \square P_n$ is called a ladder. It is denoted by L_n.*

Definition 5.1.12. *(see [28]) The graph G with the vertex set* $\{u_0, u_1, \cdots u_{n+1}, v_0, v_1, \cdots v_{n+1}\}$ *and the edge set* $\{u_iu_{i+1}, v_iv_{i+1} : 0 \leq i \leq n\} \bigcup \{u_iv_i/i = 1, 2, \cdots n\}$ *is called ladder* L_{n+2}.

Definition 5.1.13. *(see [28]) The graph G with the vertex set* $\{u_1, u_2, \cdots u_n, v_1, v_2, \cdots v_n\}$ *and edge set* $\{u_iu_{(i+1)}, v_iv_{(i+1)}, v_iu_{(i+1)} : 1 \leq i \leq n-1\} \bigcup \{u_iv_i : 1 \leq i \leq n\}$ *is called a semiladder of length n.*

5.2 Some More Cycle Related Graphs

Theorem 5.2.1. $J_{n,m} \in \mathscr{V}_a$ *if and only if both n and m are odd.*

Proof. Let $u_1, u_2, \cdots u_m$ are the m vertices of C_{nm} which is adjacent to the central vertex w and $v_{i1}, v_{i2}, \cdots v_{i(n-1)}$ are the $(n-1)$ vertices between u_i and $u_{i+1}, i = 1, 2, \cdots m$ where $u_{m+1} = u_1$. Without loss of generality assume that $u_i = u_{i(mod\ m)}$ and $v_{ij} = v_{i(mod\ m)j(mod\ n)}$. First suppose that $J_{n,m} \in \mathscr{V}_a$. Then we have $(nm+1)a = 0$. This implies that $nm+1$ is even which in turn implies that both n and m are odd. Conversely, assume that n and m are odd. Define a labeling $\ell : E(J_{n,m}) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$\text{For } i = 1, 2, \cdots m \text{ do :}$$
$$\ell(u_iw) = a,$$
$$\ell(u_iv_{i1}) = b,$$
$$\ell(v_{ij}v_{i(j+1)}) = \begin{cases} c, & j = 1, 3, \cdots n-2, \\ b, & j = 2, 4, \cdots n-3. \end{cases}$$
$$\text{end for}$$
$$\ell(u_iv_{(i-1)(n-1)}) = b, i = 1, 2, 3, \cdots m.$$

With this labeling we get $\ell^+(v) = a$ for all $v \in V(J_{n,m})$. Obviously, ℓ is an a-sum V_4-magic labeling of $J_{n,m}$. □

Theorem 5.2.2. $J_{n,m} \in \mathscr{V}_0$ *for all n and m.*

Proof. Let the vertices of $J_{n,m}$ be as in the proof of theorem 5.2.1. We consider the following cases.

Case 1: m is even.
Define a labeling $\ell : E(J_{n,m}) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$\ell(u_iw) = c, \text{ for } i = 1, 2, \cdots m,$$
$$\text{For } j = 1, 2, \cdots n-2 \text{ do :}$$
$$\ell(v_{ij}v_{i(j+1)}) = \begin{cases} a, & \text{for } i = 1, 3, \cdots m-1, \\ b, & \text{for } i = 2, 4, \cdots m. \end{cases}$$

$$\text{end for}$$

$$\ell(u_i v_{i1}) = \begin{cases} a, & \text{for } i = 1, 3, \cdots m-1, \\ b, & \text{for } i = 2, 4, \cdots m. \end{cases}$$

$$\ell(u_{i+1} v_{i(n-1)}) = \begin{cases} a, & i = 1, 3, \cdots m-1 \\ b, & i = 2, 4, \cdots m+1 \end{cases}$$

Obviously, ℓ is a zero-sum V_4-magic labeling of $J_{n,m}$.

Case 2: m is odd.
Define a labeling $\ell : E(J_{n,m}) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$\begin{aligned}
&\ell(u_i w) = a, \text{ for } i = 1, 4, 5, \cdots m, \\
&\ell(u_2 w) = b, \ \ell(u_3 w) = c, \\
&\qquad \text{For } j = 1, 2, \cdots n-2 \text{ do :} \\
&\ell(v_{2j} v_{2(j+1)}) = a, \\
&\ell(v_{ij} v_{i(j+1)}) = b, \text{ for } i = 3, 5, \cdots m, \\
&\ell(v_{ij} v_{i(j+1)}) = c, \text{ for } i = 4, 6, \cdots m-1, m+1. \\
&\qquad \text{end for} \\
&\ell(u_2 v_{21}) = \ell(u_3 v_{2(n-1)}) = a, \\
&\ell(u_i v_{i1}) = \ell(u_{(i+1)} v_{i(n-1)}) = b, \text{for } i = 3, 5, \cdots m, \\
&\ell(u_i v_{i1}) = \ell(u_{(i+1)} v_{i(n-1)}) = c, \text{for } i = 4, 6, \cdots m-1, m+1.
\end{aligned}$$

Obviously, ℓ is a 0-sum V_4-magic labeling of $J_{n,m}$. □

Theorem 5.2.3. *$J_{n,m} \in \mathscr{V}_{a,0}$ if and only if both m and n are odd.*

Proof. From theorem 5.2.1 we have $J_{n,m} \in \mathscr{V}_a$ if and only if both n and m are odd. From theorem 5.2.2 it follows that $J_{n,m} \in \mathscr{V}_0$ for all n and m. Combining this we get the result. □

Theorem 5.2.4. *The windmill graph $D_n^{(m)} \in \mathscr{V}_a$ if and only if m is odd and n is even.*

Proof. Suppose $D_n^{(m)} \in \mathscr{V}_a$. Then by lemma 3.2.1, $[m(n-1)+1]a = 0$. This implies that $m(n-1)$ is odd. This holds only when m is odd and n is even. Conversely suppose that m is odd and n is even. Let $u_1^i, u_2^i, \cdots u_{(n-1)}^i$ are the vertices of i^{th} copy of K_n in $D_n^{(m)}$ and v is the common vertex. Define a labeling $\ell : E(D_n^{(m)}) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots m \text{ do :} \\
&\ell(u_j^i v) = a, j = 1, 2, \cdots n, \\
&\ell(u_j^i u_{j+1}^i) = a, j = 1, 2, \cdots n. \\
&\text{end for}
\end{aligned}$$

Obviously ℓ is an a-sum V_4-magic labeling of $D_n^{(m)}$. □

Theorem 5.2.5. *$D_n^{(m)} \in \mathscr{V}_0$ for all n and m.*

Proof. We consider the following cases.

Case 1: n is odd.
Label all the edges by a. Then we have $\ell^+(v) = 0$ for all $v \in V(D_n^{(m)})$.

Case 2: n is even.
Define a labeling $\ell : E(D_n^{(m)}) \longrightarrow V_4 \setminus \{0\}$ by

$$\text{For } i = 1, 2, \cdots m \text{ do :}$$
$$\ell(u_j^i u_{j+1}^i) = \begin{cases} a, & j = 1, 3, \cdots n-1, \\ b, & j = 2, 4, \cdots n. \end{cases}$$
$$\ell(u_j^i u_k^i) = c, \ j, k = 1, 2, \cdots n, \ k \neq j+1.$$
$$\text{end for}$$

Thus we get $\ell^+(v) = 0$ for all $v \in V(D_n^{(m)})$. Obviously ℓ is a zero-sum V_4-magic labeling of $D_n^{(m)}$.

This completes the proof of the theorem. □

Theorem 5.2.6. *$D_n^{(m)} \in \mathscr{V}_{a,0}$ if and only if m is odd and n is even.*

Proof. From theorem 5.2.4 we have the windmill graph $D_n^{(m)} \in \mathscr{V}_a$ if and only if m is odd and n is even. From theorem 5.2.5 we get $D_n^{(m)} \in \mathscr{V}_0$ for all n and m. Combining this the result follows. □

Theorem 5.2.7. *$F_m \notin \mathscr{V}_a$ for any m.*

Proof. Observe that F_m is the one-point union of m copies of a rooted triangle. Let the vertices of the i^{th} copy be $0, u_i$ and v_i. Assume that 0 is the common apex of the triangles. If F_m admits an a-sum V_4-magic labeling, then

$$\ell^+(u_i) = \ell^+(v_i) = a.$$

This implies that for all i,

$$\ell(u_i v_i) = b, \ \ell(0u_i) = \ell(0v_i) = c$$
$$\text{or} \ \ \ell(u_i v_i) = c, \ \ell(0u_i) = \ell(0v_i) = b.$$

In both the cases, $\ell^+(0) = 2ma = 0$. This is a contradiction. □

Theorem 5.2.8. *$F_m \in \mathscr{V}_0$ for all m.*

Proof. Label all the edges by a. Obviously this is a zero-sum V_4-magic labeling of F_m. $\square$

Theorem 5.2.9. *$F_m \notin \mathscr{V}_{a,0}$ for any m.*

Proof. From theorem 5.2.7 we have $F_m \notin \mathscr{V}_a$ for any m and theorem 5.2.8 states that $F_m \in \mathscr{V}_0$ for all m. Combining both we get the result. $\square$

Theorem 5.2.10. *$C_n(t) \in \mathscr{V}_a$ if and only if n is even and t is odd.*

Proof. First assume that $C_n(t) \in \mathscr{V}_a$. Then by lemma 3.2.1, $[(n-1)t+1]a = 0$. This equation holds if and only if n is even and t is odd. Conversely suppose that n is even and t is odd. Define a labeling $\ell : C_n(t) \longrightarrow V_4 \setminus \{0\}$ as follows:

$$\text{For } j = 1, 2, \cdots t \text{ do :}$$

$$\ell(u_{ij}u_{(i+1)j}) = \begin{cases} b, & \text{for } i = 1, 3, \cdots n-1, \\ c, & \text{for } i = 2, 4, \cdots n. \end{cases}$$

$$\text{end for}$$

Obviously $\ell^+(u_{ij}) = a,\ \ell^+(v) = a$. $\square$

Theorem 5.2.11. *(see [28]) $C_n(t) \in \mathscr{V}_0$ for all n and t.*

Theorem 5.2.12. *$C_n(t) \in \mathscr{V}_{a,0}$ if and only if n is even and t is odd.*

Proof. From theorem 5.2.10, $C_n(t) \in \mathscr{V}_a$ if and only if n is even and t is odd and from theorem 5.2.11 we have $C_n(t) \in \mathscr{V}_0$ for all n and t. Combining all this the result follows. $\square$

Theorem 5.2.13. *$T_n^{(m)} \in \mathscr{V}_a$ if and only if m is even and n is odd.*

Proof. Suppose that $T_n^{(m)} \in \mathscr{V}_a$. Let $u_{ij}, i = 1, 2, \cdots n, j = 1, 2, \cdots m$ be the vertices in the graph. Without loss of generality assume that $u_{(n+1)j} = u_{1j}$. Then we have

$$\sum_{i=2}^{n}\sum_{j=2}^{m} \ell^+(u_{ij}) + \sum_{j=1}^{m} \ell^+(u_{1j}) = 0$$

This implies that $[(m-1)n+1]$ is even which again implies that m is even and n is odd. Conversely assume that m is even and n is odd. Define a labeling $\ell : E(T_n^{(m)}) \longrightarrow V_4 \setminus \{0\}$ by

$$\text{For } i = 1, 3, \cdots n \text{ do :}$$

$$\ell(u_{ij}u_{(j+1)}) = \begin{cases} b, & j = 1, 3, \cdots m-1 \\ c, & j = 2, 4, \cdots n \end{cases}$$

$$\text{end for}$$

$$\text{For } i = 2, 4, \cdots n-1 \text{ do :}$$

$$\ell(u_{ij}u_{(j+1)}) = \begin{cases} c, & j = 2, 3, \cdots m-2 \\ b, & j = 1, m-1, m \end{cases}$$

end for

Clearly $\ell^+(v) = a$ for all $v \in V(T_n^{(m)})$. □

Theorem 5.2.14. *(see [28]) $T_n^{(m)} \in \mathscr{V}_0$ for all n and m.*

Theorem 5.2.15. *$T_n^{(m)} \in \mathscr{V}_{a,0}$ if and only if m is even and n is odd.*

Proof. From theorem 5.2.13 we get $T_n^{(m)} \in \mathscr{V}_a$ if and only if m is even and n is odd and from theorem 5.2.14 we get $T_n^{(m)} \in \mathscr{V}_0$ for all n and m. Combining them the result follows. □

Theorem 5.2.16. *For all $m, n \geq 3$, $C_m@C_n \in \mathscr{V}_a$ if and only if $n(m-1)$ is even.*

Proof. Suppose that $C_m@C_n \in \mathscr{V}_a$. Then $[n(m-1)]a = 0$. This implies that $n(m-1)$ is even. Now let $u_1, u_2, \cdots u_n$ be the vertices of C_n and $v_{1j}, v_{2j}, \cdots v_{(m-2)j}$ be the vertices of $C_j, j = 1, 2, \cdots n$. We consider the following cases.

Case 1: n is even and m is odd.

For $j = 1, 2, \cdots n$ do :

$$\ell(u_ju_{(j+1)}) = a, \ \ell(u_jv_{1j}) = c,$$

$$\ell(u_{(j+1)}v_{(m-2)j}) = b,$$

$$\ell(v_{ij}v_{i(j+1)}) = \begin{cases} b, & i = 1, 3, \cdots m-4, \\ c, & i = 2, 4, \cdots m-3. \end{cases}$$

end for

Case 2: Both n and m are even.

$$\ell(u_ju_{(j+1)}) = a, \ j = 1, 2, \cdots n,$$

For $j = 1, 3, \cdots n-1$ do :

$$\ell(v_{ij}v_{i(j+1)}) = \begin{cases} c, & i = 1, 3, \cdots m-3, \\ b, & i = 2, 4, \cdots m-4. \end{cases}$$

end for

For $j = 2, 4, \cdots n$ do :

$$\ell(v_{ij}v_{i(j+1)}) = \begin{cases} b, & i = 1, 3, \cdots m-3, \\ c, & i = 2, 4, \cdots m-4. \end{cases}$$

end for

Case 3: Both n and m are odd.

$$
\begin{aligned}
&\text{For } j = 1, 2, \cdots n \text{ do :}\\
\ell(u_j u_{(j+1)}) &= a,\ \ell(u_j v_{1j}) = b,\\
\ell(u_{(j+1)} v_{(m-2)j}) &= c,\\
\ell(v_{ij} v_{i(j+1)}) &= \begin{cases} c, & i = 1, 3, \cdots m-4,\\ b, & i = 2, 4, \cdots m-3. \end{cases}\\
&\text{end for}
\end{aligned}
$$

Thus ℓ is an a-sum V_4-magic labeling of $C_m@C_n$. This completes the proof. □

Theorem 5.2.17. *(see [13]) For all $m, n \geq 3$, $C_m@C_n \in \mathscr{V}_0$.*

Theorem 5.2.18. *For all $m, n \geq 3$, $C_m@C_n \in \mathscr{V}_{a,0}$ if and only if $n(m-1)$ is even.*

Proof. From theorem 5.2.16 we have for all $m, n \geq 3$, the $C_m@C_n \in \mathscr{V}_a$ if and only if $n(m-1)$ is even and theorem 5.2.17 states that for all $m, n \geq 3$, $C_m@C_n \in \mathscr{V}_0$. Combining the theorems we get the result. □

Theorem 5.2.19. *For any $n \geq 4$, the bipyramid graph $BP(n)$ is a-sum V_4-magic if and only if n is even.*

Proof. Suppose that $BP(n) \in \mathscr{V}_a$. This implies that $(n+2)a = 0$. Thus we obtained that n is even. Conversely assume that n is even.

$$
\begin{aligned}
&\text{For } i = 1, 2 \text{ do :}\\
&\ell(v_i u_1) = c,\\
&\ell(v_i u_j) = b,\ j = 2, 3, \cdots n\\
&\text{end for}\\
\ell(u_j u_{(j+1)}) &= \begin{cases} b, & j = 1, 3, \cdots n-1,\\ c, & j = 2, 4, \cdots n. \end{cases}
\end{aligned}
$$

Obviously $BP(n) \in \mathscr{V}_a$. □

Theorem 5.2.20. *(see [13]) For any $n \geq 4, BP(n)$ is zero-sum V_4-magic.*

Theorem 5.2.21. *For any $n \geq 4, BP(n) \in \mathscr{V}_{a,0}$ if and only if n is even.*

Proof. Theorem 5.2.19 states that for any $n \geq 4$, the bipyramid graph $BP(n)$ is a-sum V_4-magic if and only if n is even and from theorem 5.2.20 we have for any $n \geq 4, BP(n)$ is zero-sum V_4-magic. Combining the two theorems the result follows. □

Theorem 5.2.22. *Consider the bipyramid graph $BP(G)$ based on G. We have the following.*

i) If G is a-sum V_4-magic and number of vertices in G is odd, then $BP(G)$ is a-sum V_4-magic.

ii) If G is a-sum V_4-magic and number of vertices in G is even, then $BP(G)$ is 0-sum V_4-magic.

iii) If G is 0-sum V_4-magic and number of vertices in G is even, then $BP(G)$ is both a-sum V_4-magic and 0-sum V_4-magic.

iv) If G is 0-sum V_4-magic and number of vertices in G is odd, then $BP(G)$ is 0-sum V_4-magic.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of G and v_1, v_2 be the remaining vertices in $BP(G)$.

i) Suppose that G is a-sum V_4-magic and number of vertices in G is odd. Then $\ell^+(u_j) = a$ for all $j = 1, 2, \ldots, n$. Since $|V(G)|$ is odd, $deg(v_i)$ is odd for $i = 1, 2$. Moreover, $deg(u_j)$ in $BP(G)$ is $[(deg(u_j)$ in $G) + 2]$. Thus by defining a labeling $\ell(u_j v_i) = a, j = 1, 2, \ldots, n,\ i = 1, 2$ the result follows.

ii) Suppose that G is a-sum V_4-magic and number of vertices in G is even. That is $\ell^+(u_j) = a$ for all $j = 1, 2, \ldots, n$. Define a labeling as follows:

$$\ell(u_j v_1) = b,\ \ell(u_j v_2) = c,\ j = 1, 2, \ldots, n.$$

Then $BP(G)$ is 0-sum V_4-magic.

iii) Suppose that G is 0-sum V_4-magic and number of vertices in G is even. By defining a labeling $\ell(u_j v_i), j = 1, 2, \ldots, n\ i = 1, 2$, $BP(G)$ becomes zero-sum V_4-magic. Moreover, $BP(G)$ is a-sum V_4-magic if we define a labeling as follows:

$$\begin{aligned} &\ell(u_1 v_1) = c,\ \ell(u_1 v_2) = b \\ &\qquad \text{For } j = 2, 3, \ldots, n \\ &\ell(u_j v_1) = b,\ \ell(u_j v_2) = c \\ &\qquad \text{end for} \end{aligned}$$

iv) Define a labeling as follows:

$$\begin{aligned} &\qquad \text{For } i = 1, 2, \\ &\ell(v_i u_1) = b,\ \ell(v_i u_2) = c, \\ &\ell(v_i u_j) = a,\ j = 1, 2, \ldots, n \\ &\qquad \text{end for} \end{aligned}$$

Then the result follows. $\square$

5.3 Ladder Graphs

Theorem 5.3.1. *Ladders L_n are a-sum V_4-magic for all n.*

Proof. Let $u_1, u_2, \cdots u_n$ and $v_1, v_2, \cdots v_n$ be the vertices of a ladder L_n such that $E(G) = \{u_i u_{(i+1)}/i = 1, 2, \cdots n-1\} \cup \{v_j v_{(j+1)}/j = 1, 2, \cdots n-1\} \cup \{u_i v_i/i = 1, 2, \cdots n\}$. Define a labeling $\ell : E(L_n) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_1v_1) &= \ell(u_nv_n) = b, \\ \ell(u_iv_i) &= a, \text{for } i = 2, 3, \cdots n-1, \\ \ell(u_iu_{(i+1)}) &= \ell(v_iv_{(i+1)}) = c, \text{for } i = 1, 2, \cdots n-1. \end{aligned}$$

Then clearly ℓ is an a-sum V_4-magic labeling of L_n. □

Theorem 5.3.2. *(see [28]) $L_n \in \mathscr{V}_0$ for all n.*

Theorem 5.3.3. *$L_n \in \mathscr{V}_{a,0}$ for all n.*

Proof. From theorem 5.3.1 we have ladders L_n are a-sum V_4-magic for all n. Theorem 5.3.2 states that $L_n \in \mathscr{V}_0$ for all n. From the two theorems the result follows. □

Theorem 5.3.4. *(see [28]) $L_{n+2} \in \mathscr{V}_a$ for all n.*

Theorem 5.3.5. *$L_{n+2} \notin \mathscr{V}_0$ for any n.*

Proof. Since the graph has pendant edges it is not zero-sum V_4-magic for any n. □

Theorem 5.3.6. *Semiladders are a-sum V_4-magic for all n.*

Proof. Let G be a semiladder of length n. We consider two cases.

Case 1: n is odd

Define a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_1v_1) &= \ell(u_nv_n) = b, \\ \ell(u_iv_i) &= a, \text{for } i = 2, 3, \cdots n-1, \\ \ell(v_iu_{(i+1)}) &= a, \text{for } i = 1, 2, \cdots n-1. \\ &\text{For } i = 1, 3, \cdots n-2 \text{ do :} \\ \ell(u_iu_{(i+1)}) &= c, \\ \ell(v_iv_{(i+1)}) &= b. \\ &\text{end for} \\ &\text{For } i = 2, 4, \cdots n-1 \text{ do :} \\ \ell(u_iu_{(i+1)}) &= b, \end{aligned}$$

$$\ell(v_i v_{(i+1)}) = c.$$
end for

Thus ℓ is an a-sum V_4-magic labeling of G.

Case 2: n is even
Define a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(u_1 v_1) &= b,\ \ell(u_n v_n) = c,\\
\ell(u_i v_i) &= a, \text{for } i = 2, 3, \cdots n-1,\\
\ell(v_i u_{(i+1)}) &= a, \text{for } i = 1, 2, \cdots n-1,\\
&\text{For } i = 1, 3, \cdots n-1 \text{ do :}\\
\ell(u_i u_{(i+1)}) &= c,\\
\ell(v_i v_{(i+1)}) &= b.\\
&\text{end for}\\
&\text{For } i = 2, 4, \cdots n-2 \text{ do :}\\
\ell(u_i u_{(i+1)}) &= b,\\
\ell(v_i v_{(i+1)}) &= c.\\
&\text{end for}
\end{aligned}$$

Thus ℓ is an a-sum V_4-magic labeling of G. □

Theorem 5.3.7. *(see [28]) Semiladders are zero-sum V_4-magic for all n.*

Theorem 5.3.8. *If G is a semiladder, then $G \in \mathscr{V}_{a,0}$.*

Proof. By theorem 5.3.6, semiladders are a-sum V_4-magic for all n. And by theorem 5.3.7, semiladders are zero-sum V_4-magic for all n. Hence the proof follows. □

5.4 Path Related Graphs

Theorem 5.4.1. *The composition $P_n[K_2^c]$ is a-sum V_4-magic for all n.*

Proof. Let $v_1, v_2, \cdots v_n$ be the vertices of P_n and x, y be that of K_2^c. Let u_i denote the vertices (v_i, x) and w_i denote (v_i, y) of $P_n[K_2^c], 1 \leq i \leq n$. Define a labeling $\ell : E(P_n[K_2^c]) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(u_i u_{(i+1)}) &= b, \text{for } i = 1, 2, \cdots n-1,\\
\ell(w_1 w_2) &= b,\\
\ell(w_i w_{(i+1)}) &= c, \text{for } i = 2, 3, \cdots n-1,
\end{aligned}$$

$$\ell(u_1w_2) = c,$$
$$\ell(u_iw_{(i+1)}) = b, \text{for } i = 2, 3, \cdots n-1,$$
$$\ell(u_{(i+1)}w_i) = c, \text{for } i = 1, 2, \cdots n-1.$$

Thus ℓ is an a-sum V_4-magic labeling of $P_n[K_2^c]$. $\square$

Theorem 5.4.2. *(see [28]) $P_n[K_2^c] \in \mathscr{V}_0$ for all n.*

Theorem 5.4.3. *$P_n[K_2^c] \in \mathscr{V}_{a,0}$ for all n.*

Proof. Combining theorems 5.4.1 and 5.4.2, we have $P_n[K_2^c] \in \mathscr{V}_{a,0}$ for all n. $\square$

Theorem 5.4.4. *The planar grid $P_m \square P_n$ is a-sum V_4-magic if and only if mn is even.*

Proof. Suppose that $P_m \square P_n \in \mathscr{V}_a$. Let $(i,j), i = 0, 1, \cdots m-1, j = 0, 1, \cdots n-1$ denote the vertices of $P_m \square P_n$. By lemma 3.2.1, we have $\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \ell^+((i,j)) = 0$. Thus we have mn is even. For the converse consider the following cases.

Case 1: Both m and n are even.
Define a labeling $\ell : E(P_m \square P_n) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$\begin{aligned}
&\text{For } i = 0, 1, \cdots m-2 \text{ do :}\\
\ell((i,j)(i+1,j)) &= b, j = 0, n-1\\
&\text{end for}\\
&\text{For } j = 0, 1, \cdots n-2 \text{ do :}\\
\ell((i,j)(i,j+1)) &= c, i = 0, m-1\\
&\text{end for}\\
&\text{For } i = 1, 2, \cdots m-2 \text{ do :}\\
\ell((i,j)(i,j+1)) &= \begin{cases} a, & j = 0, 2, \cdots n-2 \\ c, & j = 1, 3, \cdots n-3 \end{cases}\\
&\text{end for}\\
&\text{For } j = 1, 2, \cdots n-2 \text{ do :}\\
\ell((i,j)(i+1,j)) &= \begin{cases} a, & i = 0, 2, \cdots m-2 \\ b, & j = 1, 3, \cdots m-3 \end{cases}\\
&\text{end for}
\end{aligned}$$

With this labeling $P_m \square P_n$ is a-sum V_4 magic.

Case 2: m is even and n is odd.

Define a labeling $\ell : E(P_m \Box P_n) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$\ell((i,0)(i+1,0)) = b,\ i = 0,1,\cdots m-2$$

$$\ell((i,n-1)(i+1,n-1)) = \begin{cases} b, & i = 0,2,\cdots m-2 \\ a, & i = 1,3,\cdots m-3 \end{cases}$$

For $j = 0,1,\cdots n-2$ do :

$$\ell((i,j)(i,j+1)) = c,\ i = 0, m-1$$

end for

For $j = 1,2,\cdots n-2$ do :

$$\ell((i,j)(i+1,j)) = \begin{cases} a, & i = 0,2,\cdots m-2 \\ c, & i = 1,3,\cdots m-3 \end{cases}$$

end for

For $i = 1,2,\cdots m-2$ do :

$$\ell((i,j)(i,j+1)) = \begin{cases} a, & j = 0,2,\cdots n-3 \\ b, & j = 1,3,\cdots n-2 \end{cases}$$

end for

Obviously $P_m \Box P_n$ is a-sum V_4 magic.

Case 3: m is odd and n is even.

By interchanging the roles of m and n in Case 2, we get $\ell^+(v) = a$ for all $v \in V(P_m \Box P_n)$.

This completes the proof. $\square$

Theorem 5.4.5. *(see [28]) $P_m \Box P_n \in \mathscr{V}_0$ for all m and n.*

Theorem 5.4.6. *$P_m \Box P_n \in \mathscr{V}_{a,0}$ if and only if mn is even.*

Proof. From theorem 5.4.4 we have $P_m \Box P_n \in \mathscr{V}_a$ if and only if mn is even and by theorem 5.4.5, we have $P_m \Box P_n \in \mathscr{V}_0$ for all m and n. Combining the two theorems the result follows. $\square$

Theorem 5.4.7. *If the generalized Theta graph $\Theta(a_1, a_2, \cdots a_k)$ is a-sum V_4-magic then either odd number of a_i's are odd or even number of a_i's are even.*

Proof. First suppose that $\Theta(a_1, a_2, \cdots a_k)$ is a-sum V_4 magic. Then we have $\left(\sum_{i=1}^{k} a_i - k\right) a = 0$. This implies that $\left(\sum_{i=1}^{k} a_i\right) a = ka$. This is if and only if both $\sum_{i=1}^{k} a_i$ and k are odd or even simultaneously. This happens if and only if odd number of a_i's are odd or even number of a_i's are even. $\square$

Theorem 5.4.8. *Let $\Theta(a_1, a_2, \cdots a_k)$ be a generalized Theta graph. If k and even number of a_i's are even then $\Theta(a_1, a_2, \cdots a_k)$ is a-sum V_4 magic.*

Proof. Let $v_1^i, v_2^i, \cdots v_{a_i-1}^i$ be the vertices of the i^{th} path and let u,w be the common vertices. Define a labeling $\ell : \Theta(a_1, a_2, \cdots a_k) \longrightarrow V_4 \setminus \{0\}$ by

$$\text{For } i = 1, 2, \cdots k-1 \text{ do :}$$
$$\ell(uv_1^i) = b$$
$$\text{end for}$$
$$\ell(uv_1^k) = c$$
$$\text{For } i = 1, 2, \cdots k \text{ do :}$$
$$\ell(v_j^i v_{j+1}^i) = \begin{cases} c, & j = 1, 3, \cdots a_i - 2, \quad \text{if } a_i \text{ is odd} \\ & j = 1, 3, \cdots a_i - 3, \quad \text{if } a_i \text{ is even} \end{cases}$$
$$\ell(v_j^i v_{j+1}^i) = \begin{cases} b, & j = 2, 4, \cdots a_i - 2, \quad \text{if } a_i \text{ is even} \\ & j = 2, 4, \cdots a_i - 3, \quad \text{if } a_i \text{ is odd} \end{cases}$$

Now label the edge $v_{a_k-1}^i w, i = 1, 2, \cdots k$ in the following way.
If $\ell(v_{a_k-2}^i v_{a_k-1}^i) = b$, let $\ell(v_{a_k-1}^i w) = c$ and viceversa. Thus ℓ is an a-sum V_4-magic labeling of $\Theta(a_1, a_2, \cdots a_k)$. □

Theorem 5.4.9. *(see [13]) $\Theta(a_1, a_2, \cdots a_k)$ is zero-sum V_4-magic for any sequence $a_1, a_2, \cdots a_k$.*

Theorem 5.4.10. *Let $\Theta(a_1, a_2, \cdots a_k)$ be a generalized Theta graph. If k and even number of a_i's are even then $\Theta(a_1, a_2, \cdots a_k) \in \mathscr{V}_{a,0}$.*

Proof. By theorem 5.4.8, if k and even number of a_i's are even then $\Theta(a_1, a_2, \cdots a_k)$ is a-sum V_4 magic. Also from theorem 5.4.9, it follows that $\Theta(a_1, a_2, \cdots a_k)$ is zero-sum V_4-magic for any sequence $a_1, a_2, \cdots a_k$. Combining both we get the result. □

5.5 Book Graphs

Theorem 5.5.1. *For any $n \geq 3$ and $k \geq 1$, $B(n, k) \in \mathscr{V}_a$ if and only if $(n-2)k$ is even.*

Proof. First assume that $B(n, k) \in \mathscr{V}_a$. Then $[(n-2)k + 2]a = 0$. This implies that $(n-2)k$ is even. Conversely assume that $(n-2)k$ is even. We consider the following cases.

Case 1: Both n and k are even.
Define a labeling $\ell : E(B(n, k)) \longrightarrow V_4 \setminus \{0\}$ by

$$\ell(uv) = a,$$
$$\text{For } j = 1, 2, \cdots k \text{ do :}$$

$$\ell(uu_1^j) = \ell(vu_{(n-2)}^j) = b,$$
$$\ell(u_{2i}^j u_{2i+1}^j) = b, \text{for } i = 1, 2, \cdots \frac{n-4}{2},$$
$$\ell(u_{2i-1}^j u_{2i}^j) = c, \text{for } i = 1, 2, \cdots \frac{n-4}{2}.$$
end for

Then ℓ is an a-sum V_4-magic labeling of $B(n,k)$.

Case: 2 n is odd and k is even,
Define a labeling $\ell : E(B(n,k)) \longrightarrow V_4 \setminus \{0\}$ by

$$\ell(uv) = a,$$
For $j = 1, 2, \cdots k$ do :
$$\ell(uu_1^j) = b,$$
$$\ell(vu_{(n-2)}^j) = c,$$
$$\ell(u_{2i}^j u_{2i+1}^j) = b, \text{for } i = 1, 2, \cdots \frac{n-3}{2},$$
$$\ell(u_{2i-1}^j u_{2i}^j) = c, \text{for } i = 1, 2, \cdots \frac{n-3}{2}.$$
end for

Case 3: n is even and k is odd
Define a labeling $\ell : E(B(n,k)) \longrightarrow V_4 \setminus \{0\}$ by

$$\ell(uv) = c,$$
For $j = 1, 2, \cdots k$ do :
$$\ell(uu_1^j) = \ell(vu_{(n-2)}^j) = b,$$
$$\ell(u_{2i}^j u_{2i+1}^j) = b, \text{for } i = 1, 2, \cdots \frac{n-4}{2},$$
$$\ell(u_{2i-1}^j u_{2i}^j) = c, \text{for } i = 1, 2, \cdots \frac{n-2}{2}.$$
end for

Thus ℓ is an a-sum V_4-magic labeling of $B(n,k)$. □

Theorem 5.5.2. *(see [13]) For any $n \geq 3$ and $k \geq 1, B(n,k)$ is zero-sum V_4 magic.*

Theorem 5.5.3. *For any $n \geq 3$ and $k \geq 1, B(n,k) \in \mathscr{V}_{a,0}$ if and only if $(n-2)k$ is even.*

Proof. From theorem 5.5.1, $B(n,k) \in \mathscr{V}_a$ if and only if $(n-2)k$ is even and from theorem 5.5.2, for any $n \geq 3$ and $k \geq 1, B(n,k)$ is zero-sum V_4 magic. Thus we have the result. □

Theorem 5.5.4. *The book B_n is a-sum V_4-magic for all n.*

Proof. Let w_1, w_2 be the vertices of the common edge. Let $\{u_1, u_2, \cdots u_n\} \cup \{v_1, v_2, \cdots v_n\}$ be the vertices of B_n.

Case 1: n is odd.

$$\begin{aligned} \ell(w_1w_2) &= c, \\ &\text{For } i = 1, 2, \cdots n \text{ do :} \\ \ell(w_1u_i) &= \ell(w_2v_i) = b, \\ \ell(u_iv_i) &= c. \\ &\text{end for} \end{aligned}$$

Case 2: n is even.

$$\begin{aligned} \ell(w_1w_2) &= a, \\ &\text{For } i = 1, 2, \cdots n \text{ do :} \\ \ell(w_1u_i) &= \ell(w_2v_i) = b, \\ \ell(u_iv_i) &= c. \\ &\text{end for} \end{aligned}$$

Clearly ℓ is an a-sum V_4-magic labeling of B_n. $\square$

Theorem 5.5.5. *$B_n \in \mathscr{V}_0$ for all n.*

Proof. We consider the following cases:

Case 1: n is odd.
Label all the edges by a. Then we get $\ell^+(v) = 0$ for all $v \in V(G)$.

Case 2: n is even.
Define a labeling $\ell : E(B_n) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(w_1w_2) &= a, \\ \ell(u_1w_1) &= \ell(v_1w_1) = \ell(u_1v_1) = c, \\ \ell(w_1u_i) &= \ell(w_2v_i) = b, i = 2, 3, \cdots n, \\ \ell(u_iv_i) &= b, i = 2, 3, \cdots n. \end{aligned}$$

With this labeling we get $\ell^+(v) = 0$ for all $v \in V(G)$. $\square$

Corollary 5.5.6. *$B_n \in \mathscr{V}_{a,0}$ for all n.*

Proof. The proof follows from theorems 5.5.4 and 5.5.5. $\square$

5.6 $K_{m,n}$ and K_n-Related Graphs

Consider the complete bipartite graph $K_{m,n}$.

Theorem 5.6.1. *For $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is a-sum V_4-magic if and only if $m+n$ is even.*

Proof. First assume that $K_{m,n}$ is a-sum V_4 magic. Let $\{u_i, i = 1, 2, \cdots m\} \cup \{v_j, j = 1, 2, \cdots n\}$ be the vertices of the graph with $E(G) = \{u_i v_j : i = 1, 2, \cdots m, j = 1, 2, \cdots n\}$. Then we have $\sum_{i=1}^{m} \ell^+(u_i) + \sum_{j=1}^{n} \ell^+(v_j) = 0$. This implies that $m+n$ is even. Conversely suppose that $m+n$ is even. Then we have the following cases.

Case 1: m and n are odd.
Define a labeling $\ell : E(K_{m,n}) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} &\text{For } i = 1, 2, \cdots m \text{ do :} \\ &\ell(u_i v_j) = a, \ j = 1, 2, \cdots n. \\ &\text{end for} \end{aligned}$$

Thus $\ell^+(v) = a$.

Case 2: m and n are even.

$$\begin{aligned} &\ell(u_i v_2) = c, i = 1, 3, 4, \cdots m, \\ &\ell(u_2 v_j) = c, j = 1, 3, 4, \cdots n, \\ &\ell(u_2 v_2) = b, \\ &\qquad \text{For } i = 1, 3, 4, \cdots m \text{ do :} \\ &\ell(u_i v_j) = b, j = 1, 3, 4, \cdots n. \\ &\qquad \text{end for} \end{aligned}$$

Obviously, $K_{m,n}$ is a-sum V_4 magic. □

Theorem 5.6.2. *(see [13]) $K_{m,n}$ is zero-sum V_4-magic for all m and n.*

Theorem 5.6.3. *$K_{m,n} \in \mathscr{V}_{a,0}$ if and only if $m+n$ is even.*

Proof. The proof follows from theorems 5.6.1 and 5.6.2. □

Now we discuss the classification of certain V_4-magic graphs obtained from K_n.
Consider the complete graph K_n of order $n \geq 4$ with $V(K_n) = \{v_1, v_2, \cdots v_n\}$ and for each r such that $2 \leq r \leq n-2$, let $\mathcal{G}(n, r)$ be a spanning subgraph of K_n with $E(\mathcal{G}(n, r)) = E(K_n) - \{v_i v_j : 1 \leq i < j \leq r\}$. In [13], Sin Min Lee et al. proved that, for any $n > 3$, $K_n \setminus e$ the complete graph with one edge removed is V_4-magic. Here we generalize this result as follows.

Theorem 5.6.4. *The graph $\mathcal{G}(n,r)$ is a-sum V_4-magic if and only if n is even.*

Proof. Suppose $\mathcal{G}(n,r) \in \mathscr{V}_a$. Then by lemma 3.2.1, $\sum_{i=1}^{n} \ell^+(v_i) = 0$ where $v_1, v_2, \cdots, v_n$ are the vertices of the graph $\mathcal{G}(n,r)$. That is $na = 0$. This implies that n is even. Conversely suppose that n is even.

Case 1: r is odd
Label all the edges by a. With this labeling we have $\ell^+(v) = a$ for all $v \in V(\mathcal{G}(n,r))$.

Case 2: r is even
Define a labeling $\ell : E(\mathcal{G}(n,r)) \longrightarrow V_4 \setminus \{0\}$ by

$$
\begin{aligned}
&\text{For } i = 1, 2, \cdots, r \text{ do:}\\
\ell(v_i v_j) = &\begin{cases} b, j = r+1, r+2, \cdots, n-2, n \\ c, j = n-1 \end{cases}\\
&\text{end for}\\
&\text{For } i = r+1, r+2, \cdots, n \text{ do:}\\
\ell(v_i v_j) = &\; a, j = r+2, r+3, \cdots, n\\
&\text{end for}
\end{aligned}
$$

One can easily verify that, $\ell^+(v) = a$ for all $v \in V$. That is, $\ell^+(v) = a$ for all $v \in V(\mathcal{G}(n,r))$. Thus ℓ is an a-sum V_4-magic labeling of $\mathcal{G}(n,r)$. This completes the proof. □

Theorem 5.6.5. *The graph $\mathcal{G}(n,r)$ is zero-sum V_4-magic for all n.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of $\mathcal{G}(n,r)$.

Case 1: r is even.

Subcase 1: n is odd.
Define a labeling $\ell : E(\mathcal{G}(n,r)) \longrightarrow V_4 \setminus \{0\}$ by

$$
\begin{aligned}
&\text{For } i = 1, 2, \cdots, r\\
\ell(v_i v_n) &= a\\
\ell(v_i v_{n-1}) &= b\\
\ell(v_i v_{n-2}) &= c\\
\ell(v_i v_j) &= a, j = r+1, \cdots, n-3\\
&\text{end for}\\
&\text{For } i = r+1, \cdots, n\\
\ell(v_i v_j) &= a, j = r+2, \cdots, n\\
&\text{end for}
\end{aligned}
$$

With this labeling we get ℓ is a zero sum V_4-magic labeling of $\mathcal{G}(n,r)$.

Subcase 2: n is even.

Define a labeling $\ell : E(\mathcal{G}(n,r)) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} &\text{For } j = r+1, \cdots, n \\ \ell(v_1v_j) &= a \\ \ell(v_2v_j) &= b \\ &\text{end for} \\ \ell(v_iv_j) &= c, i = 3, \cdots, n \end{aligned}$$

With this labeling we have ℓ is a zero sum V_4-magic labeling of $\mathcal{G}(n,r)$.

Case 2: r is odd.

Subcase 1 : n is odd.

Label all the edges by a. then we get $\ell^+(v) = 0$ for all $v \in V(\mathcal{G}(n,r))$.

Subcase 2 : n is even.

The edge labeling of the graph is shown in the following table.

	v_1	v_2	v_3	v_4	$\cdots$	v_r	v_{r+1}	v_{r+2}	v_{r+3}	v_{r+4}	$\cdots$	v_{n-3}	v_{n-2}	v_{n-1}	v_n
v_1							a	a	a	a	$\ldots$	a	c	b	a
v_2							b	b	b	b	$\ldots$	b	a	c	b
v_3							c	c	c	c	$\ldots$	c	b	a	c
v_4							a	a	a	a	$\ldots$	a	c	b	a
$\vdots$							$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_r							a	a	a	a	$\ldots$	a	c	b	a
v_{r+1}	a	b	c	a	$\ldots$	a		a	a	a	$\ldots$	a	a	a	a
v_{r+2}	a	b	c	a	$\ldots$	a	a		a	a	$\ldots$	a	a	a	a
v_{r+3}	a	b	c	a	$\ldots$	a	a	a		a	$\ldots$	a	a	a	a
v_{r+4}	a	b	c	a	$\ldots$	a	a	a	a		$\ldots$	a	a	a	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{n-3}	a	b	c	a	$\ldots$	a	a	a	a	a	$\ldots$		a	a	a
v_{n-2}	c	a	b	c	$\ldots$	c	a	a	a	a	$\ldots$	a		a	a
v_{n-1}	b	c	a	b	$\ldots$	b	a	a	a	a	$\ldots$	a	a		a
v_n	a	b	c	a	$\ldots$	a	a	a	a	a	$\ldots$	a	a	a	

With this labeling we get $\ell^+(v) = 0$ for all $v \in V(G)$. Thus ℓ is a zero-sum V_4-magic labeling of $\mathcal{G}(n,r)$. This completes the proof. □

Theorem 5.6.6. *The graph $\mathcal{G}(n,r) \in \mathscr{V}_{a,0}$ if and only if n is even.*

Proof. The proof follows from theorems 5.6.4 and 5.6.5. □

Consider the graph $\mathcal{G}(n,r)$ together with pendant edges at the vertices $v_1, v_2, \cdots, v_r$ and triangles at the vertices $v_{r+1}, v_{r+2}, \cdots, v_n$. We denote this graph by $\mathcal{G}_r^n(G)$.

Theorem 5.6.7. *$\mathcal{G}_r^n(G)$ is a-sum V_4-magic if and only if both n and r are of the same parity.*

Proof. First suppose that $\mathcal{G}_r^n(G)$ is a-sum V_4 magic. Then by lemma 3.2.1, $\sum_{i=1}^{n} \ell^+(v_i) + \sum_{i=1}^{r} \ell^+(u_i) + \sum_{i=r+1}^{n} \sum_{j=1}^{2} \ell^+(w_i^j) = 0$. That is $[n + r + 2(n - r)]a = 0$. This implies that either both n and r are even or both are odd. Conversely assume that both n and r are of same parity.

Case 1: n and r are even.

Define a labeling $\ell : E(\mathcal{G}_r^n(G)) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(v_i u_i) &= a, i = 1, 2, \cdots, r \\
&\text{For } i = 1, 2, \cdots, n \\
\ell(v_i v_j) &= a, j = r + 1, \cdots, n \\
&\text{end for} \\
&\text{For } i = r + 1, \cdots, n \\
\ell(v_i v_j) &= a, j = r + 1, \cdots, n \\
\ell(v_i w_i^j) &= b, j = 1, 2 \\
\ell(w_i^1 w_i^2) &= c \\
&\text{end for}
\end{aligned}$$

With the above labeling we have $\ell^+(v) = a$ for all $v \in V(\mathcal{G}_r^n(G))$. Thus ℓ is an a-sum V_4-magic labeling of $\mathcal{G}_r^n(G)$.

Case 2: n and r are odd.

Subcase 1: $r = n - 2$

Define a labeling $\ell : E(\mathcal{G}_r^n(G)) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(v_i u_i) &= a, i = 1, 2, \cdots, r \\
&\text{For } j = n - 1, n \\
\ell(v_i v_j) &= a, i = 1, 2, \cdots, r - 1 \\
\ell(v_r v_j) &= b \\
&\text{end for} \\
\ell(v_{n-1} v_n) &= c \\
&\text{For } i = r + 1, \cdots, n \\
\ell(v_i w_i^j) &= b, j = 1, 2 \\
\ell(w_i^1 w_i^2) &= c \\
&\text{end for}
\end{aligned}$$

With the above labeling ℓ is an a-sum V_4-magic labeling of $\mathcal{G}_r^n(G)$.

Subcase 2: $r \neq n-2$. Define a labeling $\ell : E(\mathcal{G}_r^n(G)) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(v_i u_i) &= a, i = 1, 2, \cdots, r \\
&\text{For } j = r+1, \cdots, n \\
\ell(v_i v_j) &= a, i = 1, 2, \cdots, r-1 \\
\ell(v_r v_j) &= b \\
&\text{end for} \\
&\text{For } j = r+2, \cdots, n \\
\ell(v_i v_j) &= a, i = r+2, \cdots, n \\
\ell(v_{r+1} v_j) &= c \\
&\text{end for} \\
&\text{For } i = r+1, \cdots, n \\
\ell(v_i w_i^j) &= b, j = 1, 2 \\
\ell(w_i^1 w_i^2) &= c \\
&\text{end for}
\end{aligned}$$

With the above labeling we have ℓ is an a-sum V_4-magic labeling of $\mathcal{G}_r^n(G)$.

□

Theorem 5.6.8. *$\mathcal{G}_r^n(G) \notin \mathscr{V}_0$ for any n and r.*

Proof. Since the graph has pendant edges it can never be zero-sum V_4-magic for any n and r. □

Chapter 6

V_4-Barycentric Magic Graphs

In the first section of this chapter, we introduce a-sum and zero sum V_4 barycentric magic graphs. In the second section of this chapter, we discuss V_4 barycentric magic star graph and complete bipartite graph. In the third section of this chapter we characterize a-sum V_4 barycentric magic trees. In the fourth section we discuss V_4 barycentric magic complete graph and K_n-related graph which is not V_4 barycentric magic. In the fifth section we include the definitions of splitting graphs and mycielski graphs. Furthermore, we discuss V_4 barycentric magic splitting graphs and mycielski graphs of certain graphs. In the last section we discuss some cycle related and some more special V_4 barycentric magic graphs.

6.1 Introduction

A graph G is said to be A-barycentric magic if there exists a labeling $\ell : E(G) \longrightarrow A \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \longrightarrow A$ defined by

$$\ell^+(v) = \sum\{\ell(uv) : uv \in E(G)\}$$

is a constant map and also satisfies $\ell^+(v) = \deg(v)\ell(u_v v)$ for all $v \in V$, and for some vertex u_v adjacent to v [34].
In the definition of an A-barycentric magic graph instead of any abelian group we particularly choose the Klein 4-group V_4 and then define the following:

Definition 6.1.1. *A graph G is said to be V_4- barycentric magic if there exists a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \longrightarrow V_4$ defined by*

$$\ell^+(v) = \sum\{\ell(uv) : uv \in E(G)\}$$

is a constant map and also satisfies $\ell^+(v) = deg(v)\ell(u_v v)$ *for all* $v \in V$*, and for some vertex* u_v *adjacent to* v*. If this constant is some* $a \in V_4$*, the graph* G *is said to be* a*-sum* V_4 *barycentric magic. When* $a = 0$*,* G *is said to be zero-sum* V_4 *barycentric magic.*

In this chapter we investigate graphs that belong to the following classes:

(i) $\mathscr{BV}_a$: the class of graphs that are a-sum V_4 barycentric magic.

(ii) $\mathscr{BV}_0$: the class of graphs that are zero-sum V_4 barycentric magic.

(iii) $\mathscr{BV}_{a,0}$: the class of graphs that are both a-sum and zero-sum V_4 barycentric magic.

And finally we come to the conclusion that the class $\mathscr{BV}_{a,0}$ is empty.

This chapter is mainly built upon the following lemmas.

Lemma 6.1.2. *If a graph* G *has a vertex of even degree, then* G *is not* a*-sum* V_4*-barycentric magic.*

Proof. Suppose that G is a-sum V_4-barycentric magic. Let v be a vertex of even degree, say k. Then we have $k\ell(u_v v) = a$, for some vertex u_v adjacent to v. This implies that $a = 0$. This is a contradiction to the fact that G is a-sum V_4-barycentric magic. $\square$

Lemma 6.1.3. *If a graph* G *has a vertex of odd degree, then* G *is not zero-sum* V_4*-barycentric magic.*

Proof. Suppose that G is zero-sum V_4-barycentric magic. Let v be a vertex of odd degree, say k. Then we have $k\ell(u_v v) = 0$, for some vertex u_v adjacent to v. This implies that $\ell(u_v v) = 0$ which is not possible. $\square$

Lemma 6.1.4. *Let* G *be a graph such that all of its vertices are of even degree then* G *is zero-sum* V_4*-barycentric magic.*

Proof. If we label all the edges of G by a, then $\ell^+(v) = 0$. Moreover, $\deg(v)\ell(vu_v) = 0$. Thus, G is zero-sum V_4-barycentric magic. $\square$

Lemma 6.1.5. *Let* G *be a graph such that all of its vertices are of odd degree then* G *is* a*-sum* V_4*-barycentric magic.*

Proof. If we label all the edges of G by a, then $\ell^+(v) = a$. Moreover, $\deg(v)\ell(vu_v) = a$. Thus, G is a-sum V_4-barycentric magic. $\square$

6.2 Star and Complete bipartite graph

Theorem 6.2.1. *The star $K_{1,n} \in \mathscr{BV}_a$ if and only if n is odd.*

Proof. Assume that $K_{1,n} \in \mathscr{BV}_a$. Then by lemma 3.2.1, we have $(n+1)a = 0$. This implies that n is odd.

Conversely, assume that n is odd. Then $\deg(v)$ is odd for all $v \in V(K_{1,n})$. Then by lemma 6.1.5, we have $K_{1,n} \in \mathscr{BV}_a$. ☐

Theorem 6.2.2. *The star $K_{1,n} \notin \mathscr{BV}_0$ for any n.*

Proof. Since $K_{1,n}$ has a vertex of degree 1, the proof follows from lemma 6.1.3. ☐

Corollary 6.2.3. *The star $K_{1,n} \notin \mathscr{BV}_{a,0}$ for any n.*

Proof. The proof follows from theorems 6.2.1 and 6.2.2. ☐

Theorem 6.2.4. *For $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is a-sum V_4-barycentric magic if and only if both m and n are odd.*

Proof. Assume that $K_{m,n}$ is a-sum V_4-barycentric magic. Then by lemma 3.2.1, we have $(m+n)a = 0$. This implies that $(m+n)$ is even. That is, m and n are of the same parity. Since $K_{m,n}$ is a-sum V_4-barycentric magic, we have, $\deg(v)\ell(u_v v) = a$ for all $v \in V$. This implies that $\deg(v)$ is odd for all $v \in V$ and $\ell(u_v v) = a$. Hence m and n are odd. Conversely assume that m and n are odd. Label all the edges by a. Then the result follows. ☐

Theorem 6.2.5. *For $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is zero-sum V_4-barycentric magic if and only if both m and n are even.*

Proof. Assume that $K_{m,n}$ is zero-sum V_4-barycentric magic. Then we have, $\deg(v)\ell(u_v v) = 0$ for all $v \in V$ which implies that $\deg(v)$ is even for all $v \in V$. That is $m+n, m$ and n are even. Hence the proof. Conversely assume that m and n are even. Label all the edges by a. Then the result follows. ☐

6.3 Trees

We define the following.

Definition 6.3.1. *Let t be a (p,q) tree. For each vertex v_i we define a new tree $t * v_i * P_3$ obtained by identifying v_i with the end vertex of P_3. Note that for each vertex v_i we have a tree $t * v_i * P_3$. Now consider all such trees $t * v_i * P_3, i = 1, 2, \ldots p$. These trees form a class of trees with $(p+2)$ vertices and $(q+2)$ edges. This class is denoted by $t * P_3$.*

A similar class of trees $t * MP_3$ is obtained by identifying v_i with the middle vertex of P_3.

Definition 6.3.2. *Let t be a (p, q) tree. Consider two copies of P_2. For each of the vertices v_i, v_j of t, we define a new tree $t * v_i * v_j * P_2$ obtained by identifying v_i with a vertex of one copy of P_2 and v_j with a vertex of the other copy. Note that for each of the vertices v_i, v_j we obtain a tree $t * v_i * v_j * P_2$. Now consider all such trees $t * v_i * v_j * P_2, i,\ j = 1, 2, \ldots p,\ i \neq j$. We denote this class by $t * P_2 * P_2$. They form a class of $(p + 2, q + 2)$ trees.*

Theorem 6.3.3. *A tree t is a-sum V_4-barycentric magic if and only if the number of vertices of t is even and all its vertices have odd degrees.*

Proof. If all the vertices of a tree t have odd degrees, then by lemma 6.1.5, $t \in \mathscr{BV}_a$.

Conversely, assume that t is a-sum V_4-barycentric magic. Then by lemma 3.2.1, we have $na = 0$. This implies that n is even. That is the number of edges of t say q is odd. We will prove the theorem by induction on q. The statement is true for $q = 1$. Consider trees with 3 edges. The possibilities are $t = P_4$ or $t = K_{1,3}$. Since P_4 have vertices of even degree, by lemma 6.1.2, P_4 is not a-sum V_4-barycentric magic. Obviously all the vertices of $K_{1,3}$ are of odd degree. Suppose the statement is true for all nontrivial trees t with at most $2k + 1$ edges, and let t' be the class of all trees with $2k + 3$ edges. Then $t' = t * P_3$ or $t * MP_3$ or $t * P_2 * P_2$. Note that all these family of trees have $2k + 3$ edges. Moreover, both the families $t * P_3$ and $t * P_2 * P_2$ have vertex of degree two. So, by lemma 6.1.2 these families are not a-sum V_4 magic. Now consider the class $t * MP_3$. By induction hypothesis all the vertices of this class of trees have odd degrees. This completes the proof. ☐

Theorem 6.3.4. *A tree is not zero-sum V_4-barycentric magic.*

Proof. Since a tree has pendant edges it cannot be zero-sum V_4-barycentric magic. ☐

6.4 K_n and K_n-related graph

Theorem 6.4.1. *For $n \geq 2$, the complete graph $K_n \in \mathscr{BV}_a$ if and only if n is even.*

Proof. Suppose n is even. Then $\deg(v)$ is odd for all $v \in V(K_n)$. Then by lemma 6.1.5, $K_n \in \mathscr{BV}_a$. Conversely assume that $K_n \in \mathscr{BV}_a$. Then by lemma 3.2.1, n is even. ☐

Theorem 6.4.2. *For $n \geq 2$, the complete graph $K_n \in \mathscr{BV}_0$ if and only if n is odd.*

Proof. Suppose that n is odd. Then $\deg(v)$ is even for all $v \in V(K_n)$. Then by lemma 6.1.4, $K_n \in \mathscr{BV}_0$. Conversely assume that $K_n, n \geq 2 \in \mathscr{BV}_0$. Then we have $\ell^+(v) = 0 = \deg(v)\ell(u_v v)$. Without loss of generality assume that $\ell(u_v v) = a$. That is, $(n - 1)a = 0$ which implies that n is odd. ☐

Corollary 6.4.3. *For any $n \geq 2$, the complete graph $K_n \notin \mathscr{BV}_{a,0}$.*

Proof. From theorems 6.4.1 and 6.4.2, the result follows. □

Theorem 6.4.4. *For any $n > 3$, $K_n \setminus e$, the complete graph with one edge removed, is not a-sum V_4-barycentric magic for any n.*

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We consider the following cases:

Case 1: Suppose n is even. Without loss of generality, we may remove the edge v_1v_2. Then $\deg(v_1) = n-2$. This implies that degree of v_1 is even. So by lemma 6.1.2, $K_n \setminus e$ is not a-sum V_4-barycentric magic.

Case 2: Suppose n is odd. In this case, all vertices except v_1 and v_2 are of even degree. So, by lemma 6.1.2, $K_n \setminus e$ is not a-sum V_4-barycentric magic.

This completes the proof. □

Theorem 6.4.5. *For any $n > 3$, $K_n \setminus e$, the complete graph with one edge removed, is not zero-sum V_4-barycentric magic for any n.*

Proof. We consider the following cases:

Case 1: Suppose n is even. Without loss of generality, we may remove the edge v_1v_2. Then $\deg(v) = n-1$ for all v except v_1 and v_2. This implies that degree of v is odd for all v except v_1 and v_2. In this case, $\deg v\ell(vu_v) = 0$ is not satisfied for any $\ell(vu_v)$. So by lemma 6.1.3, $K_n \setminus e$ is not zero-sum V_4-barycentric magic.

Case 2: Suppose n is odd. Without loss of generality, we may remove the edge v_1v_2. Then $\deg(v_1) = \deg(v_2) = n-1$. This implies that degree of v_i is odd for $i = 1, 2$. Then $\deg v_i\ell(v_iu_{v_i}) = 0$ is not satisfied for any $\ell(v_iu_{v_i})$. So by lemma 6.1.3, $K_n \setminus e$ is not zero-sum V_4-barycentric magic.

This completes the proof. □

6.5 Splitting graphs and Mycielski graphs

Definition 6.5.1. *(see [23]) For any graph G, the splitting graph $S(G)$ is obtained by adding to each vertex u_i in G a new vertex v_i such that v_i is adjacent to the neighbours of u_i in G.*

Definition 6.5.2. *(see [27]) The Mycielski graph $\mu(G)$ is obtained by adding to each vertex u_i a new vertex v_i such that v_i is adjacent to the neighbours of u_i. Finally, add a new vertex u such that u is adjacent to each and every u_i.*

Theorem 6.5.3. *$S(K_{m,n})$ is zero-sum V_4-magic for all $m, n \geq 3$ and zero-sum V_4-barycentric magic if and only if m and n are even.*

Proof. Let $S_1 = \{u_1, u_2, \ldots, u_m\}$ and $S_2 = \{v_1, v_2, \ldots, v_n\}$ be the partite sets and $S_1' = \{u_1', u_2', \ldots, u_m'\}$ and $S_2' = \{v_1', v_2', \ldots, v_n'\}$ be the new vertices. The edge set is given by $E(S(K_{(m,n)})) = \{u_iv_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_i^{'}v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_iv_j^{'} : 1 \leq i \leq m, 1 \leq j \leq n\}$. Without loss of generality assume that $m \geq n$. We consider 4 cases:

Case 1: Assume that both m and n are odd. Define a labeling $\ell : E(S(K_{m,n})) \to V_4 \setminus \{0\}$ by:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	v_n
u_1'	a	b	c	a	a	a	a	$\cdots$	a
u_2'	b	c	a	a	a	a	a	$\cdots$	a
u_3'	c	a	b	a	a	a	a	$\cdots$	a
u_4'	a	b	c	a	a	a	a	$\cdots$	a
u_5'	a	b	c	a	a	a	a	$\cdots$	a
u_6'	a	b	c	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m'	a	b	c	a	a	a	a	$\cdots$	a
	v_1'	v_2'	v_3'	v_4'	v_5'	v_6'	v_7'	$\cdots$	v_n'
u_1	a	a	a	a	a	a	a	$\cdots$	a
u_2	b	b	b	b	b	b	b	$\cdots$	b
u_3	c	c	c	c	c	c	c	$\cdots$	c
u_4	a	a	a	a	a	a	a	$\cdots$	a
u_5	a	a	a	a	a	a	a	$\cdots$	a
u_6	a	a	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	a	a	a	a	a	a	a	$\cdots$	a

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	$\cdots$	u_m
v_1	a	b	c	a	a	a	a	$\cdots$	a
v_2	a	b	c	a	a	a	a	$\cdots$	a
v_3	a	b	c	a	a	a	a	$\cdots$	a
v_4	a	b	c	a	a	a	a	$\cdots$	a
v_5	a	b	c	a	a	a	a	$\cdots$	a
v_6	a	b	c	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_n	a	b	c	a	a	a	a	$\cdots$	a

Obviously $\ell^+(v) = 0$ for all $v \in V$. Also by lemma 6.1.3, $S(K_{(m,n)})$ is not zero sum V_4-barycentric magic.

Case 2: Assume that m and n are even. In this case, label all the edges by a. Obviously this is a zero sum V_4-magic labeling of $S(K_{m,n})$ and by lemma 6.1.4, $S(K_{m,n}) \in \mathscr{BV}_0$.

Case 3: Assume that m is odd and n is even. Define a labeling $\ell : E(S(K_{m,n})) \to V_4 \setminus \{0\}$ by:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	v_n
u'_1	a	a	a	a	a	a	a	$\cdots$	a
u'_2	a	a	a	a	a	a	a	$\cdots$	a
u'_3	a	a	a	a	a	a	a	$\cdots$	a
u'_4	a	a	a	a	a	a	a	$\cdots$	a
u'_5	a	a	a	a	a	a	a	$\cdots$	a
u'_6	a	a	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u'_m	a	a	a	a	a	a	a	$\cdots$	a

	v'_1	v'_2	v'_3	v'_4	v'_5	v'_6	v'_7	$\cdots$	v'_n
u_1	a	a	a	a	a	a	a	$\cdots$	a
u_2	b	b	b	b	b	b	b	$\cdots$	b
u_3	c	c	c	c	c	c	c	$\cdots$	c
u_4	a	a	a	a	a	a	a	$\cdots$	a
u_5	a	a	a	a	a	a	a	$\cdots$	a
u_6	a	a	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	a	a	a	a	a	a	a	$\cdots$	a

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	$\cdots$	u_m
v_1	a	a	a	a	a	a	a	$\cdots$	a
v_2	a	a	a	a	a	a	a	$\cdots$	a
v_3	a	a	a	a	a	a	a	$\cdots$	a
v_4	a	a	a	a	a	a	a	$\cdots$	a
v_5	a	a	a	a	a	a	a	$\cdots$	a
v_6	a	a	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_n	a	a	a	a	a	a	a	$\cdots$	a

Obviously $\ell^+(v) = 0$ for all $v \in V$. Also by lemma 6.1.3, $S(K_{m,n}) \notin \mathscr{BV}_0$.

Case 4**:** m is even and n is odd.
This case is similar to case 3.

This completes the proof. □

Theorem 6.5.4. *$S(K_{m,n})$ is a-sum V_4-magic for all $m, n \geq 3$ and a-sum V_4-barycentric magic if and only if m and n are odd.*

Proof. We consider the following cases:

Case 1**:** Assume that m is odd and n is even. Define $\ell : S(K_{m,n}) \to V_4 \setminus \{0\}$ by:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	v_n
u'_1	b	c	a	a	a	a	a	$\cdots$	a
u'_2	b	c	a	a	a	a	a	$\cdots$	a
u'_3	b	c	a	a	a	a	a	$\cdots$	a
u'_4	b	c	a	a	a	a	a	$\cdots$	a
u'_5	b	c	a	a	a	a	a	$\cdots$	a
u'_6	b	c	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u'_{m-2}	b	c	a	a	a	a	a	$\cdots$	a
u'_{m-1}	b	c	b	b	b	b	b	$\cdots$	b
u'_m	c	b	c	c	c	c	c	$\cdots$	c

	v'_1	v'_2	v'_3	v'_4	v'_5	v'_6	v'_7	$\cdots$	v'_n
u_1	a	a	a	a	a	a	a	$\cdots$	a
u_2	a	a	a	a	a	a	a	$\cdots$	a
u_3	a	a	a	a	a	a	a	$\cdots$	a
u_4	a	a	a	a	a	a	a	$\cdots$	a
u_5	a	a	a	a	a	a	a	$\cdots$	a
u_6	a	a	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	a	a	a	a	a	a	a	$\cdots$	a

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	$\cdots$	u_m
v_1	b	b	b	b	b	b	b	$\cdots$	b
v_2	c	c	c	c	c	c	c	$\cdots$	c
v_3	a	a	a	a	a	a	a	$\cdots$	a
v_4	a	a	a	a	a	a	a	$\cdots$	a
v_5	a	a	a	a	a	a	a	$\cdots$	a
v_6	a	a	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_n	a	a	a	a	a	a	a	$\cdots$	a

Obviously, $\ell^+(v) = a$ for all $v \in V(S(K_{m,n}))$. By lemma 6.1.2, $S(K_{m,n}) \notin \mathscr{BV}_a$.

Case 2**:** Assume that m is even and n is odd. In this case the labeling is similar to case 1. By lemma 6.1.2, $S(K_{m,n}) \notin \mathscr{BV}_a$.

Case 3**:** Assume that m and n are odd. Define $\ell : E(S(K_{m,n})) \to V_4 \setminus \{0\}$ by:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	v_n
u_1'	a	a	a	a	a	a	a	$\cdots$	a
u_2'	a	a	a	a	a	a	a	$\cdots$	a
u_3'	a	a	a	a	a	a	a	$\cdots$	a
u_4'	a	a	a	a	a	a	a	$\cdots$	a
u_5'	a	a	a	a	a	a	a	$\cdots$	a
u_6'	a	a	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m'	a	a	a	a	a	a	a	$\cdots$	a

	v_1'	v_2'	v_3'	v_4'	v_5'	v_6'	v_7'	$\cdots$	v_n'
u_1	a	a	a	a	a	a	a	$\cdots$	a
u_2	a	a	a	a	a	a	a	$\cdots$	a
u_3	a	a	a	a	a	a	a	$\cdots$	a
u_4	a	a	a	a	a	a	a	$\cdots$	a
u_5	a	a	a	a	a	a	a	$\cdots$	a
u_6	a	a	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	a	a	a	a	a	a	a	$\cdots$	a

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	v_n
u_1	a	b	c	a	a	a	a	$\cdots$	a
u_2	b	c	a	a	a	a	a	$\cdots$	a
u_3	c	a	b	a	a	a	a	$\cdots$	a
u_4	a	a	a	a	a	a	a	$\cdots$	a
u_5	a	a	a	a	a	a	a	$\cdots$	a
u_6	a	a	a	a	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{m-2}	a	a	a	a	a	a	a	$\cdots$	a
u_{m-1}	a	b	c	b	b	b	b	$\cdots$	b
u_m	a	b	c	c	c	c	c	$\cdots$	c

Obviously $\ell^+(v) = a$ for all $v \in V$. By lemma 6.1.5, $S(K_{m,n}) \in \mathscr{BV}_a$.

Case 4**:** Assume that m and n are even.

Define $\ell : E(S(K_{m,n})) \to V_4 \setminus \{0\}$ by:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	v_n
u'_1	b	c	b	b	b	b	b	$\cdots$	b
u'_2	c	b	c	c	c	c	c	$\cdots$	c
u'_3	b	c	b	b	b	b	b	$\cdots$	b
u'_4	b	c	b	b	b	b	b	$\cdots$	b
u'_5	b	c	b	b	b	b	b	$\cdots$	b
u'_6	b	c	b	b	b	b	b	$\cdots$	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u'_m	b	c	b	b	b	b	b	$\cdots$	b

	v'_1	v'_2	v'_3	v'_4	v'_5	v'_6	v'_7	$\cdots$	v'_n
u_1	b	c	b	b	b	b	b	$\cdots$	b
u_2	c	b	c	c	c	c	c	$\cdots$	c
u_3	b	c	b	b	b	b	b	$\cdots$	b
u_4	b	c	b	b	b	b	b	$\cdots$	b
u_5	b	c	b	b	b	b	b	$\cdots$	b
u_6	b	c	b	b	b	b	b	$\cdots$	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	b	c	b	b	b	b	b	$\cdots$	b

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	$\cdots$	u_m
v_1	b	b	b	b	b	b	b	$\cdots$	b
v_2	b	b	b	b	b	b	b	$\cdots$	b
v_3	b	b	b	b	b	b	b	$\cdots$	b
v_4	b	b	b	b	b	b	b	$\cdots$	b
v_5	b	b	b	b	b	b	b	$\cdots$	b
v_6	b	b	b	b	b	b	b	$\cdots$	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_n	b	b	b	b	b	b	b	$\cdots$	b

Obviously ℓ is an a-sum V_4-magic labeling of $S(K_{m,n})$ and by lemma 6.1.2, $S(K_{m,n}) \notin \mathscr{B}\mathscr{V}_a$.

$\square$

Theorem 6.5.5. *$\mu(K_{m,n})$ is not a-sum V_4-magic and not a-sum V_4-barycentric magic for any* $m, n \geq 2$.

Proof. Suppose that $\mu(K_{m,n})$ is a-sum V_4-magic. Then by lemma 3.2.1, we have $[2(m+n)+1]a = 0$ which is a contradiction. Thus $\mu(K_{m,n})$ is not a-sum V_4-magic and hence not a-sum V_4-barycentric magic for any $m, n \geq 2$. $\square$

Theorem 6.5.6. *$\mu(K_{m,n})$ is zero-sum V_4-magic and zero-sum V_4-barycentric magic if and only if m and n are odd.*

Proof. Let $V(\mu(K_{m,n})) = V(S(K_{m,n})) \cup \{u\}$ and $E(\mu(K_{m,n})) = E(S(K_{m,n})) \cup \{uu'_i : 1 \leq i \leq m\} \cup \{uv'_j : 1 \leq j \leq n\}$. We consider the following cases:

Case 1: Assume that m and n are even.

For $j = 1, 2, \cdots, n$, do:
$\ell(u_i v'_j) = a, i = 1, 4, \cdots, m,$
$\ell(u_2 v'_j) = b,\ \ell(u_3 v'_j) = c.$
end for
For $i = 1, 2, \cdots, m$, do:
$\ell(u'_i v_j) = a, i = 1, 4, \cdots, n,$
$\ell(u'_i v_2) = b,\ \ell(u'_i v_3) = c.$
end for
For $i = 1, 2, \cdots, m$
For $j = 1, 2, \cdots, n$
$\ell(u'_i u) = \ell(v'_j u) = \ell(u_i v_j) = a.$
end for
end for

Case 2: Assume that m and n are odd.
Label all the edges by a.

Case 3: Assume that m is odd and n is even.

For $i = 1, 2, \cdots, m$
For $j = 1, 2, \cdots, n$
$\ell(v'_j u_i) = a.$
end for
end for
For $j = 1, 2, \cdots, n$
$\ell(u_i v_j) = a, i = 1, 4, 5, \cdots, m$
$\ell(u_2 v_j) = b,\ \ell(u_3 v_j) = c.$
end for
For $j = 1, 4, \cdots, n$
$\ell(u'_i v_j) = a, i = 1, 4, 5, \cdots, m$
$\ell(u'_2 v_j) = b,\ \ell(u'_3 v_j) = c.$

$$
\begin{aligned}
&\text{end for}\\
&\ell(v_2u_1') = \ell(v_3u_3') = b\\
&\ell(v_2u_2') = \ell(v_3u_1') = c\\
&\ell(v_2u_i') = a, \text{ for } i = 3, 4, \cdots, m\\
&\ell(v_3u_i') = a, \text{ for } i = 2, 4, 5, \cdots, m\\
&\ell(u_i'u) = a, i = 1, 4, 5, \cdots, m\\
&\ell(u_2'u) = b, \ \ell(u_3'u) = c.
\end{aligned}
$$

Case 4**:** Assume that m is even and n is odd.

Interchange the roles of m and n in case 3.

This completes the proof. □

Theorem 6.5.7. *$S(K_{1,n})$ is a-sum V_4-magic if n is even.*

Proof. Let $V(S(K_{1,n})) = \{u\} \cup \{u'\} \cup \{u_i : 1 \le i \le n\} \cup \{u_i' : 1 \le i \le n\}$ and edge set $E((S(K_{1,n}))) = \{uu_i : 1 \le i \le n\} \cup \{uu_i' : 1 \le i \le n\} \cup \{u'u_i : 1 \le i \le n\}$. Define a labeling $\ell : E((S(K_{1,n}))) \longrightarrow V_4 \setminus \{0\}$ by

$$\ell(uu_i') = a, 1 \le i \le n$$

$$\ell(uu_i) = \begin{cases} c, & \text{for } i = 1, \ 3 \le i \le n, \\ b, & \text{for } i = 2. \end{cases}$$

$$\ell(u_iu') = \begin{cases} b, & \text{for } i = 1, \ 3 \le i \le n, \\ c, & \text{for } i = 2. \end{cases}$$

With this labeling $\ell^+(v) = a$ for all $v \in V$. □

Theorem 6.5.8. *$S(K_{1,n})$ is not a-sum V_4-barycentric magic for any n.*

Proof. By lemma 6.1.2, $S(K_{1,n})$ is not a-sum V_4-barycentric magic for any n. □

Theorem 6.5.9. *$S(K_{1,n})$ is not zero-sum V_4-magic and not zero-sum V_4-barycentric magic for any n.*

Proof. Since it has pendant edges, $S(K_{1,n})$ is not zero-sum V_4-magic and hence not zero-sum V_4-barycentric magic for any n. □

Theorem 6.5.10. *$\mu(K_{1,n})$ is zero sum V_4-magic if n is odd.*

Proof. Label all edges by a. The result follows. □

Theorem 6.5.11. *$\mu(K_{1,n})$ is zero sum V_4-barycentric magic if n is odd.*

Proof. By lemma 6.1.4, the result follows. □

Theorem 6.5.12. *$\mu(K_{1,n})$ is not a-sum V_4-magic and is not a-sum V_4-barycentric magic for any n.*

Proof. Suppose that $\mu(K_{1,n})$ is a-sum V_4-magic. Then by lemma 3.2.1, $(2n+3)a = 0$ which is a contradiction. Then the result follows. □

Theorem 6.5.13. *$C_n, S(C_n) \in \mathscr{BV}_0$ for all n.*

Proof. Label all the edges by a. □

Theorem 6.5.14. *$S(C_n)$ is a-sum V_4-magic for all n but $C_n, S(C_n) \notin \mathscr{BV}_a$ for any n.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Then the vertex set and edge set of $S(C_n)$ are respectively given by:

$$V(S(C_n)) = \{v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\},$$
$$E(S(C_n)) = \{v'_i v_{i+1} : 1 \leq i \leq n\} \cup \{v_i v'_{i+1} : 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq n\}$$

Without loss of generality assume that $v_{n+1} = v_1, v'_{n+1} = v'_1$. Define $\ell : E(S(C_n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
\ell(v_i v_{i+1}) &= a, \text{ for } 1 \leq i \leq n,\\
\ell(v'_i v_{i+1}) &= b, \text{ for } 1 \leq i \leq n,\\
\ell(v_i v'_{i+1}) &= c, \text{ for } 1 \leq i \leq n.
\end{aligned}$$

Observe that, $\ell^+(v_i) = \ell^+(v'_i) = a$ for $1 \leq i \leq n$. Thus $S(C_n)$ is a-sum V_4 magic. But by lemma 6.1.2, C_n and $S(C_n)$ are not a-sum V_4-barycentric magic. □

Theorem 6.5.15. *$\mu(C_n)$ is zero sum V_4-magic for all n and is not zero sum V_4-barycentric magic for any n.*

Proof. The vertex set of $\mu(C_n)$, $V(\mu(C_n)) = V(S(C_n)) \cup \{w\}$ and edge set is given by $E(\mu(C_n)) = E(S(C_n)) \cup \{wv'_i : 1 \leq i \leq n\}$. We consider two cases:

Case 1: When n is even. Define $\ell : E(\mu(C_n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
\ell(v_i v_{i+1}) &= \begin{cases} b, \text{ for } i = 1, 3, 5, \ldots, n-1,\\ c, \text{ for } i = 2, 4, 6, \ldots, n. \end{cases}\\
\ell(v'_i v_{i+1}) &= b, \text{ for } 1 \leq i \leq n,\\
\ell(v_i v'_{i+1}) &= c, \text{ for } 1 \leq i \leq n,\\
\ell(wv'_i) &= a, \text{ for } 1 \leq i \leq n.
\end{aligned}$$

Observe that $\ell^+(v_i) = \ell^+(v_i') = 0$ for $1 \leq i \leq n$ and $\ell^+(w) = 0$.

Case 2: When n is odd. Define $\ell(E(\mu(C_n)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_iv_{i+1}) = \begin{cases} c, & \text{for } i = 1, n-2, n \\ a, & \text{for } i = 2 \\ b, & \text{for } 3, 4, \ldots, n-3, n-1, \end{cases}$$

$$\ell(v_i'v_{i+1}) = \begin{cases} c, & \text{for } i = 1, 4, 5, \ldots, n \\ a, & \text{for } i = 2, \\ b, & \text{for } i = 3, \end{cases}$$

$$\ell(v_iv_{i+1}') = \begin{cases} c, & \text{for } i = 1, \\ a, & \text{for } i = 2, \\ b, & \text{for } i = 3, \ldots, n-1, n, \end{cases}$$

$$\ell(wv_i') = \begin{cases} a, & \text{for } i = 1, 4, 5, \ldots, n, \\ b, & \text{for } i = 2, \\ c, & \text{for } i = 3. \end{cases}$$

Obviously, $\ell^+(v) = 0$ for all $v \in V$. Also by lemma 6.1.3, $\mu(C_n) \notin \mathscr{BV}_0$.

This completes the proof. ☐

Theorem 6.5.16. *$\mu(C_n)$ is not a-sum V_4-magic and not a-sum V_4-barycentric magic for any n.*

Proof. Suppose $\mu(C_n)$ is a-sum V_4 magic. Then $(2n+1)a = 0$. This implies $a = 0$ which is a contradiction. Also by lemma 6.1.2, $\mu(C_n) \notin \mathscr{BV}_a$. ☐

Theorem 6.5.17. *$P_n \notin \mathscr{BV}_a$ and $\mathscr{BV}_0$ for any n.*

Proof. By lemmas 6.1.2 and 6.1.3, the result follows. ☐

Corollary 6.5.18. *$P_n \notin \mathscr{BV}_{a,0}$ for any n.*

Proof. By theorem 6.5.17, the proof follows. ☐

Theorem 6.5.19. *$S(P_n)$ is a-sum V_4-magic for all n and does not belongs to the class $\mathscr{BV}_a$.*

Proof. We have $V(S(P_n)) = \{v_i : 1 \leq i \leq n\} \cup \{v_i' : 1 \leq i \leq n\}$ and $E(S(P_n)) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1}' : 1 \leq i \leq n-1\} \cup \{v_i'v_{i+1} : 1 \leq i \leq n-1\}$. Define $\ell : E(S(P_n)) \to V_4 \setminus \{0\}$

by

$$\ell(v_iv_{i+1}) = \begin{cases} b, & \text{for } i = 1, \\ c, & \text{for } i = n-1, \\ a, & \text{for } 2 \leq i \leq n-2. \end{cases}$$

$$\ell(v_iv'_{i+1}) = \begin{cases} c, & \text{for } 1 \leq i \leq n-2, \\ a, & \text{for } i = n-1. \end{cases}$$

$$\ell(v'_iv_{i+1}) = \begin{cases} a, & \text{for } i = 1, \\ b, & \text{for } 2 \leq i \leq n-1. \end{cases}$$

Thus we get ℓ is an a-sum V_4-magic labeling of $S(P_n)$. Moreover, lemma 6.1.2 implies that $S(P_n) \notin \mathscr{BV}_a$. □

Theorem 6.5.20. *$S(P_n)$ is not zero-sum V_4-magic and not zero-sum V_4-barycentric magic for any n.*

Proof. Since it has pendant edges, $S(P_n)$ is not zero-sum V_4-magic and hence not zero-sum V_4-barycentric magic for any n. □

Theorem 6.5.21. *$\mu(P_n)$ is zero-sum V_4-magic for all n and not zero-sum V_4-barycentric magic for any n.*

Proof. $V(\mu(P_n)) = V(S(P_n)) \cup \{u\}$ and $E(\mu(P_n)) = E(S(P_n)) \cup \{uv'_i : 1 \leq i \leq n\}$. We consider two cases:

Case 1: Suppose n is odd. Define $\ell : E(\mu(P_n)) \to V_4 \setminus \{0\}$ by

$$\ell(v_iv_{i+1}) = \begin{cases} a, & \text{for } i = 1, 3, \cdots, n-2, \\ c, & \text{for } i = 1, 2, \cdots, n-1, \end{cases}$$

$$\ell(v_iv'_{i+1}) = a, \text{ for } i = 1, 2, \ldots, n-1,$$
$$\ell(v'_iv_{i+1}) = c, \text{ for } i = 1, 2, \ldots, n-1,$$
$$\ell(v'_iu) = b, \text{ for } i = 2, 3, \ldots, n-1,$$
$$\ell(v'_1u) = c, \ \ell(v'_nu) = a.$$

We have $\ell^+(u) = 0$ and $\ell^+(v_i) = \ell^+(v'_i) = 0$ for all i.

Case 2: Suppose n is even. Define $\ell : E(\mu(P_n)) \to V_4 \setminus \{0\}$ by

$$\ell(v'_iu) = \begin{cases} b, & \text{for } i = 1, n, \\ a, & \text{for } 2 \leq i \leq n-1, \end{cases}$$

$$\ell(v_iv'_{i+1}) = b, \text{ for } 1 \leq i \leq n-1,$$

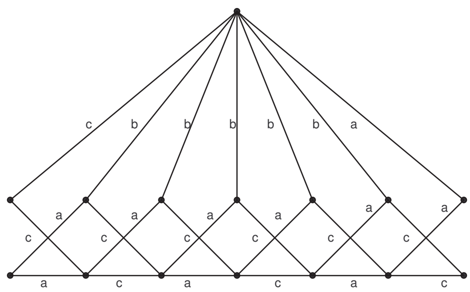


Figure 6.1: A zero-sum V_4-magic labeling of $\mu(P_7)$

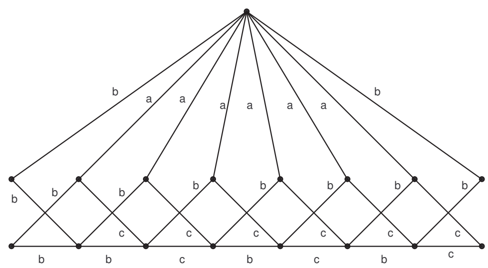


Figure 6.2: A zero-sum V_4-magic labeling of $\mu(P_8)$

$$\ell(v_i'v_{i+1}) = \begin{cases} b, & \text{for } i = 1, \\ c, & \text{for } 2 \leq i \leq n-1, \end{cases}$$

$$\ell((v_iv_{i+1}) = \begin{cases} b, & \text{for } 1, 2, 4, 6, \ldots, n-2 \\ c, & \text{for } 3, 5, \ldots, n-1. \end{cases}$$

Obviously, $\ell^+(u) = 0$ and $\ell^+(v_i') = \ell^+(v_i) = 0$ for all i.

By lemma 6.1.3, $\mu(P_n) \notin \mathscr{BV}_0$. $\square$

Theorem 6.5.22. *$\mu(P_n)$ is not a-sum V_4-magic and not a-sum V_4-barycentric magic for any n.*

Proof. Suppose that $\mu(P_n)$ is a-sum V_4-magic. Then by lemma 3.2.1 we have, $(2n+1)a = 0$ which is not possible. Hence the proof. $\square$

6.6 Cycle related and some other graphs

Theorem 6.6.1. *The sun graph $C_n \odot K_1$ is a-sum V_4-barycentric magic for all n.*

Proof. If we label all the edges by $a \in V_4$, we obtain an a-sum V_4-barycentric magic labeling of $C_n \odot K_1$. □

Theorem 6.6.2. *The sun graph $C_n \odot K_1$ is not zero-sum V_4-barycentric magic for any n.*

Proof. By lemma 6.1.3, the sun graph $C_n \odot K_1$ is not zero-sum V_4-barycentric magic for any n. □

Theorem 6.6.3. *The wheel W_n is a-sum V_4-barycentric magic if and only if n is odd.*

Proof. Suppose that n is odd. Then by lemma 6.1.5, W_n is a-sum V_4-barycentric magic. Conversely suppose that W_n is a-sum V_4-barycentric magic. Then we have $(n-3)a = 0$ which implies that n is odd. □

Theorem 6.6.4. *The wheel W_n is not zero-sum V_4-barycentric magic for any n.*

Proof. By lemma 6.1.3, the result follows. □

Theorem 6.6.5. *For any $n \geq 3$ and $k \geq 1$, the n-gon book of k pages is zero-sum V_4-barycentric magic if and only if k is odd.*

Proof. Suppose that k is odd. Then by lemma 6.1.4, the n-gon book of k pages is zero-sum V_4-barycentric magic. Conversely suppose that the n-gon book of k pages is zero-sum V_4-barycentric magic. Let the edges be labeled a. Then we have $(k+1)a = 2a = 0$ implies k is odd. □

Theorem 6.6.6. *For any $n \geq 3$ and $k \geq 1$, the n-gon book of k pages is not a-sum V_4-barycentric magic.*

Proof. By lemma 6.1.2, the result follows. □

Theorem 6.6.7. *For any $n \geq 3$, the bipyramid graph $BP(n)$ is zero-sum V_4-barycentric magic if and only if n is even.*

Proof. Suppose that n is even. Then by lemma 6.1.4, $BP(n)$ is zero-sum V_4-barycentric magic. Conversely suppose that $BP(n)$ is zero-sum V_4-barycentric magic. If we label the edges by a, then we have $na = 2a = 0$ which implies that n is even. □

Theorem 6.6.8. *For any $n \geq 3$, the bipyramid graph $BP(n)$ is not a-sum V_4-barycentric magic.*

Proof. By lemma 6.1.2, $BP(n)$ is not a-sum V_4-barycentric magic. $\square$

Theorem 6.6.9. *The sunflower graph SF_n is zero-sum V_4-magic for all n but is not zero-sum V_4-barycentric magic for any n.*

Proof. We consider two cases:

Case 1: n is even.

Define a labeling $\ell : E(SF_n) \to V_4 \setminus \{0\}$ by

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n \text{ do:}\\
\ell(v_0 v_i) &= a,\\
\ell(w_i v_i) &= \ell(w_i v_{i+1}) = a.\\
&\text{end for}\\
\ell(v_i v_{i+1}) &= \begin{cases} b, & i = 1, 3, \cdots, n-1,\\ c, & i = 2, 4, \cdots, n.\end{cases}
\end{aligned}$$

Case 2: n is odd.

$$\begin{aligned}
\ell(v_0 v_i) &= a, \text{ for } i = 1, 4, 5, \cdots, n\\
\ell(v_0 v_2) &= b,\ \ell(v_0 v_3) = c.\\
&\text{For } i = 1, 2, \cdots, n \text{ do:}\\
\ell(w_i v_i) &= \ell(w_i v_{i+1}) = a.\\
&\text{end for}\\
\ell(v_1 v_2) &= c,\ \ell(v_2 v_3) = a,\\
\ell(v_i v_{i+1}) &= \begin{cases} b, & i = 3, 5, \cdots, n,\\ c, & i = 4, 6, \cdots, n-1.\end{cases}
\end{aligned}$$

Thus ℓ is a zero-sum V_4-magic labeling of SF_n. But by lemma 6.1.3, SF_n is not zero-sum V_4-barycentric magic for any n. $\square$

Theorem 6.6.10. *SF_n is neither a-sum V_4-magic nor a-sum V_4-barycentric magic for any n.*

Proof. Suppose that SF_n is a-sum V_4-magic. Then we have $(2n+1)a = 0$ which implies that $a = 0$. This is a contradiction. Hence the proof. $\square$

Now consider the complete tripartite graph K_{n_1,n_2,n_3}.

Theorem 6.6.11. *K_{n_1,n_2,n_3} is zero-sum V_4-magic for all n_1, n_2, n_3 and is zero-sum V_4-barycentric magic if and only if n_1, n_2, n_3 are of same parity.*

Proof. We consider the following cases:

Case 1: Assume that n_1, n_2 and n_3 are even.

Label all the edges by a. Then K_{n_1,n_2,n_3} is zero-sum V_4 magic. By lemma 6.1.4, K_{n_1,n_2,n_3} is zero-sum V_4-barycentric magic.

Case 2: Assume that n_1, n_2 and n_3 are odd.

The case is similar to case 1.

Case 3: n_1, n_2 are odd and n_3 is even. Define a labeling $\ell : E(K_{n_1,n_2,n_3}) \to V_4 \setminus \{0\}$ by:

	u_1	u_2	u_3	u_4	$\cdots$	u_{n_1}
v_1	a	b	c	a	$\cdots$	a
v_2	b	c	a	b	$\cdots$	b
v_3	c	a	b	c	$\cdots$	c
v_4	a	b	c	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
v_{n_2}	a	b	c	a	$\cdots$	a

	w_1	w_2	w_3	$\cdots$	w_{n_3}
u_1	a	a	a	$\cdots$	a
u_2	b	b	b	$\cdots$	b
u_3	c	c	c	$\cdots$	c
u_4	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
u_{n_1}	a	a	a	$\cdots$	a
v_1	a	a	a	$\cdots$	a
v_2	b	b	b	$\cdots$	b
v_3	c	c	c	$\cdots$	c
v_4	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
v_{n_2}	a	a	a	$\cdots$	a

Case 4: n_1, n_2 are even and n_3 is odd.

The case is similar to case 3.

Case 5: n_1, n_3 are odd and n_2 is even.

Take $n_2 = n_3$ in case 3.

Case 6: n_1, n_3 are even and n_2 is odd.

Take $n_2 = n_3$ in case 4.

Case 7: n_2, n_3 are odd and n_1 is even.

Take $n_1 = n_3$ in case 3.

Case 8: n_2, n_3 are even and n_1 is odd.

Take $n_1 = n_3$ in case 4.

Hence K_{n_1,n_2,n_3} is zero-sum V_4-magic for all $n_i, i = 1, 2, 3$ and is zero-sum V_4-barycentric magic if and only if n_1, n_2, n_3 are of same parity. $\square$

Theorem 6.6.12. *K_{n_1,n_2,n_3} is a-sum V_4-magic if and only if $n_1 + n_2 + n_3$ is even and is not a-sum V_4-barycentric magic for any $n_i, i = 1, 2, 3$.*

Proof. First assume that K_{n_1,n_2,n_3} is a-sum V_4 magic. Then by lemma 3.2.1, we have $n_1 + n_2 + n_3$ is even.

Conversely assume that $n_1 + n_2 + n_3$ is even. Then the following cases arise:

Case 1: n_1, n_2, n_3 are even.
Define a labeling $\ell : E(K_{n_1,n_2,n_3}) \to V_4 \setminus \{0\}$ by:

$$
\begin{aligned}
&\text{For } i = 1, 2, \ldots, n_1, \text{ do:}\\
\ell(u_i v_j) &= a, \text{ for } j = 1, 4, \ldots, n_2,\\
\ell(u_i v_2) &= b, \ \ell(u_i v_3) = c.\\
\ell(u_i w_k) &= a, \text{ for } k = 1, 2, \ldots, n_3.\\
&\text{end for}\\
&\text{For } j = 1, 3, \ldots, n_2, \text{ do:}\\
\ell(w_k v_j) &= b, \text{ for } k = 1, 3, \ldots, n_3,\\
\ell(w_2 v_j) &= c.\\
&\text{end for}\\
\ell(w_k v_2) &= c, \text{ for } k = 1, 3, \ldots, n_3,\\
\ell(w_2 v_2) &= b.
\end{aligned}
$$

Case 2: n_1, n_2 are odd and n_3 is even.
Define a labeling $\ell : E(K_{n_1,n_2,n_3}) \to V_4 \setminus \{0\}$ by:

$$
\begin{aligned}
&\text{For } i = 1, 2, \ldots, n_1, \text{ do:}\\
\ell(u_i v_j) &= a, \text{ for } j = 1, 2, \ldots, n_2,\\
\ell(u_i w_k) &= a, \text{ for } k = 1, 2, \ldots, n_3.\\
&\text{end for}\\
&\text{For } k = 1, 2, \ldots, n_3, \text{ do:}\\
\ell(v_j w_k) &= a, \text{ for } j = 1, 4, \ldots, n_2,\\
\ell(v_2 w_k) &= b, \ \ell(v_3 w_k) = c.\\
&\text{end for.}
\end{aligned}
$$

Case 3: n_2, n_3 are odd and n_1 is even.
Interchange the roles of n_1 and n_3 in case 2.

Case 4: n_1, n_3 are odd and n_2 is even.
Interchange the roles of n_2 and n_3 in case 2.

Thus ℓ is an a-sum V_4-magic labeling of K_{n_1,n_2,n_3}. Also by lemma 6.1.2, K_{n_1,n_2,n_3} is not a-sum V_4-barycentric magic for any $n_i, i = 1, 2, 3$. $\square$

In previous chapters we have proved that there exist graphs which are both a-sum V_4-magic and zero-sum V_4 magic. That is the class $\mathscr{V}_{a,0}$ is nonempty. From lemmas 6.1.2 and 6.1.3, we can conclude that all the vertices of the graph must be of same parity in order to become a-sum V_4-barycentric magic or zero-sum V_4-barycentric magic. Hence a graph cannot be a-sum V_4-barycentric magic and zero-sum V_4-barycentric magic simultaneously. Thus we have the following theorem:

Theorem 6.6.13. *No graph belong to the class $\mathscr{BV}_{a,0}$. That is, $\mathscr{BV}_{a,0} = \phi$.*

Chapter 7

Some Special V_4-Barycentric Magic Graphs

In this chapter, we consider some special V_4 barycentric magic graphs. Line, middle and total graphs of some wellknown graphs are considered.

7.1 Introduction

In this chapter we investigate the line, middle and total graphs of some wellknown graphs which are a-sum and zero-sum V_4-barycentric magic.

7.2 Line, Middle and Total Graphs

Theorem 7.2.1. *A cubic graph is a-sum V_4-magic and a-sum V_4-barycentric magic.*

Proof. Label all the edges by a. The result follows. □

A cubic graph and its line graph are shown in figure 7.1.

Theorem 7.2.2. *A cubic graph is zero-sum V_4-magic but not zero-sum V_4-barycentric magic.*

Proof. A zero-sum V_4-magic labeling of a cubic graph is shown in figure 7.1. By lemma 6.1.3, it follows that the cubic graph is not zero-sum V_4-barycentric magic. □

Theorem 7.2.3. *The line graph of a cubic graph is not a-sum V_4-magic and not a-sum V_4-barycentric magic.*

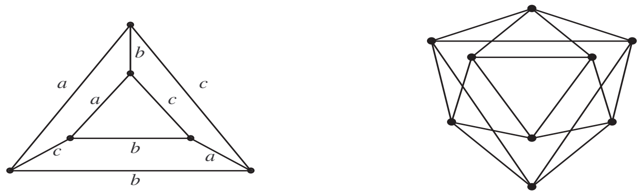


Figure 7.1: Cubic graph and its line graph

Proof. Since the number of vertices is odd, the line graph of cubic graph is not a-sum V_4-magic and by lemma 6.1.2, it is not a-sum V_4-barycentric magic. □

Theorem 7.2.4. *The line graph of a cubic graph is zero-sum V_4-magic and zero-sum V_4-barycentric magic.*

Proof. Since all the vertices have degree 2 or 4, by labeling all the edges by a the proof follows. □

The middle graph and total graph of a cubic graph is shown in figure 7.2.

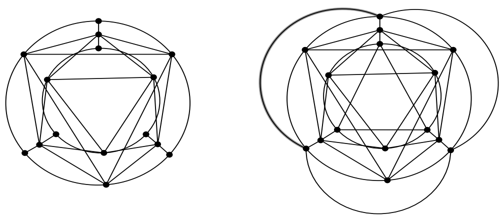


Figure 7.2: Middle graph of cubic graph (left), Total graph of cubic graph (right)

Theorem 7.2.5. *The middle graph of a cubic graph is not a-sum V_4-magic and not a-sum V_4-barycentric magic.*

Proof. Since the number of vertices is odd, the middle graph of cubic graph is not a-sum V_4-magic and by lemma 6.1.2, it is not a-sum V_4-barycentric magic. □

The middle graph of a cubic graph is not zero-sum V_4-barycentric magic since it has vertices of odd degree.

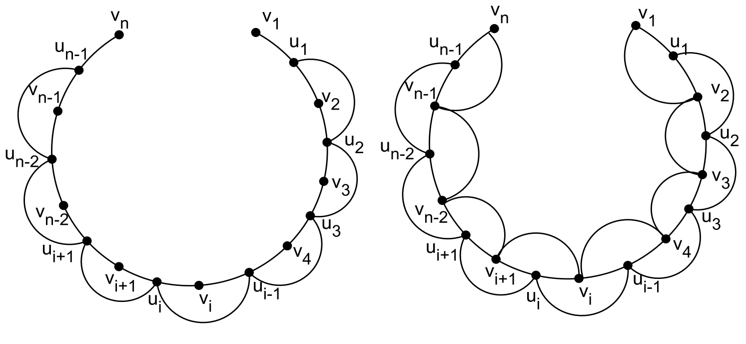


Figure 7.3: Middle Graph of Path $M(P_n)$ (left), Total Graph of Path $T(P_n)$ (right)

Theorem 7.2.6. *The total graph of a cubic graph is not a-sum V_4-magic and not a-sum V_4-barycentric magic.*

Proof. Since the number of vertices is odd, the total graph of cubic graph is not a-sum V_4-magic and by lemma 6.1.2, it is not a-sum V_4-barycentric magic. □

The total graph of a cubic graph is not zero-sum V_4-barycentric magic since it has vertices of odd degree.

Consider the path graph P_n.

Theorem 7.2.7. *The middle graph of P_n, $M(P_n)$ is not a-sum V_4-magic and is not a-sum V_4-barycentric magic for any n.*

Proof. Suppose $M(P_n)$ is a-sum V_4-magic. Then we have $(2n-1)a = 0$ which implies $a = 0$. This is a contradiction. Hence the result follows. □

Theorem 7.2.8. *The middle graph of $P_n, M(P_n)$ is not zero-sum V_4-magic and is not zero-sum V_4-barycentric magic for any n.*

Proof. Since it has pendant edges, the result follows. □

Theorem 7.2.9. *The total graph of $P_n, T(P_n)$ is not a-sum V_4-magic and is not a-sum V_4-barycentric magic for any n.*

Proof. Suppose $T(P_n)$ is a-sum V_4-magic. Then we have $(2n-1)a = 0$ which implies $a = 0$. This is a contradiction. Hence the result follows. □

Theorem 7.2.10. *$T(P_n)$ is zero-sum V_4-magic for all n and it is not zero-sum V_4-barycentric magic for any n.*

Proof. Let the vertices of $T(P_n)$ be labeled as in figure 7.3. Define a labeling $\ell : E(T(P_n)) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n \text{ do:}\\
\ell(u_i v_i) &= \ell(u_i v_{i+1}) = c\\
\ell(v_i v_{i+1}) &= a\\
&\text{end for}\\
\ell(u_i u_{i+1}) &= i = 1, 2, \cdots, n-2
\end{aligned}$$

This is a zero-sum V_4-magic labeling of $T(P_n)$. By lemma 6.1.3, $T(P_n)$ is not zero-sum V_4-barycentric magic for any n. $\square$

Theorem 7.2.11. *$M(C_n)$ is a-sum V_4-magic for all $n \geq 3$ and is not a-sum V_4-barycentric magic for any n.*

Proof. The vertices of $M(C_n)$ are labeled as given in figure 7.4. Define an edge labeling $\ell : E(M(C_n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n \text{ do :}\\
\ell(u_i v_i) &= b\\
\ell(u_i v_{i+1}) &= c\\
\ell(u_i u_{i+1}) &= a\\
&\text{end for}
\end{aligned}$$

This gives an a-sum V_4-magic labeling of $M(C_n)$. By lemma 6.1.2, it is not a-sum V_4-barycentric magic for any n. $\square$

Theorem 7.2.12. *$M(C_n)$ is zero-sum V_4-magic for all $n \geq 3$ and zero-sum V_4-barycentric magic for all $n \geq 3$.*

Proof. Label all the edges by a. Then it is easy to see that $M(C_n)$ is zero-sum V_4-magic for all $n \geq 3$. By lemma 6.1.4, $M(C_n)$ is zero-sum V_4-barycentric magic for all $n \geq 3$. $\square$

Theorem 7.2.13. *$T(C_n)$ is a-sum V_4-magic for all $n \geq 3$ and is not a-sum V_4-barycentric magic for any n.*

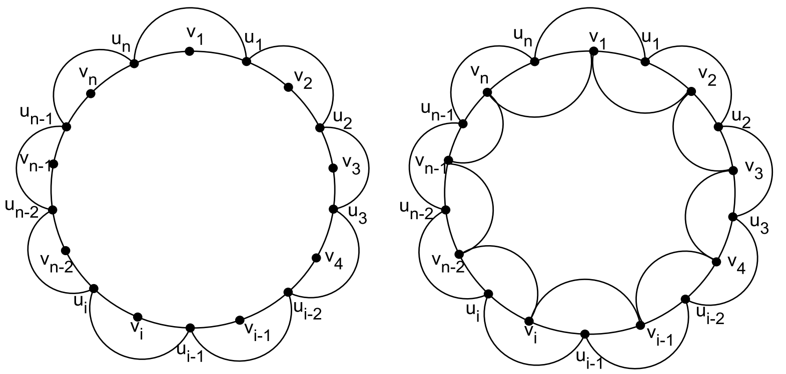


Figure 7.4: Middle Graph of Cycle $M(C_n)$ (left), Total Graph of Cycle $T(C_n)$ (right)

Proof. The vertices of $T(C_n)$ are labeled as given in figure 7.4. Define an edge labeling $\ell : E(T(C_n)) \to V_4 \setminus \{0\}$ by:

$$\begin{gathered}\text{For } i = 1, 2, \cdots, n \text{ do :}\\ \ell(u_iv_i) = b,\ \ell(u_iv_{i+1}) = c\\ \ell(u_iu_{i+1}) = \ell(v_iv_{i+1}) = a\\ \text{end for}\end{gathered}$$

This gives an a-sum V_4-magic labeling of $T(C_n)$. By lemma 6.1.2, it is not a-sum V_4-barycentric magic for any n. ☐

Theorem 7.2.14. *$T(C_n)$ is zero-sum V_4-magic for all $n \geq 3$ and zero-sum V_4-barycentric magic for all $n \geq 3$.*

Proof. Label all the edges by a. Then it is easy to see that $T(C_n)$ is zero-sum V_4-magic for all $n \geq 3$. By lemma 6.1.4, $T(C_n)$ is zero-sum V_4-barycentric magic for all $n \geq 3$. ☐

Consider the sun graph S_n.

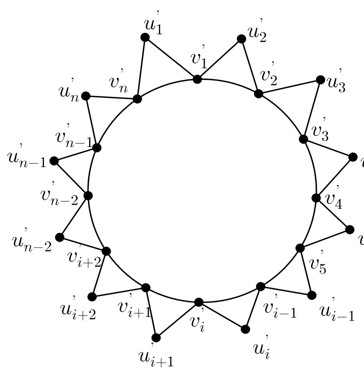
Theorem 7.2.15. *$L(S_n)$ is not a-sum V_4-barycentric magic for any n.*

Proof. The proof follows from lemma 6.1.2. ☐

Theorem 7.2.16. *$L(S_n)$ is zero-sum V_4-barycentric magic for all n.*

Proof. The proof follows from lemma 6.1.4. ☐

Next we prove similar results for middle and total graphs of sun graph.



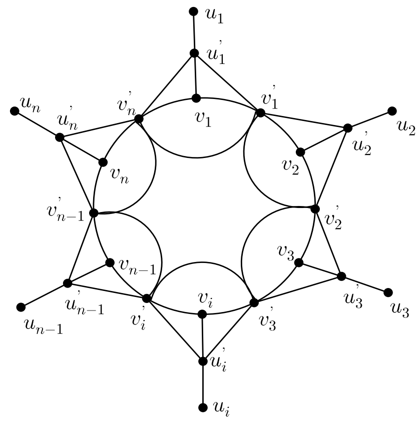


Figure 7.5: Line Graph of Sun Graph $L(S_n)$ (left), Middle Graph of Sun Graph $M(S_n)$ (right)

Theorem 7.2.17. *$M(S_n)$ is a-sum V_4-magic for all n and is not a-sum V_4-barycentric magic for any n.*

Proof. The graph of $M(S_n)$ with labeled vertices is shown in figure 7.5. Define a labeling $\ell : E(M(S_n)) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n \text{ do:}\\
\ell(u_i u_i') &= \ell(v_i u_i') = a\\
\ell(v_i v_i') &= \ell(v_i' v_{i+1}) = a\\
\ell(v_i' v_{i+1}') &= a\\
\ell(v_i' u_{i+1}') &= b\\
\ell(v_i' u_i') &= c\\
&\text{end for}
\end{aligned}$$

Then with this labeling $M(S_n)$ is a-sum V_4-magic for all n and by lemma 6.1.2, it is not a-sum V_4-barycentric magic for any n. $\square$

Theorem 7.2.18. *$M(S_n)$ is not zero-sum V_4-magic for any n and is not zero-sum V_4-barycentric magic for any n.*

Proof. Since $M(S_n)$ has pendant edges, it is not zero-sum V_4-magic for any n and hence not zero-sum V_4-barycentric magic for any n. $\square$

Theorem 7.2.19. *$T(S_n)$ is a-sum V_4-magic for all n and is not a-sum V_4-barycentric magic for any n.*

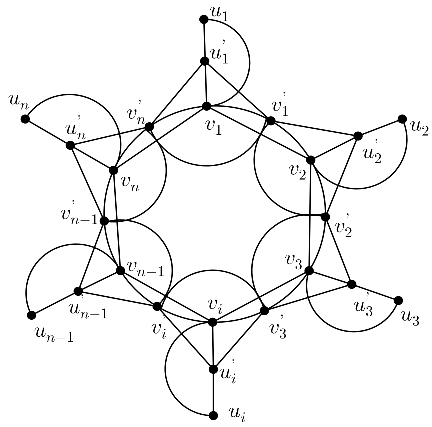


Figure 7.6: Total Graph of Sun Graph $T(S_n)$

Proof. The graph of $T(S_n)$ with labeled vertices is shown in figure 7.6. Define a labeling $\ell : E(T(S_n)) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n \text{ do:}\\
\ell(u_i u_i') &= b\\
\ell(v_i u_i') &= \ell(v_i u_i) = c\\
\ell(v_i v_i') &= b\\
\ell(v_i' v_{i+1}) &= c\\
\ell(v_i' v_{i+1}') &= \ell(v_i v_{i+1}) = a\\
\ell(v_i' u_{i+1}') &= \ell(v_i' u_i') = a\\
&\text{end for}
\end{aligned}$$

Then with this labeling $T(S_n)$ is a-sum V_4-magic for all n and by lemma 6.1.2, it is not a-sum V_4-barycentric magic for any n. □

Theorem 7.2.20. *$T(S_n)$ is zero-sum V_4-magic for all n and is zero-sum V_4-barycentric magic for all n.*

Proof. Label all the edges by a. Then the graph is zero-sum V_4-magic for all n. Moreover by lemma 6.1.4, $T(S_n)$ is zero-sum V_4-barycentric magic for all n. □

Now we consider the wheel graph W_n.

Theorem 7.2.21. *$L(W_n)$ is not a-sum V_4-barycentric magic for any n.*

Proof. By lemma 6.1.2, the theorem holds. □

Theorem 7.2.22. *$L(W_n)$ is zero-sum V_4-barycentric magic if and only if n is odd.*

Proof. Suppose that n is odd. Then all the vertices of $L(W_n)$ are of even degree. Then by lemma 6.1.4, $L(W_n)$ is zero-sum V_4-barycentric magic. For the converse we assume that n is even. Then the graph has vertices of odd degree. By lemma 6.1.3, $L(W_n)$ cannot be zero-sum V_4-barycentric magic. This proves the theorem. □

The middle graph and total graph of W_n with labeled vertices are shown in figure 7.7.

Theorem 7.2.23. *$M(W_n)$ is a-sum V_4-magic if and only if n is even and is not a-sum V_4-barycentric magic for any n.*

Proof. First suppose that $M(W_n)$ is a-sum V_4 magic. Then by lemma 3.2.1, $3na = 0$ which implies n is even. Conversely assume that n is even. Define a labeling $\ell : E(M(W_n)) \to V_4 \backslash \{0\}$ by

$$\begin{aligned} \ell(v_i u_i) &= b \\ \ell(u_i v_{i+1}) &= c \\ \ell(e_i e_j) &= \ell(e_i v_i) = \ell(u_i u_{i+1}) = a \\ \ell(e_i u_{i-1}) &= \ell(e_i u_i) = a \end{aligned}$$

This gives an a-sum V_4-magic labeling of $M(W_n)$. By lemma 6.1.2, $M(W_n)$ is not a-sum V_4-barycentric magic for any n. □

Theorem 7.2.24. *$M(W_n)$ is zero-sum V_4-magic if n is even and is not zero-sum V_4-barycentric magic for any n.*

Proof. For n even, label the edges as follows:

$$\begin{aligned} &\text{For } i = 1, 2, \cdots, n \text{ do:} \\ \ell(v_i u_i) &= a \\ \ell(u_i v_{i+1}) &= b \\ \ell(e_i v_i) &= c \\ \ell(e_i u_i) &= \ell(e_{i+1} u_i) = \ell(e_i v) = a \\ \ell(e_i e_j) &= a, j = 1, 2, \cdots, n, i \neq j, i+1 \neq j \\ \ell(u_i u_{i+1}) &= \begin{cases} a, & i = 1, 3, \cdots, n-1 \\ b, & i = 2, 4, \cdots, n \end{cases} \\ \ell(e_i e_{i+1}) &= \begin{cases} a, & i = 1, 3, \cdots, n-1 \\ c, & i = 2, 4, \cdots, n \end{cases} \end{aligned}$$

With this labeling, $M(W_n)$ is zero-sum V_4-magic. It follows from lemma 6.1.3 that $M(W_n)$ is not zero-sum V_4-barycentric magic for any n. □

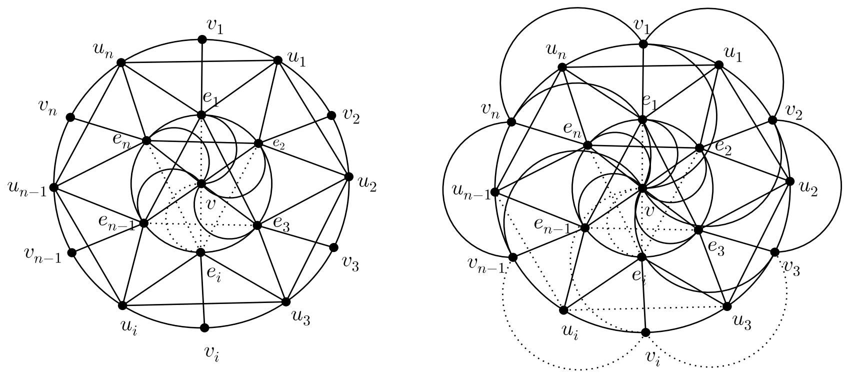


Figure 7.7: Middle Graph of Wheel Graph $M(W_n)$ (left), Total Graph of Wheel Graph $T(W_n)$ (right)

Theorem 7.2.25. *$T(W_n)$ is a-sum V_4-magic if and only if n is odd and is not a-sum V_4-barycentric magic for any n.*

Proof. Suppose that $T(W_n)$ is a-sum V_4-magic. Then by lemma 3.2.1, $(3n+1)a = 0$ which implies n is odd. Conversely suppose that n is odd.

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n \text{ do:}\\
\ell(u_iv_i) &= \ell(u_iv_{i+1}) = a\\
\ell(u_iu_{i+1}) &= \ell(v_iv_{i+1}) = a\\
\ell(e_iv_i) &= \ell(e_iv) = \ell(e_iu_i) = b\\
\ell(v_iv) &= \ell(e_iu_{i-1}) = c\\
\ell(e_ie_j) &= a, j = 1, 2, \cdots, n, i \neq j\\
&\text{end for}
\end{aligned}$$

With this labeling, $T(W_n)$ is a-sum V_4-magic. By lemma 6.1.2, $T(W_n)$ is not a-sum V_4-barycentric magic for any n. $\square$

Theorem 7.2.26. *$T(W_n)$ is zero-sum V_4-magic for all n and is zero-sum V_4-barycentric magic if n is odd.*

Proof. We consider the following cases:

Case 1: n is odd.

Label all the edges by a. This gives a zero-sum V_4-magic labeling of $T(W_n)$ and by lemma 6.1.4, $T(W_n)$ is zero-sum V_4-barycentric magic.

Case 2: n is even.

Define a labeling $\ell : E(T(W_n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n \text{ do:}\\
\ell(e_i e_j) &= a\\
\ell(e_i v) &= \ell(e_i v_i) = a\\
\ell(u_{i+1} v_i) &= \ell(u_i v_i) = a\\
\ell(u_i u_{i+1}) &= \ell(v_i v_{i+1}) = a\\
\ell(e_i u_{i-1}) &= \begin{cases} b, & i = 1, 3, \cdots, n\\ c, & i = 2, 4, \cdots, n-1 \end{cases}\\
\ell(e_i u_i) &= \begin{cases} c, & i = 1, 3, \cdots, n\\ b, & i = 2, 4, \cdots, n-1 \end{cases}
\end{aligned}$$

With this labeling $T(W_n)$ is zero-sum V_4-magic. Furthermore, it follows from lemma 6.1.3, that $T(W_n)$ is not zero-sum V_4-barycentric magic.

□

Theorem 7.2.27. *The helm graph H_n is not a-sum V_4-barycentric magic for any n.*

Proof. By lemma 6.1.2, H_n is not a-sum V_4-barycentric magic for any n. □

Theorem 7.2.28. *H_n is not zero-sum V_4-barycentric magic for any n.*

Proof. Since the graph has pendant edges, it is not zero-sum V_4-barycentric magic for any n. □

The line and middle graphs of H_n are shown in figure 7.8.

Theorem 7.2.29. *$L(H_n)$ is a-sum V_4-magic if and only if n is even and is not a-sum V_4-barycentric magic for any n.*

Proof. Suppose that $L(H_n)$ is a-sum V_4-magic. Then by lemma 3.2.1, $(3n)a = 0$ which implies n is even. Conversely assume that n is even. Define a labeling $\ell : E(L(H_n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n \text{ do:}\\
\ell(e'_{i-1} s_i) &= \ell(e'_i s_i) = a\\
\ell(e_i s_i) &= \ell(e'_i e'_{i+1}) = a\\
\ell(e_i e_j) &= a, j = 1, 2, \cdots, n, i \neq j\\
\ell(e_i e'_i) &= b\\
\ell(e_i e'_{i-1}) &= c\\
&\text{end for}
\end{aligned}$$

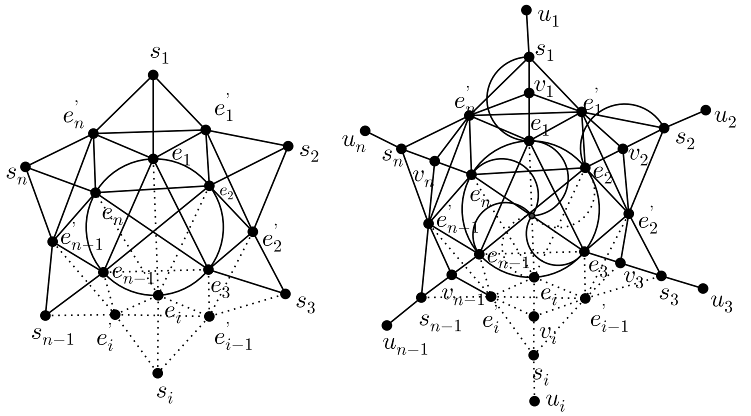


Figure 7.8: Line Graph of Helm Graph $L(H_n)$ (left), Middle Graph of Helm Graph $M(H_n)$ (right)

By lemma 6.1.2, it is not a-sum V_4-barycentric magic for any n. $\square$

Theorem 7.2.30. *$L(H_n)$ is zero-sum V_4-magic for all n and is not zero-sum V_4-barycentric magic for any n.*

Proof. We consider the following cases:

Case 1: n is even.

$$\ell(s_ie'_{i-1}) = \begin{cases} b, & i = 1, 3, \cdots, n-1 \\ c, & i = 2, 4, \cdots, n \end{cases}$$

$$\ell(s_ie'_i) = \begin{cases} c, & i = 1, 3, \cdots, n-1 \\ b, & i = 2, 4, \cdots, n \end{cases}$$

Label the remaining edges by a.

Case 2: n is odd.

$$\ell(s_ie_i) = \ell(e'_ie'_{i+1}) = a$$
$$\ell(e'_{i-1}s_i) = \ell(e_ie'_i) = b$$
$$\ell(e'_is_i) = \ell(e_{i+1}e'_i) = c$$
$$\ell(e_ie_j) = a, j = 1, 2, \cdots, n, i \neq j$$

Therefore, $L(H_n)$ is zero-sum V_4-magic for all n. By lemma 6.1.3, $L(H_n)$ is not zero-sum V_4-barycentric magic for any n.

□

Theorem 7.2.31. *$M(H_n)$ is a-sum V_4-magic if and only if n is odd and is not a-sum V_4-barycentric magic for any n.*

Proof. Suppose that $M(H_n)$ is a-sum V_4-magic. Then $(5n+1)a = 0$ which implies n is odd. Conversely suppose that n is odd. Define a labeling $\ell : E(M(H_n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n \text{ do:}\\
&\ell(e'_{i-1}v_i) = b\\
&\ell(e'_i v_i) = c\\
&\text{end for}
\end{aligned}$$

Label the remaining edges by a. By lemma 6.1.2, it is not a-sum V_4-barycentric magic for any n. □

Theorem 7.2.32. *$M(H_n)$ is not zero-sum V_4-magic and is not zero-sum V_4-barycentric magic for any n.*

Proof. Since the graph has pendant edges, it is not zero-sum V_4-barycentric magic for any n. □

Definition 7.2.33. *(see [29]) A gear graph G_n is obtained from the wheel graph by adding a vertex between every pair of adjacent vertices of the cycle where $V(G_n) = \{v\} \cup \{v_1, v_2, \cdots, v_{2n}\}$.*

Theorem 7.2.34. *The line graph of gear graph $L(G_n)$ is a-sum V_4-magic if and only if n is even and is a-sum V_4-barycentric magic for all n.*

Proof. Suppose that $L(G_n)$ is a-sum V_4-magic. Then by lemma 3.2.1, n is even. Conversely assume that n is even. Label all the edges by a. Then $L(G_n)$ is a-sum V_4-magic. By lemma 6.1.5, $L(G_n)$ is a-sum V_4-barycentric magic for all n. □

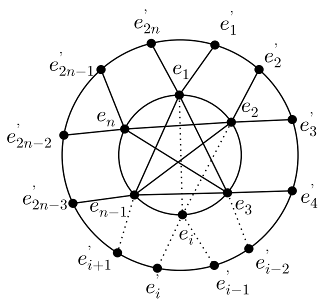
The line and middle graphs of G_n are shown in figure 7.9.

Theorem 7.2.35. *$L(G_n)$ is zero-sum V_4-magic for all n and is not zero-sum V_4-barycentric magic for any n.*

Proof. We consider the following cases:

Case 1: n is odd.

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, n, \text{ do:}\\
&\ell(e_ie_j) = a, j = 1, 2, \cdots, n, i \neq j\\
&\ell(e_ie'_j) = \ell(e_ie'_{j+1}) = a,\ j = 1, 2, \cdots, 2n \text{ where } e_i \text{ is adjacent to } e'_j \text{ and } e'_{j+1}
\end{aligned}$$



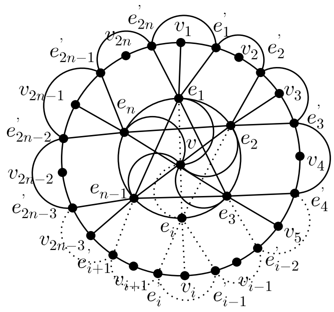


Figure 7.9: Line Graph of Gear Graph $L(G_n)$ (left), Middle Graph of Gear Graph $M(G_n)$ (right)

$$\ell(e'_k e'_{k+1}) = \begin{cases} b, & k = 1, 3, \cdots, 2n-1 \\ c, & k = 2, 4, \cdots, 2n \end{cases}$$

end for

Case 2: n is even.

For $i = 1, 2, \cdots, n,$ do:

$\ell(e_i e'_j) = \ell(e_i e'_{j+1}) = a, j = 1, 2, \cdots, 2n$ where e_i is adjacent to e'_j and e'_{j+1}

$$\ell(e_i e_{i+1}) = \begin{cases} b, & i = 1, 3, \cdots, n-1 \\ c, & i = 2, 4, \cdots, n \end{cases}$$

$$\ell(e'_k e'_{k+1}) = \begin{cases} b, & k = 1, 3, \cdots, 2n-1 \\ c, & k = 2, 4, \cdots, 2n \end{cases}$$

$\ell(e_i e_j) = a, j = 1, 2, \cdots, n, \ j \neq i, i+1$

end for

With this labeling, $L(G_n)$ is zero-sum V_4-magic for all n. By lemma 6.1.3, $L(G_n)$ is not zero-sum V_4-barycentric magic for any n. □

Now consider the middle graph of G_n.

Theorem 7.2.36. *$M(G_n)$ is a-sum V_4-magic if and only if n is odd and is not a-sum V_4-barycentric magic for any n.*

Proof. Suppose that $M(G_n)$ is a-sum V_4-magic. Then by lemma 3.2.1, $(5n+1)a = 0$ which implies n is odd. Let the vertices of $M(G_n)$ be as shown in figure 7.9. For the converse we define an edge labeling as follows:

$$\ell(v_i e'_i) = \begin{cases} a, & i = 1, 3, \cdots, 2n-1 \\ c, & i = 2, 4, \cdots, 2n \end{cases}$$

$$\ell(v_{i+1} e'_i) = \begin{cases} b, & i = 1, 3, \cdots, 2n-1 \\ a, & i = 2, 4, \cdots, 2n \end{cases}$$

$$\ell(e'_i e'_{i+1}) = a, i = 1, 2, \cdots, 2n$$

For $i = 1, 2, \cdots, n$ do:

$$\ell(v_i e_{2i-1}) = \ell(e_i v) = a$$

$$\ell(e_i e'_{2i-1}) = b$$

$$\ell(e_i e'_{2i-2}) = c$$

$$\ell(e_i e_j) = a, j = 1, 2, \cdots, n,\ i \neq j$$

end for

Thus $M(G_n)$ admits an a-sum V_4-magic labeling and by lemma 6.1.2, it is not a-sum V_4-barycentric magic for any n. □

Theorem 7.2.37. *$M(G_n)$ is zero-sum V_4-magic for all n and is not zero-sum V_4-barycentric magic for any n.*

Proof. We consider two cases:

Case 1: n is even.

For $i = 1, 2, \cdots, n$ do:

$$\ell(e_i v) = \ell(e_i e_j) = \ell(e_i v_{2i-1}) = a$$

For $k = 1, 2, \cdots, 2n$ do:

$$\ell(v_i e'_k) = b$$

$$\ell(e'_k v_{i+1}) = \begin{cases} b, & k = 1, 3, \cdots, 2n-1 \\ c, & k = 2, 4, \cdots, 2n \end{cases}$$

$$\ell(e'_i e'_{i+1}) = \begin{cases} c, & k = 1, 3, \cdots, 2n-1 \\ a, & k = 2, 4, \cdots, 2n \end{cases}$$

end for

end for

Case 2: n is odd.

$$\ell(e_i v) = \begin{cases} a, & i = 1, 4, 5, \cdots, n \\ b, & i = 2 \\ c, & i = 3 \end{cases}$$

$$\ell(e_iv_{2i-1}) = \begin{cases} a, & i = 1, 4, 5, \cdots, n \\ b, & i = 2 \\ c, & i = 3 \end{cases}$$

For $i = 1, 2, \cdots, n$ do:

$$\ell(e_ie_j) = a, j = 1, 2, \cdots, n,\ i \neq j$$

end for

$$\ell(e'_ie'_{i+1}) = \begin{cases} c, & i = 3 \\ a, & \text{otherwise} \end{cases}$$

$$\ell(v_1e'_1) = c$$

$$\ell(v_2e'_2) = b$$

$$\ell(v_3e'_3) = a$$

$$\ell(v_ie'_i) = \ell(v_{i+1}e_{i+1'}) = \begin{cases} b, & i = 4, 8, \cdots, 2n-1 \\ c, & i = 6, 10, \cdots, 2n \end{cases}$$

$$\ell(e'_1v_2) = \ell(e'_3v_4) = b$$

$$\ell(e'_2v_3) = \ell(e'_5v_6) = c$$

$$\ell(e'_4v_5) = a$$

$$\ell(e'_iv_{i+1}) = \ell(e'_{i+1}v_{i+2}) = \begin{cases} b, & i = 6, 10, \cdots, 2n \\ c, & i = 8, 12, \cdots, 2n-2 \end{cases}$$

Thus $M(G_n)$ is zero-sum V_4-magic for all n. Moreover, by lemma 6.1.3, $M(G_n)$ is not zero-sum V_4-barycentric magic for any n.

□

Chapter 8

V_4-Bimagic Graphs

In the first section of this chapter, we introduce a-sum and zero sum V_4-bimagic graphs and define sixteen different classes of graphs. In the second section of this chapter, we discuss the classification of star graph. In the third section we discuss the classification of some cycle related graphs. In the last section of this chapter we discuss the classification of ladders and some special graphs.

8.1 Introduction

If a graph G and its line graph $L(G)$ are both a-sum V_4-magic, then G is called an a-sum V_4-bimagic graph. If a graph G and its line graph $L(G)$ are both zero-sum V_4-magic, then G is called a zero-sum V_4-bimagic graph. A graph G is called a V_4-bimagic graph if G and its line graph $L(G)$ are both a-sum V_4-magic or zero-sum V_4-magic. An example of a V_4-bimagic graph is the cycle C_n. If G or $L(G)$ is not V_4-magic, then G is called a non V_4-bimagic graph. In this chapter we define the following classes of graphs:

(1) If both the graph G and its line graph $L(G)$ are a-sum V_4-magic, we say that G belongs to the class $\mathscr{A}$.

(2) If G is a-sum V_4-magic and $L(G)$ is zero-sum V_4-magic, we say that G belongs to the class $\mathscr{B}$.

3) If both the graph G and its line graph $L(G)$ are zero-sum V_4-magic, we say that G belongs to the class $\mathscr{C}$.

4) If G is zero-sum V_4-magic and $L(G)$ is a-sum V_4-magic, we say that G belongs to the class $\mathscr{D}$.

5) If G is a-sum V_4-magic and $L(G)$ is not a-sum V_4-magic, we say that G belongs to the class $\mathscr{E}$.

6) If G is a-sum V_4-magic and $L(G)$ is not zero-sum V_4-magic, we say that G belongs to the class $\mathscr{F}$.

7) If G is not a-sum V_4-magic and $L(G)$ is a-sum V_4-magic, we say that G belongs to the class $\mathscr{G}$.

8) If G is not a-sum V_4-magic and $L(G)$ is zero-sum V_4-magic, we say that G belongs to the class $\mathscr{H}$.

9) If G is zero-sum V_4-magic and $L(G)$ is not a-sum V_4-magic, we say that G belongs to the class $\mathscr{I}$.

10) If G is zero-sum V_4-magic and $L(G)$ is not zero-sum V_4-magic, we say that G belongs to the class $\mathscr{J}$.

11) If G is not zero-sum V_4-magic and $L(G)$ is a-sum V_4-magic, we say that G belongs to the class $\mathscr{K}$.

12) If G is not zero-sum V_4-magic and $L(G)$ is zero-sum V_4-magic, we say that G belongs to the class $\mathscr{L}$.

13) If both G and $L(G)$ are not a-sum V_4-magic, we say that G belongs to the class $\mathscr{M}$.

14) If G is not a-sum V_4-magic and $L(G)$ is not zero-sum V_4-magic, we say that G belongs to the class $\mathscr{N}$.

15) If G is not zero-sum V_4-magic and $L(G)$ is not a-sum V_4-magic, we say that G belongs to the class $\mathscr{O}$.

16) If both G and $L(G)$ are not zero-sum V_4-magic, we say that G belongs to the class $\mathscr{P}$.

A graph is V_4-bimagic if it belongs to the class $\mathscr{A} \cup \mathscr{C}$.

8.2 Basic Results

Theorem 8.2.1. *Let G be a V_4-bimagic (p, q) graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_p\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_q\}$. Then*

$$\sum_{i=1}^{p} \ell^+(u_i) = \sum_{i=1}^{q} \ell^+(e_i) = 0.$$

Proof. Proof follows from lemma 3.2.1. □

Theorem 8.2.2. *Let G be an a-sum V_4-bimagic (p, q) graph. Then both p and q are even.*

Proof. Proof follows from theorem 8.2.1. $\square$

Theorem 8.2.3. *The star $K_{1,n}$ is not a-sum V_4-bimagic for all $n > 1$.*

Proof. Proof follows from theorem 8.2.2. $\square$

Theorem 8.2.4. *(see [13]) The star $K_{1,n}$ is a-sum V_4-magic if and only if n is odd.*

Theorem 8.2.5. *$K_{1,n}$ is not zero-sum V_4-magic for any n.*

Proof. Since the graph has pendant edges, it is not zero-sum V_4-magic for any n. $\square$

Theorem 8.2.6. *$L(K_{1,n}) = K_n$ is zero-sum V_4-magic for all n.*

Proof. Let $v_1, v_2, \cdots v_n$ be the vertices of K_n. We consider the following cases.

Case 1: n is odd. Label all the edges by a. Then we get $\ell^+(v) = 0$ for all $v \in V(K_n)$.

Case 2: n is even.

$$\text{For } i = 1, 2, \cdots, n \text{ do:}$$
$$\ell(v_i v_j) = a, j = 1, 2, \cdots, n, j \neq i+1, j \neq i$$
$$\text{end for}$$
$$\ell(v_i v_{i+1}) = \begin{cases} b, & i = 1, 3, \cdots n-1 \\ c, & i = 2, 4, \cdots n \end{cases}$$

Thus $\ell^+(v) = 0$ for all $v \in V(L(K_{1,n}))$. $\square$

Theorem 8.2.7. *$L(K_{1,n})$ is a-sum V_4-magic if and only if n is even.*

Proof. $L(K_{1,n}) = K_n \in \mathscr{V}_a \Rightarrow na = 0 \Rightarrow n$ is even. If n is even, label all the edges by a. The result follows. $\square$

Theorem 8.2.8. *$K_{1,n}$ belongs to the class $\mathscr{B}$ if and only if n is odd.*

Proof. The proof follows from theorems 8.2.4 and 8.2.6. $\square$

Theorem 8.2.9. *$K_{1,n}$ belongs to the class $\mathscr{K}$ if and only if n is even.*

Proof. The proof follows from theorems 8.2.5 and 8.2.7. $\square$

Theorem 8.2.10. *$K_{1,n}$ belongs to the class $\mathscr{L}$ for all n.*

Proof. The proof follows from theorems 8.2.5 and 8.2.6. $\square$

8.3 Cycle Related Graphs

Let $S_n = C_n \odot K_1$ be the sun graph on $2n$ vertices. Let $V(S_n) = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ where v_i' s are the vertices of the cycle taken in cyclic order and u_i' s are the pendant vertices such that each v_iu_i is a pendant edge. Let $E(S_n) = \{e_i' : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$ where e_i is the edge v_iv_{i+1} and e_i' is the edge $v_iu_i(1 \leq i \leq n)$. By the definition of line graph $V(L(S_n)) = E(S_n) = \{u_i' : 1 \leq i \leq n\} \cup \{v_i' : 1 \leq i \leq n\}$ where v_i' and u_i' represents the edge e_i and $e_i'(1 \leq i \leq n)$ respectively. The edge set $E(L(S_n)) = \{v_i'v_{i+1}' : 1 \leq i \leq n\} \cup \{v_i'u_{i+1}' : 1 \leq i \leq n\} \cup \{v_i'u_i' : 1 \leq i \leq n\}$. Hence the line graph of the sun graph Sun_n is the graph $C_n @ C_3$.

Theorem 8.3.1. *$L(S_n)$ is zero-sum V_4-magic for all n.*

Proof. Label all the edges by a. Then we have the graph $L(S_n)$ is zero-sum V_4-magic. □

Theorem 8.3.2. *$L(S_n)$ is a-sum V_4-magic for all n.*

Proof. Define a labeling $\ell : E(L(S_n)) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned} &\text{For } i = 1, 2, \cdots n \text{ do:} \\ &\ell(v_i'v_{i+1}') = a,\ \ell(u_i'v_i') = b \\ &\ell(u_i'v_{i+1}') = c \\ &\text{end for} \end{aligned}$$

With this labeling $\ell^+(v) = a$ for all $v \in V(\mathrm{L(G)})$. Hence the graph $L(S_n)$ is a-sum V_4-magic for all n. □

Theorem 8.3.3. *Sun_n belongs to the classes $\mathscr{A}$, $\mathscr{B}$, $\mathscr{K}$ and $\mathscr{L}$ for all n.*

Proof. The proof follows from theorems 3.2.9, 3.2.10, 8.3.1 and 8.3.2. □

$BSun(n,k)$ denote the broken sun on n vertices. Let $V(BSun(n,k)) = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_k\}$ where v_i' s are the vertices of C_n and u_j' s are the pendant vertices such that $v_{i_j}u_j$ is a pendant edge where v_{i_j} denote the particular vertex v_i adjacent to u_j. Let $E(BSun(n,k)) = \{e_i : 1 \leq i \leq n\} \cup \{e_j' : 1 \leq j \leq k\}$ where e_i is the edge $v_iv_{i+1}(1 \leq i \leq n-1)$ and e_j' is the edge $v_{i_j}u_j$. The vertex v_{i_j} is a point of intersection of the edges e_{i-1}, e_i and e_j'. By the definition of line graph, $V(L(BSun(n,k))) = \{v_1', v_2', \cdots, v_n'\} \cup \{u_1', u_2', \cdots, u_k'\}$ where v_i' represents the edge $e_i, 1 \leq i \leq n$ and u_j' represents the edge $e_j', 1 \leq j \leq k$ and $E(L(BSun(n,k))) = \{v_i'v_{i+1}' : 1 \leq i \leq n\} \cup \{v_{i_j-1}'u_j' : v_{i_j-1}'$ is the edge e_{i-1} which is incident to $v_{i_j}, j = 1, 2, \cdots, k\} \cup \{v_{i_j}'u_j' : v_{i_j}'$ is the edge e_i which is incident to $v_{i_j}, j = 1, 2, \cdots, k\}$.

Theorem 8.3.4. *$BSun(n,k)$ is not zero-sum V_4-magic for any n.*

Proof. Since it has pendant edges the result follows. □

Theorem 8.3.5. *The line graph of a broken sun $BSun(n,k)$ is a-sum V_4-magic if and only if $n+k$ is even.*

Proof. Suppose that $L(BSun(n,k))$ is a-sum V_4-magic. Then by lemma 3.2.1, $(n+k)a=0$ which implies that $n+k$ is even. Conversely suppose that $n+k$ is even.
Define a labeling $\ell : E(L(BSun(n,k))) \longrightarrow V_4 \setminus \{0\}$ as follows:

Case 1: Both n and k are even.

Subcase i: All the pendant edges $v_{i_j}u_j$ are on adjacent vertices of $BSun(n,k)$.

$$
\begin{aligned}
&\text{For } i = 1, 2, \cdots, n\\
&\text{For } j = 1, 2, \cdots, k\\
\ell(v'_{i_j-1}v'_{i_j}) &= a,\ \ \ell(v'_{i_j-1}u'_j) = b\\
\ell(v'_{i_j}u'_j) &= c\\
&\text{end for}\\
&\text{end for}
\end{aligned}
$$

If $\ell(v'_iu'_j) = b$, then $v'_iv'_{i+1} = b$ and viceversa. If $\ell(v'_iv'_{i+1}) = b$ then $\ell(v'_{i+1}v'_{i+2}) = c$ and if $\ell(v'_iv'_{i+1}) = c$ then $\ell(v'_{i+1}v'_{i+2}) = b$.

Subcase ii: All the pendant edges are on alternate vertices of $BSun(n,k)$.

$$
\begin{aligned}
&\text{For } i = 1, 2, \cdots, n\\
&\text{For } j = 1, 3, \cdots, k-1\\
\ell(v'_{i_j-1}v'_{i_j}) &= a,\ \ \ell(v'_{i_j-1}u'_j) = b\\
\ell(v'_{i_j}u'_j) &= c\\
&\text{end for}\\
&\text{end for}\\
&\text{For } i = 1, 2, \cdots, n\\
&\text{For } j = 2, 4, \cdots, k\\
\ell(v'_{i_j-1}v'_{i_j}) &= a,\ \ \ell(v'_{i_j-1}u'_j) = c\\
\ell(v'_{i_j}u'_j) &= b\\
&\text{end for}\\
&\text{end for}
\end{aligned}
$$

If $\ell(v'_iu'_j) = b$, then $v'_iv'_{i+1} = b$ and viceversa.

Subcase iii: The pendant edges $v_{i_j}u_j$ are on random vertices.

$$
\begin{aligned}
&\text{For } i = 1, 2, \cdots, n\\
&\text{For } j = 1, 2, \cdots, k
\end{aligned}
$$

$$\ell(v'_{i_j-1}v'_{i_j}) = a,\ \ell(v'_{i_j-1}u'_j) = b$$
$$\ell(v'_{i_j}u'_j) = c$$
end for

end for

Depending on the labeling of the edge v_iv_{i+1}, for some edges

For $i = 1, 2, \cdots, n$

For $j = 1, 2, \cdots, k$

$$\ell(v'_{i_j-1}u'_j) = b,\ \ell(v'_{i_j}u'_j) = c$$

end for

If $\ell(v'_iu'_j) = b$, then $v'_iv'_{i+1} = c$ and viceversa. If $\ell(v'_iv'_{i+1}) = b$ then $\ell(v'_{i+1}v'_{i+2}) = c$ and viceversa.

Case 2: Both n and k are odd.

The proof is similar to case 1.

This completes the proof. □

Theorem 8.3.6. *$L(BSun(n, k))$ is zero-sum V_4-magic for all n and k.*

Proof. Define a labeling $\ell : E(L(BSun(n, k))) \longrightarrow V_4 \setminus \{0\}$ as follows:

For $i = 1, 2, \cdots, n$, do:

For $j = 1, 2, \cdots, k$, do:

$$\ell(v'_{i_j}u_j) = \ell(v'_{i_j+1}u_j) = a,$$
$$\ell(v'_{i_j}v'_{i_j+1}) = b,\ \ell(v_iv_{i+1}) = c$$

end for

end for

Hence the proof. □

Theorem 8.3.7. *$BSun(n, k)$ belongs to the classes $\mathscr{A}, \mathscr{B}$ and $\mathscr{K}$ if and only if $n + k$ is even.*

Proof. The proof follows from theorems 3.2.11, 8.3.4, 8.3.5 and 8.3.6. □

Theorem 8.3.8. *$BSun(n, k)$ belongs to the $\mathscr{L}$-class for all n and k.*

Proof. The proof follows from theorems 8.3.4 and 8.3.6. □

For $n > 2$ and $0 < k < n$, a consecutive broken sun, denoted by $CBSun(n, k)$ is the graph belonging to $BSun(n, k)$ such that the subgraph induced by the vertices of degree 2 is a path on $n - k$ vertices.

For $n > 2$ and $0 < k < n$, the line graph of $CBSun(n, k)$, denoted by $L(CBS(n, k))$ is the graph belonging to the class $L(BSun(n, k))$ such that the subgraph induced by the vertices of degree 2 is a path on $n - k - 1$ vertices.
The vertex set and edge set are similar to that of the line graph of $BSun(n, k)$.

Theorem 8.3.9. *$CBSun(n, k)$ is not zero-sum V_4-magic for any n.*

Proof. Since it has pendant edges the result follows. □

Theorem 8.3.10. *$L(CBSun(n, k))$ is a-sum V_4-magic if and only if $n + k$ is even.*

Proof. It is already proved as subcase (i) in the proof of theorem 8.3.5. □

Theorem 8.3.11. *The line graph of $CBSun(n, k)$ is zero-sum V_4-magic for all n and k.*

Proof. The proof is similar to the proof of theorem 8.3.6. □

Theorem 8.3.12. *$CBSun(n, k)$ belongs to the classes $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{K}$ if and only if $n + k$ is even.*

Proof. The proof follows from theorems 3.2.12, 8.3.9, 8.3.10 and 8.3.11. □

Theorem 8.3.13. *$CBSun(n, k)$ belongs to the $\mathscr{L}$-class for all n and k.*

Proof. The proof follows from theorems 8.3.9 and 8.3.11. □

W_n denotes the wheel graph on $n+1$ vertices. Let $V(W_n) = \{u_1, u_2, \cdots, u_n\} \cup \{u\}$ where u is the central vertex and $E(W_n) = \{e_i : 1 \le i \le n\} \cup \{e'_i : 1 \le i \le n\}$ where e_i is the edge $uu_i, 1 \le i \le n$ and e'_i is the edge $u_iu_{i+1}, 1 \le i \le n$. Consider the line graph $L(W_n)$. Then $V(L(W_n)) = \{v'_1, v'_2, \cdots, v'_n\} \cup \{v_1, v_2, \cdots, v_n\}$ where v_i represents the edge $e_i, 1 \le i \le n$ and v'_i represents the edge $e'_i, 1 \le i \le n$ and $E(L(W_n)) = \{v_iv_j : i, j = 1, 2, \cdots, n, i \ne j\} \cup \{v'_iv_{i+1} : 1 \le i \le n\} \cup \{v'_iv_i : 1 \le i \le n\} \cup \{v'_iv'_{i+1} : 1 \le i \le n\}$.

Theorem 8.3.14. *$L(W_n)$ is a-sum V_4-magic for all n.*

Proof. Define a labeling $\ell : E(L(W_n)) \longrightarrow V_4 \setminus \{0\}$ by

Case 1: n is odd.

$$\text{For } i = 1, 2, \cdots, n \text{ do :}$$
$$\ell(v_iv'_i) = b.\ \ell(v_{i+1}v'_i) = c$$

$$\ell(v_i'v_{i+1}') = a$$
$$\ell(v_iv_j) = a, j = 1, 2, \cdots, n$$
$$\text{end for}$$

Case 2: n is even.

$$\text{For } i = 1, 2, \cdots, n \text{ do :}$$
$$\ell(v_iv_j) = a, j = 1, 2, \cdots, n,\ i \neq j$$
$$\ell(v_i'v_{i+1}') = a$$
$$\text{end for}$$
$$\text{For } i = 1, 3, \cdots, n-1 \text{ do :}$$
$$\ell(v_iv_i') = b,\ \ell(v_{i+1}v_i') = c,$$
$$\text{end for}$$
$$\text{For } i = 2, 4, \cdots, n \text{ do :}$$
$$\ell(v_iv_i') = c,\ \ell(v_{i+1}v_i') = b,$$
$$\text{end for}$$

We can easily verify that ℓ is an a-sum V_4-magic labeling of $L(W_n)$. □

Theorem 8.3.15. *$L(W_n)$ is zero-sum V_4-magic for all n.*

Proof. Define a labeling $\ell : E(L(W_n)) \longrightarrow V_4 \setminus \{0\}$ as follows:

Case 1: n is odd.

Label all the edges by a. Then we get $\ell^+(v) = 0$ for all $v \in V(L(W_n))$.

Case 2: n is even.

$$\text{For } i = 1, 2, \cdots, n \text{ do:}$$
$$\ell(v_i'v_i) = \ell(v_i'v_{i+1}) = a$$
$$\ell(v_i'v_{i+1}') = a$$
$$\ell(v_iv_j) = a, j = 1, 2, \cdots, n, j \neq i+1$$
$$\text{end for}$$
$$\ell(v_iv_{i+1}) = \begin{cases} b, & i = 1, 3, \cdots, n-1 \\ c, & i = 2, 4, \cdots, n \end{cases}$$

Thus ℓ is a zero-sum V_4-magic labeling of $L(W_n)$. □

Theorem 8.3.16. *W_n belongs to the classes $\mathscr{A}$ and $\mathscr{B}$ if and only if n is odd.*

Proof. The proof follows from theorems 3.3.1, 8.3.14 and 8.3.15. □

Theorem 8.3.17. *W_n belongs to the classes $\mathscr{C}$ and $\mathscr{D}$ for all n.*

Proof. The proof follows from theorems 3.3.2, 3.3.3, 8.3.14 and 8.3.15. □

Consider the graph $C_m@C_n$. Let $V(C_m@C_n) = \{u_i, 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m-2\}$ and $E(C_m@C_n) = \{u_iu_{i+1} : 1 \leq i \leq n\} \cup \{u_{ij}u_{i(j+1)} : 1 \leq i \leq n, 1 \leq j \leq m-2\} \cup \{u_iu_{i1} : 1 \leq i \leq n\} \cup \{u_{i(m-2)}u_{i+1} : 1 \leq i \leq n\}$. Consider the line graph $L(C_m@C_n)$. Then $V(L(C_m@C_n)) = \{v_i : 1 \leq i \leq n\} \cup \{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$ and $E(L(C_m@C_n)) = \{v_iv_{i+1} : 1 \leq i \leq n\} \cup \{v_iv_{i1} : 1 \leq i \leq n\} \cup \{v_iv_{i(m-1)} : 1 \leq i \leq n\} \cup \{v_{ij}v_{i(j+1)} : 1 \leq i \leq n, 1 \leq j \leq m-2\} \cup \{v_iv_{(i+1)1} : 1 \leq i \leq n\} \cup \{v_{i+1}v_{i(m-1)} : 1 \leq i \leq n\}$.

Theorem 8.3.18. *(see [13]) For all $m, n \geq 3$, $C_m@C_n$ is zero-sum V_4 magic.*

Theorem 8.3.19. *$L(C_m@C_n)$ is a-sum V_4-magic if and only if mn is even.*

Proof. Suppose that $L(C_m@C_n)$ is a-sum V_4 magic. Then by lemma 3.2.1, $n + n(m-1)$ is even. This implies that mn is even. Conversely suppose that mn is even.
We consider the following cases:

Case 1: m is odd and n is even.

Define a labeling $\ell : E(L(C_m@C_n)) \longrightarrow V_4 \setminus \{0\}$ as follows:

$$\ell(v_iv_{i+1}) = \begin{cases} b, & i = 1, 3, \cdots, n-1 \\ c, & i = 2, 4, \cdots, n \end{cases}$$

For $i = 1, 2, \cdots, n$ do:

$$\ell(v_iv_{i(m-1)}) = \ell(v_iv_{(i+1)1}) = \ell(v_{i+1}v_{i(m-1)}) = \ell(v_iv_{i1}) = \ell(v_{i(m-1)}v_{(i+1)1}) = b$$

$$\ell(v_{ij}v_{i(j+1)}) = \begin{cases} c, & j = 1, 3, \cdots, m-2 \\ b, & j = 2, 4, \cdots, m-3 \end{cases}$$

end for

Case 2: m is even and n is odd.

Define a labeling $\ell : E(L(C_m@C_n)) \longrightarrow V_4 \setminus \{0\}$ as follows:

For $i = 1, 2, \cdots, n$ do:

$$\ell(v_iv_{i+1}) = \ell(v_{(i+1)}v_{i(m-1)}) = \ell(v_iv_{(i+1)1}) = \ell(v_{(i+1)1}v_{i(m-1)}) = a$$

$$\ell(v_iv_{i1}) = b,\ \ell(v_iv_{i(m-1)}) = c$$

$$\ell(v_{ij}v_{i(j+1)}) = \begin{cases} c, & i = 1, 3, \cdots, m-3 \\ b, & i = 2, 4, \cdots, m-2 \end{cases}$$

end for

Case 3: Both m and n are even.

The proof is similar to case 2.

With the above defined labeling $L(C_m@C_n)$ is a-sum V_4 magic. □

Theorem 8.3.20. *$L(C_m@C_n)$ is zero-sum V_4-magic for all $m, n \geq 3$.*

Proof. The proof is quite trivial. □

Theorem 8.3.21. *$C_m@C_n$ belongs to the $\mathscr{A}$-class if and only if either m is odd and n is even or both m and n are even.*

Proof. The proof follows from theorems 5.2.16 and 8.3.19. □

Theorem 8.3.22. *$C_m@C_n$ belongs to the $\mathscr{B}$-class if and only if $n(m-1)$ is even.*

Proof. The proof follows from theorems 5.2.16 and 8.3.20. □

Theorem 8.3.23. *$C_m@C_n$ belongs to the $\mathscr{C}$-class for all $m, n \geq 3$.*

Proof. The proof follows from theorems 8.3.18 and 8.3.20. □

Theorem 8.3.24. *$C_m@C_n$ belongs to the $\mathscr{D}$-class if and only if mn is even.*

Proof. The proof follows from theorems 8.3.18 and 8.3.19. □

$C_n(t)$ denote the one point union of t copies of cycle C_n. Let $V(C_n(t)) = \{w\} \cup \{w_{ij} : 1 \leq i \leq t, 1 \leq j \leq n-1\}$ and $E(C_n(t)) = \{w_{ij}w_{i(j+1)} : 1 \leq i \leq t, 1 \leq j \leq n-2\} \cup \{ww_{i1} : 1 \leq i \leq t\} \cup \{ww_{i(n-1)} : 1 \leq i \leq t\}$. Consider the line graph $L(C_n(t))$. Then the vertex set $V(L(C_n(t))) = \{v_{ij} : 1 \leq i \leq t, j = 1, 2\} \cup \{u_{ij} : 1 \leq i \leq t, 1 \leq j \leq n-2\}$ where v_{i1} denotes the edge ww_{i1}, v_{i2} denotes the edge $ww_{i(n-1)}$ and u_{ij} denotes the edge $w_{ij}w_{i(j+1)}$. The edge set is given by $E(L(C_n(t))) = \{v_{ij}v_{kl} : 1 \leq i, k \leq t, j, l = 1, 2$ except for the case $i = k, j = l\} \cup \{v_{i1}u_{i1} : 1 \leq i \leq t\} \cup \{v_{i2}u_{i(n-2)} : 1 \leq i \leq t\} \cup \{u_{ij}u_{i(j+1)} : 1 \leq i \leq t, 1 \leq j \leq n-3\}$.

Theorem 8.3.25. *(see [28]) $C_n(t)$ is zero-sum V_4-magic for all n and t.*

Theorem 8.3.26. *$L(C_n(t))$ is a-sum V_4-magic if and only if nt is even.*

Proof. $L(C_n(t))$ is a-sum V_4-magic implies that $[2t + t(n-2)]a = 0$. This implies that nt is even. Conversely suppose that nt is even. Then the following three cases arise.

i) Both n and t are even.

ii) n is even and t is odd.

iii) n is odd and t is even

i) Both n and t are even.

$$
\begin{aligned}
&\text{For } i = 1, 2, \cdots, t \text{ do:}\\
&\ell(v_{i1}v_{i2}) = \ell(v_{i2}v_{(i+1)1}) = b\\
&\ell(v_{i1}u_{i1}) = \ell(v_{i2}u_{i(n-2)}) = b\\
&\ell(u_{ij}u_{i(j+1)}) = \begin{cases} c, & j = 1, 3, \cdots, n-3 \\ b, & j = 2, 4, \cdots, n-4 \end{cases}\\
&\text{For } k = 1, 2, \cdots, t,\ j, l = 1, 2, \text{ do:}\\
&\ell(v_{ij}v_{kl}) = c,\ v_{ij} \neq v_{kl},\ k \neq i \text{ and } l \neq j+1,\ k \neq i+1 \text{ and } l \neq j-1\\
&\text{end for}\\
&\text{end for}
\end{aligned}
$$

ii) n is even and t is odd.

$$
\begin{aligned}
&\text{For } i = 1, 2, \cdots, t \text{ do:}\\
&\ell(v_{ij}v_{kl}) = b,\ k = 1, 2, \cdots, t,\ j, l = 1, 2,\ v_{ij} \neq v_{kl}\\
&\ell(u_{ij}u_{i(j+1)}) = \begin{cases} b, & j = 1, 3, \cdots, n-3 \\ c, & j = 2, 4, \cdots, n-4 \end{cases}\\
&\ell(v_{i1}u_{i1}) = \ell(v_{i2}u_{i(n-2)}) = c\\
&\text{end for}
\end{aligned}
$$

iii) n is odd and t is even

$$
\begin{aligned}
&\ell(v_{i1}v_{i2}) = \begin{cases} b, & i = 1, 3, \cdots, t-1 \\ c, & i = 2, 4, \cdots, t \end{cases}\\
&\ell(v_{i2}v_{(i+1)1}) = \begin{cases} c, & i = 1, 3, \cdots, t-1 \\ b, & i = 2, 4, \cdots, t \end{cases}\\
&\text{For } i = 1, 2, \cdots, t, \text{ do:}\\
&\ell(v_{i1}u_{i1}) = b\\
&\ell(v_{i2}u_{i(n-2)}) = c\\
&\ell(u_{ij}u_{i(j+1)}) = \begin{cases} c, & j = 1, 3, \cdots, n-3 \\ b, & j = 2, 4, \cdots, n-4 \end{cases}\\
&\text{For } k = 1, 2, \cdots, t,\ j, l = 1, 2, \text{ do:}\\
&\ell(v_{ij}v_{kl}) = c,\ v_{ij} \neq v_{kl},\ k \neq i \text{ and } l \neq j+1,\ k \neq i+1 \text{ and } l \neq j-1\\
&\text{end for}
\end{aligned}
$$

With the labeling defined above $L(C_n(t))$ is a-sum V_4-magic. □

Theorem 8.3.27. *$L(C_n(t))$ is zero-sum V_4-magic for all n and t.*

Proof. Label all the edges by a. Since all the vertices have even degree, $\ell^+(v) = 0$ for all $v \in V(L(C_n(t)))$. $\square$

Theorem 8.3.28. *$C_n(t)$ belongs to the classes $\mathscr{A}$ and $\mathscr{B}$ if and only if n is even and t is odd.*

Proof. The proof follows from theorems 5.2.10, 8.3.26 and 8.3.27. $\square$

Theorem 8.3.29. *$C_n(t)$ belongs to the $\mathscr{C}$-class for all n and t.*

Proof. The proof follows from theorems 8.3.25 and 8.3.27. $\square$

Theorem 8.3.30. *$C_n(t)$ belongs to the $\mathscr{D}$-class if and only if nt is even.*

Proof. The proof follows from theorems 8.3.25 and 8.3.26. $\square$

$J_{n,m}$ denotes the Jahangir graph on $nm+1$ vertices. Let $V(J_{n,m}) = \{u\} \cup \{u_i : 1 \le i \le m\} \cup \{v_{ij} : 1 \le i \le m, 1 \le j \le n-1\}$ and edge set is given by $E(J_{n,m}) = \{uu_i : 1 \le i \le m\} \cup \{u_iv_{i1} : 1 \le i \le m\} \cup \{u_{i+1}v_{i(n-1)} : 1 \le i \le m\} \cup \{v_{ij}v_{i(j+1)} : 1 \le i \le m, 1 \le j \le n-2\}$. By the definition of the line graph, the vertex set of the line graph of $J_{n,m}$ is given by $V(L(J_{n,m})) = \{v_i : i = 1, 2, \cdots, m\} \cup \{u_{i1}, u_{i2} : i = 1, 2, \cdots, m\} \cup \{w_{ij} : i = 1, 2, \cdots, m, j = 1, 2, \cdots, n-2\}$ where v_i is the edge $uu_i, i = 1, 2, \cdots, m$. u_{i1} denotes the edge $u_iv_{(i-1)(m-1)}, u_{i2}$ denotes the edge u_iv_{i1}, w_{ij} denotes the edge $v_{ij}v_{i(j+1)}$. The edge set of the line graph $E(L(J_{n,m}))$ is given by $\{v_iv_j : 1 \le i, j \le m, i \ne j\} \cup \{v_iu_{ij} : 1 \le i \le m, j = 1, 2\} \cup \{u_{i1}u_{i2} : 1 \le i \le m\} \cup \{w_{ij}w_{i(j+1)} : 1 \le i \le m, 1 \le j \le n-3\} \cup \{u_{i2}w_{i1} : 1 \le i \le m\} \cup \{w_{i(n-2)}u_{(i+1)1} : 1 \le i \le m\}$.

Theorem 8.3.31. *$L(J_{n,m})$ is a-sum V_4-magic if and only if $m(n+1)$ is even.*

Proof. First assume that $L(J_{n,m})$ is a sum V_4 magic.

$$\begin{aligned} L(J_{n,m}) \text{ is } a \text{ sum } V_4\text{-magic} &\Rightarrow [m + 2m + m(n-2)]a = 0 \\ &\Rightarrow [m + mn]a = 0 \\ &\Rightarrow m(n+1) \text{ is even.} \end{aligned}$$

Conversely assume that $m(n+1)$ is even.

Case 1: m and n are even. Define a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} &\text{For } i = 1, 2, \cdots, m \text{ do:} \\ \ell(v_iv_j) &= a, j = 1, 2, \cdots, m, \ i \ne j \\ \ell(v_iu_{ij}) &= a, j = 1, 2 \\ \ell(u_{i1}u_{i2}) &= b \\ \ell(w_{ij}w_{i(j+1)}) &= \begin{cases} c, & j = 1, 3, \cdots, n-3 \\ b, & j = 2, 4, \cdots, n-4 \end{cases} \end{aligned}$$

$$\ell(u_{i2}w_{i1}) = b,\ \ell(w_{i(n-2)}u_{(i+1)1}) = b$$
end for

With this labeling defined above $\ell^+(v) = a$ for all $v \in V(L(J_{n,m}))$.

Case 2: m is even and n is odd. Define a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ by

For $i = 1, 2, \cdots, m$ do:

$$\ell(v_iv_j) = a, j = 1, 2, \cdots, m,\ i \neq j$$

$$\ell(v_iu_{ij}) = a, j = 1, 2$$

end for

$$\ell(u_{i1}u_{i2}) = \begin{cases} b, & i = 1, 3, \cdots, m-1 \\ c, & i = 2, 4, \cdots, m \end{cases}$$

$$\ell(u_{i2}w_{i1}) = \begin{cases} b, & i = 1, 3, \cdots, m-1 \\ c, & i = 2, 4, \cdots, m \end{cases}$$

$$\ell(w_{i(n-2)}u_{(i+1)1}) = \begin{cases} c, & i = 1, 3, \cdots, m-1 \\ b, & i = 2, 4, \cdots, m \end{cases}$$

For $i = 1, 3, \cdots, m-1$ do:

$$\ell(w_{ij}w_{i(j+1)}) = \begin{cases} c, & j = 1, 3, \cdots, n-2 \\ b, & i = 2, 4, \cdots, n-3 \end{cases}$$

end for

For $i = 2, 4, \cdots, m$ do:

$$\ell(w_{ij}w_{i(j+1)}) = \begin{cases} b, & j = 1, 3, \cdots, n-2 \\ c, & i = 2, 4, \cdots, n-3 \end{cases}$$

end for

With this labeling defined above $\ell^+(v) = a$ for all $v \in V(L(J_{n,m}))$.

Case 3: m and n are odd. Define a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ by

For $i = 1, 2, \cdots, m$ do:

$$\ell(v_iv_j) = a, j = 1, 2, \cdots, m,\ i \neq j$$

$$\ell(v_iu_{i1}) = b,\ \ell(v_iu_{i2}) = c$$

$$\ell(u_{i1}u_{i2}) = a$$

$$\ell(w_{ij}w_{i(j+1)}) = \begin{cases} b, & j = 1, 3, \cdots, n-4 \\ c, & j = 2, 4, \cdots, n-3 \end{cases}$$

$$\ell(u_{i2}w_{i1}) = c$$

$$\ell(w_{i(n-2)}u_{(i+1)1}) = b$$

end for

With this labeling defined above $\ell^{+}(v) = a$ for all $v \in V(L(J_{n,m}))$.

Thus ℓ is an a-sum V_4-magic labeling of $L(J(n,m))$. $\square$

Theorem 8.3.32. *$L(J(n,m))$ is zero-sum V_4-magic for all n and m.*

Proof. **Case 1:** Both m and n are odd.

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, m \text{ do:}\\
\ell(v_i v_j) &= a, j = 1, 2, \cdots, m,\ i \neq j\\
\ell(v_i u_{i1}) &= \ell(v_i u_{i2}) = a\\
\ell(u_{i1} u_{i2}) &= b,\ \ell(u_{i2} w_{i1}) = c\\
\ell(w_{i(n-2)} u_{(i+1)1}) &= c\\
\ell(w_{ij} w_{i(j+1)}) &= c,\ j = 1, 2, \cdots, n-2\\
&\text{end for}
\end{aligned}$$

Case 2: m and n are even.

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots, m \text{ do:}\\
\ell(v_i v_j) &= a, j = 1, 2, \cdots, m,\ i \neq j\\
&\text{end for}\\
\ell(v_i u_{i1}) = \ell(u_{i2} w_{i1}) &= \begin{cases} b, & i = 1, 3, \cdots, m-1 \\ c, & i = 2, 4, \cdots, m \end{cases}\\
\ell(v_i u_{i2}) &= \begin{cases} c, & i = 1, 3, \cdots, m-1 \\ b, & i = 2, 4, \cdots, m \end{cases}\\
\ell(u_{i1} u_{i2}) &= a\\
&\text{For } j = 1, 2, \cdots, n-3 \text{ do:}\\
\ell(w_{ij} w_{i(j+1)}) &= \begin{cases} b, & i = 1, 3, \cdots, m-1 \\ c, & i = 2, 4, \cdots, m \end{cases}\\
&= \ell(w_{i(n-2)} u_{(i+1)1})\\
&\text{end for}
\end{aligned}$$

Case 3: m is odd and n is even.

The proof is similar to case 1.

Case 4: m is even and n is odd.

The proof is similar to case 2.

One can easily verify that with the labeling defined above, ℓ is a zero-sum V_4-magic labeling of $L(J_{n,m})$. $\square$

Theorem 8.3.33. *$J_{n,m}$ belongs to the classes $\mathscr{A}$ and $\mathscr{B}$ if and only if n and m are odd.*

Proof. The proof follows from theorems 5.2.1, 8.3.31 and 8.3.32. □

Theorem 8.3.34. *$J_{n,m}$ belongs to the $\mathscr{C}$-class for all n and m.*

Proof. The proof follows from theorems 5.2.2 and 8.3.32. □

Theorem 8.3.35. *$J_{n,m}$ belongs to the $\mathscr{D}$-class if and only if $m(n+1)$ is even.*

Proof. The proof follows from theorems 5.2.2 and 8.3.31. □

8.4 Ladder Graphs

The line graph of the ladder L_n is a graph with vertex set $V(G) = \{u_i : i = 1, 2, \cdots, n-1\} \cup \{v_i : i = 1, 2, \cdots, n\} \cup \{u'_i : i = 1, 2, \cdots, n-1\}$ and edge set given by $E(G) = \{u_i v_i : i = 1, 2, \cdots, n-1\} \cup \{u'_i v_i : i = 1, 2, \cdots, n-1\} \cup \{u_i v_{i+1} : i = 1, 2, \cdots, n-1\} \cup \{u'_i v_{i+1} : i = 1, 2, \cdots, n-1\} \cup \{u_i u_{i+1} : i = 1, 2, \cdots, n-2\} \cup \{u'_i u'_{i+1} : i = 1, 2, \cdots, n-2\}$.

Theorem 8.4.1. *(see [28]) L_n is zero-sum V_4-magic for all n.*

Theorem 8.4.2. *$L(L_n)$ is a-sum V_4-magic if and only if n is even.*

Proof. Suppose that $L(L_n)$ is a-sum V_4 magic. Then $(3n-2)a = 0$. This implies that $3n$ is even which again implies that n is even. Conversely assume that n is even. Define a labeling $\ell : E(L(L_n)) \to V_4 \setminus \{0\}$ by

$$
\begin{aligned}
&\text{For } i = 1, 2, \cdots, n-2 \text{ do:} \\
\ell(u_i v_i) &= \ell(u_i u_{i+1}) = b \\
\ell(u'_i u'_{i+1}) &= \ell(v_i u'_i) = c \\
&\text{end for} \\
\ell(u_{n-1} v_n) &= b, \ \ell(u'_{n-1} v_n) = c \\
\ell(u_1 v_2) &= \ell(u'_1 v_2) = \ell(u_{n-1} v_{n-1}) = \ell(u'_{n-1} v_{n-1}) = a \\
\ell(u_i v_{i+1}) &= c, \ i = 2, 3, \cdots, n-2 \\
\ell(u'_i v_{i+1}) &= \begin{cases} b, & i = 2, 4, \cdots, n-2 \\ c, & i = 3, 5, \cdots, n-3 \end{cases}
\end{aligned}
$$

Thus ℓ is an a-sum V_4-magic labeling of $L(L_n)$. □

Theorem 8.4.3. *$L(L_n)$ is zero-sum V_4-magic for all n.*

Proof. Define a labeling $\ell : E(L(L_n)) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(u_1v_1) &= \ell(u'_1v_1) = a \\
\ell(u_{n-1}v_n) &= \ell(u'_{n-1}v_n) = a \\
&\text{For } i = 1, 2, \cdots, n-1, \text{ do:} \\
\ell(u_iu_{i+1}) &= \ell(u'_iu'_{i+1}) = c \\
&\text{end for} \\
&\text{For } i = 2, 3, \cdots, n-1, \text{ do:} \\
\ell(u_iv_i) &= \ell(u'_iv_i) = b \\
&\text{end for} \\
&\text{For } i = 1, 2, \cdots, n-2, \text{ do:} \\
\ell(u_iv_{i+1}) &= \ell(u'_iv_{i+1}) = b \\
&\text{end for}
\end{aligned}$$

Hence the labeling ℓ defined above makes the graph $L(L_n)$ zero-sum V_4 magic. □

Theorem 8.4.4. *L_n belongs to the $\mathscr{A}$-class if and only if n is even.*

Proof. The proof follows from theorems 5.3.1 and 8.4.2. □

Theorem 8.4.5. *L_n belongs to the classes $\mathscr{B}$ and $\mathscr{C}$ for all n.*

Proof. The proof follows from theorems 5.3.1, 8.4.1 and 8.4.3. □

Theorem 8.4.6. *L_n belongs to the $\mathscr{D}$-class if and only if n is even.*

Proof. The proof follows from theorems 8.4.1 and 8.4.2. □

The line graph of L_{n+2} is a graph with vertex set $V(L(L_{n+2})) = \{v_i/i = 1, 2, \cdots, n\} \cup \{u_i/i = 0, 1, \cdots, n\} \cup \{u'_i/i = 0, 1, \cdots, n\}$ and edge set $E(L(L_{n+2})) = \{u_iv_{i+1}/i = 0, 1, \cdots, n-1\} \cup \{u'_iv_{i+1}/i = 0, 1, \cdots, n-1\} \cup \{u_iv_i/i = 1, 2, \cdots, n\} \cup \{u'_iv_i/i = 1, 2, \cdots, n\} \cup \{u_iu_{i+1}/i = 0, 1, \cdots, n-1\} \cup \{u'_iu'_{i+1}/i = 0, 1, \cdots, n-1\}$.

Theorem 8.4.7. *(see [28]) L_{n+2} is a-sum V_4-magic for all n.*

Theorem 8.4.8. *$L(L_{n+2})$ is a-sum V_4-magic if and only if n is even.*

Proof. Suppose that $L(L_{n+2})$ is a-sum V_4 magic. This implies that $(3n+2)a = 0$ which implies that n is even. Conversely suppose that n is even. Define a labeling $\ell : E(L(L_{n+2})) \to V_4 \setminus \{0\}$ by

$$\ell(u_iu_{i+1}) = \begin{cases} c, & i = 0, 2, \cdots, n-2 \\ b, & i = 1, 3, \cdots, n-1 \end{cases}$$

$$\ell(u_i'u_{i+1}') = \begin{cases} c, & i = 0, 2, \cdots, n-2 \\ b, & i = 1, 3, \cdots, n-1 \end{cases}$$

$$\ell(u_i v_{i+1}) = b, i = 0, 1, \cdots, n-1$$

$$\ell(u_i' v_{i+1}) = \begin{cases} b, & i = 0, 2, \cdots, n-2 \\ c, & i = 1, 3, \cdots, n-1 \end{cases}$$

$$\ell(u_i v_i) = b, i = 1, 2, \cdots, n-1$$

$$\ell(u_n v_n) = c$$

$$\ell(u_i' v_i) = \begin{cases} c, & i = 1, 3, \cdots, n-1, n \\ b, & i = 2, 4, \cdots, n-2 \end{cases}$$

With this labeling $L(L_{n+2})$ is a-sum V_4 magic. □

Theorem 8.4.9. *$L(L_{n+2})$ is zero-sum V_4-magic for all n.*

Proof. Label all the edges by a. Since all the vertices have degree either 2 or 4, it is easy to verify that $\ell^+(v) = 0$ for all $v \in V(L(L_{n+2}))$. □

Theorem 8.4.10. *L_{n+2} belongs to the classes $\mathscr{A}$ and $\mathscr{K}$ if and only if n is even.*

Proof. The proof follows from theorems 8.4.7, 5.3.5 and 8.4.8. □

Theorem 8.4.11. *L_{n+2} belongs to the classes $\mathscr{B}$ and $\mathscr{L}$ for all n.*

Proof. The proof follows from theorems 8.4.7, 5.3.5 and 8.4.9. □

Theorem 8.4.12. *Path P_n is non V_4-bimagic for $n \geq 4$.*

Proof. The path graph P_n is neither a-sum V_4-magic nor zero-sum V_4-magic. The line graph of P_n is P_{n-1} which is again neither a-sum V_4-magic nor zero-sum V_4-magic. Hence the proof. □

$\mathscr{A}$	$\mathscr{C}$	$\mathscr{A} \cap \mathscr{C}$
$C_m@C_n$	$C_m@C_n$	$C_m@C_n$
$C_n(t)$	$C_n(t)$	$C_n(t)$
$J_{n,m}$	$J_{n,m}$	$J_{n,m}$
L_n	L_n	L_n
W_n	W_n	W_n
Sun_n		
$BSun(n,k)$		
$CBSun(n,k)$		
L_{n+2}		
$K_{1,n}$		

In this chapter, we have classified a few cycle related graphs into a-sum V_4-bimagic, zero-sum V_4-bimagic and graphs which are both a-sum and zero-sum V_4-bimagic. The classification is shown in the above table. The graphs which we have discussed mainly belong to the classes $\mathscr{A}, \mathscr{B}, \mathscr{C}$ and $\mathscr{D}$. A few graphs which belong to some of the other classes are discussed below. Moreover, from the above table we have $\mathscr{A} \cap \mathscr{C} = \mathscr{C}$. That is $\mathscr{C} \subset \mathscr{A}$. Then a question arises whether this is always true. The following theorems provide an answer to this question.

The labelings of diamond graph and its line graph are shown in Figure 8.1.

Theorem 8.4.13. *The diamond graph belongs to the classes $\mathscr{B}, \mathscr{C}, \mathscr{E}$ and $\mathscr{I}$.*

Proof. A zero-sum V_4 magic labeling of diamond graph is shown in figure 8.1. An a-sum V_4 magic labeling is shown in brackets in the same figure. Also a zero-sum V_4 magic labeling of line graph of diamond graph is depicted in figure 8.1. Moreover suppose it is a-sum V_4 magic. Then we have $5a = 0$ which is a contradiction. Then the result follows. $\square$

The labelings of kite graph and its line graph are shown in Figure 8.2.

Theorem 8.4.14. *The kite graph belongs to the classes $\mathscr{G}, \mathscr{H}, \mathscr{K}$ and $\mathscr{L}$.*

Proof. Since the kite graph has a pendant edge, it is not zero-sum V_4 magic and it is not a-sum V_4 magic as it has 5 vertices. A zero-sum V_4 magic labeling of its line graph is shown in figure 8.2 and its a-sum V_4 magic labeling is shown in brackets in the same figure. Obviously the result follows. $\square$

Figure 8.1: Diamond graph and its line graph

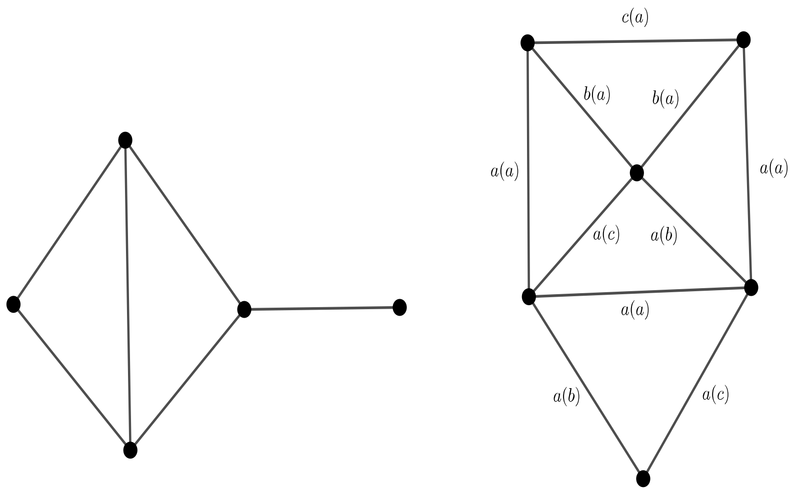


Figure 8.2: Kite graph and its line graph

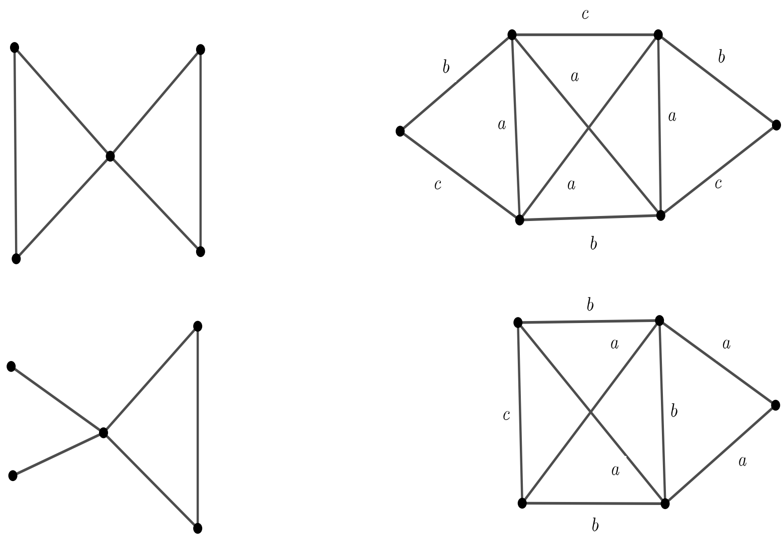


Figure 8.3: Butterfly graph and its line graph(above), Cricket graph and its line graph(below)

The labelings of butterfly graph and its line graph are given in Figure 8.3.

Theorem 8.4.15. *The butterfly graph belongs to the classes $\mathscr{C}$, $\mathscr{D}$, $\mathscr{G}$ and $\mathscr{H}$.*

Proof. Since the number of vertices is odd, the butterfly graph is not a-sum V_4 magic. By labeling all the edges by a, we get a zero-sum V_4 magic labeling of the graph and its line graph. An a-sum V_4 magic labeling of the line graph is given in figure 8.3. Then clearly the result follows. □

The labelings of cricket graph and its line graph are shown in Figure 8.3.

Theorem 8.4.16. *The cricket graph belongs to the classes $\mathscr{H}, \mathscr{L}, \mathscr{M}$ and $\mathscr{O}$.*

Proof. By similar arguments in the above theorems, the cricket graph is neither a-sum V_4 magic nor zero-sum V_4 magic. Its line graph is also not a-sum V_4 magic. A zero-sum V_4 magic labeling of the line graph is depicted in figure 8.3. Then the proof follows. □

The labelings of moth graph and its line graph are shown in Figure 8.4.

Theorem 8.4.17. *The moth graph belongs to the classes $\mathscr{B}, \mathscr{E}, \mathscr{L}$ and $\mathscr{O}$.*

Proof. An a-sum V_4 magic labeling of moth graph and a zero-sum V_4 magic labeling of its line graph are shown in figure 8.4. As it has pendant edges the graph is not zero-sum V_4 magic and its line graph is not a-sum V_4 magic as it has odd number of vertices. Then we can easily verify the result. □

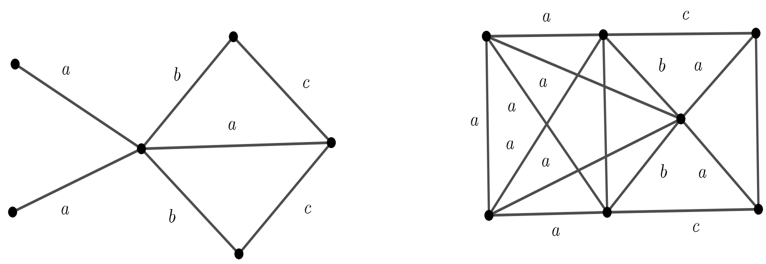


Figure 8.4: Moth graph and its line graph

From theorems 8.4.13 and 8.4.15 we can conclude that $\mathscr{C} \nsubseteq \mathscr{A}$ since diamond graph and butterfly graph are zero-sum V_4-bimagic but not a-sum V_4-bimagic.

Chapter 9

On $\mathbb{Z}_p$-Barycentric Ring Magic Graphs

In the first section of this chapter, we define $k-$barycentric sequence in a commutative ring R with unity and introduce the concept of R-barycentric ring magic graphs. In the second section of this chapter, we characterize 2-barycentric and 3-barycentric sequence in fields. In the last section of this chapter we characterize a class of $\mathbb{Z}_p$-barycentric ring magic graphs.

9.1 Introduction

Here we define $k-$barycentric sequence in a commutative ring R with unity.

Definition 9.1.1. *Let $a_1, a_2, \cdots, a_k$ be k not necessarily distinct nonzero elements of a commutative ring R with unity. This sequence is $k-$barycentric if there exist i such that $a_1 + a_2 + \cdots + a_i + \cdots + a_k = ka_i$ and $a_1a_2 \cdots a_i \cdots a_k = a_i^k$. The element a_i is called a barycenter.*

Definition 9.1.2. *Let R be a commutative ring with unity. A graph $G = (V, E)$ is said to be R-barycentric ring magic if there exists a labeling $\ell : E(G) \to R \setminus \{0\}$ of the edges of G by nonzero elements of R such that the induced vertex labelings $\ell^+ : V(G) \to R$ defined by $\ell^+(v) = \sum \ell(uv)$ where $(u, v) \in E$and $\ell^\times : V(G) \to R$ defined by $\ell^\times(v) = \prod \ell(uv)$ where $(u, v) \in E$ are constant maps and satisfies:*

i) $\ell^+(v) = deg(v)\ell(u_vv)$, *for all $v \in V(G)$, and for some vertex u_v adjacent to v.*

ii) $\ell^\times(v) = \ell(u_vv)^{deg(v)}$, *for all $v \in V(G)$, and for some vertex u_v adjacent to v.*

9.2 Basic Results

Here we need the following:

Lemma 9.2.1. *In every commutative ring with unity R, a sequence $a_1, a_2, \cdots, a_k$ where $a_1 = a_2 = \cdots = a_k = a \in R$ is $k-$barycentric.*

Proof. We have

$$a_1 + a_2 + \cdots + a_i + \cdots + a_k = \underbrace{a + a + \cdots + a + \cdots + a}_{k \text{ terms}} = ka.$$

Moreover,

$$a_1 a_2 \cdots a_i \cdots a_k = \underbrace{aa \cdots a \cdots a}_{k \text{ terms}} = a^k.$$

This proves the lemma. □

Lemma 9.2.2. *Let R be a field. Any sequence in R with 2 elements is barycentric if and only if the elements are equal.*

Proof. Let a_1, a_2 be any two elements in the field R which are barycentric. Without loss of generality assume that $a_1 + a_2 = 2a_1$ which implies $a_1 = a_2$. Again, $a_1 a_2 = {a_1}^2$ implies ${a_1}^{-1} a_1 a_2 = {a_1}^{-1} {a_1}^2$. That is, $a_1 = a_2$. Conversely assume that $a_1 = a_2$. Then clearly,

$$\begin{aligned} a_1 + a_2 = a + a = 2a \text{ and} \\ a_1 a_2 = aa = a^2. \end{aligned}$$

This proves the lemma. □

Lemma 9.2.3. *Let R be a field. A 3-sequence in R is barycentric if and only if all the elements are equal.*

Proof. Suppose that a_1, a_2, a_3 be a 3-sequence in R in which all the elements are equal. That is $a_i = a$ for $i = 1, 2, 3$. Then it is clear that the sequence is barycentric. Conversely assume that the 3-sequence in R is barycentric. Then consider the following three cases:

Case 1 : Suppose $a_1 + a_2 + a_3 = 3a_1$ and $a_1 a_2 a_3 = {a_1}^3$.
This implies

$$a_2 + a_3 = 2a_1 \text{ and} \tag{9.1}$$

$$a_2 a_3 = {a_1}^2. \tag{9.2}$$

From equation 8.1 we have $a_3 = 2a_1 - a_2$. Substituting this in equation 8.2 we get,

$$\begin{aligned} a_2(2a_1 - a_2) = {a_1}^2 &\Rightarrow 2a_1a_2 - {a_2}^2 = {a_1}^2 \\ &\Rightarrow {a_1}^2 - 2a_1a_2 + {a_2}^2 = 0 \\ &\Rightarrow (a_1 - a_2)^2 = 0 \\ &\Rightarrow a_1 = a_2. \end{aligned}$$

Substituting this in equation 8.1 we get, $a_1 = a_2 = a_3$.

Case 2: $a_1 + a_2 + a_3 = 3a_2$ and $a_1a_2a_3 = {a_2}^3$.
The proof is similar to case 1.

Case 3: $a_1 + a_2 + a_3 = 3a_3$ and $a_1a_2a_3 = {a_3}^3$.
The proof is similar to case 1.

This completes the proof. □

Since the last two lemmas work for only fields, in the foregoing sections we focus mainly on $\mathbb{Z}_p$-barycentric ring magic graphs where p is prime.

9.3 Main Results

Theorem 9.3.1. *Any graph G is $\mathbb{Z}_2$-barycentric ring magic with same additive and multiplicative constant 1 if and only if all the vertices of G have odd degrees.*

Proof. First suppose that all the vertices of G have odd degrees.

$$deg(v_i) = k_i = 2s + 1 \text{ for all } v_i \in V \text{ and for some } s \in \mathbb{N}.$$

Then we have,

$$\ell^+(v_i) = \sum \ell(uv) = 1, \ \ell^\times(v_i) = \prod \ell(uv) = 1.$$

Moreover,

$$deg(v)\ell(u_v v) = 1 \text{ and } \ell(u_v v)^{deg(v)} = 1.$$

Therefore G is $\mathbb{Z}_2$ barycentric ring magic with the same additive and multiplicative constant 1. Conversely assume that G is $\mathbb{Z}_2$-barycentric ring magic with the same additive and multiplicative constant 1. Obviously $deg(v) = k$ for all $v \in V$ where k is odd. This proves the theorem. □

Corollary 9.3.2. *Any graph G is $\mathbb{Z}_2$-barycentric ring magic with same additive and multiplicative constant 1 if and only if G is k-regular with k odd.*

Proof. First suppose that G is $\mathbb{Z}_2$-barycentric ring magic with same additive and multiplicative constant 1. Then by theorem 9.3.1, all the vertices of G have odd degrees. This is true if G is k-regular with k odd. Conversely assume that G is k-regular with k odd. Then clearly all the vertices of G have odd degrees. Hence by theorem 9.3.1, the result follows. $\square$

Theorem 9.3.3. *A regular graph G is R-barycentric ring magic for any ring R.*

Proof. Label all the edges by $a \in R \setminus \{0\}$. Then ℓ^+ and $\ell^\times$ are constant maps and also $\ell^+(v) = deg(v)\ell(u_v v)$ and $\ell^\times(v) = \ell(u_v v)^{deg(v)}$. $\square$

Theorem 9.3.4. *For every commutative ring R with unity, P_2 is R-barycentric ring magic and $P_n, n \geq 3$ is not R-barycentric ring magic.*

Proof. Label the edge by a. Then P_2 is R-barycentric ring magic for every commutative ring R with unity. $P_n, n \geq 3$ is in fact not even barycentric magic. Hence it cannot be R-barycentric ring magic. $\square$

Theorem 9.3.5. *Every $\mathbb{Z}_2$-ring magic graph is $\mathbb{Z}_2$-barycentric ring magic.*

Proof. Suppose that G is $\mathbb{Z}_2$-ring magic. Then the only possible edge label is 1. Thus we have $\ell^+(v) = \sum \ell(uv) = \sum 1 = deg(v)\ell(uv)$ and $\ell^\times(v) = \prod \ell(uv) = \prod 1 = \ell(uv)^{deg(v)}$. The proof follows. $\square$

Theorem 9.3.6. *For $n \geq 3$, if $K_{1,n}$ is $\mathbb{Z}_h$-ring magic then $K_{1,n}$ is $\mathbb{Z}_h$-barycentric ring magic.*

Proof. Let $v_1, \cdots, v_n$ be the pendant vertices and v_0 be the central vertex of $K_{1,n}$. First suppose that $K_{1,n}$ is $\mathbb{Z}_h$ ring magic. We can easily see that all the pendant edges must have the same label. That is, $\ell(v_0 v_i) = a, i = 1, 2, \cdots, n$. Also $deg(v_0) = n$ and $deg(v_i) = 1$ for all i. Thus we have,

$$\ell^+(v_i) = a,\ \ell^+(v_0) = na,\ \ell^\times(v_i) = a,\ \ell^\times(v_0) = a^n$$

Furthermore,

$$\begin{gathered} deg(v_i)\ell(u_{v_i}v_i) = 1.a = a,\ deg(v_0)\ell(u_{v_0}v_0) = n.a \\ \ell(u_{v_i}v_i)^{deg(v_i)} = a = a^1,\ \ell(u_{v_0}v_0)^{deg(v_0)} = a^n \end{gathered}$$

Hence the proof. $\square$

Theorem 9.3.7. *The cycle C_n is R-barycentric ring magic for every commutative ring R with unity.*

Proof. Label all the edges by the same element. The result follows. $\square$

Theorem 9.3.8. *(see [34]) $C(n_1, n_2)$ is $\mathbb{Z}_h$-barycentric magic if and only if h is even.*

Theorem 9.3.9. *(see [34]) $C(n_1, n_2, \cdots, n_k)$ is $\mathbb{Z}_h$-barycentric magic if and only if h is even.*

Theorem 9.3.10. *$C(n_1, n_2)$ is $\mathbb{Z}_p$-barycentric ring magic if and only if $p = 2$.*

Proof. By lemma 9.2.2 and theorem 9.3.8, the result follows. □

The generalized form of the theorem 9.3.10 is as follows:

Theorem 9.3.11. *$C(n_1, n_2, \cdots, n_k)$ is $\mathbb{Z}_p$-barycentric ring magic if and only if $p = 2$.*

Proof. By lemma 9.2.2 and theorem 9.3.9, the result follows. □

Theorem 9.3.12. *$K_{2,3}$ is not $\mathbb{Z}_p$-barycentric ring magic for any prime p.*

Proof. Let $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$ be the two partite sets of $K_{2,3}$. By lemmas 9.2.2 and 9.2.3, the edges incident to u_i must have the same labels and those incident to v_j must also have the same labels. Let us label all the edges incident to u_1 by a. Then all the edges incident to u_2 must also be labeled a since all the edges incident to v_j must have the same label. Therefore we have,

$$\ell^+(u_i) = 3a, \ \ell^+(v_j) = 2a,$$
$$\ell^\times(u_i) = a^3, \ \ell^\times(v_j) = a^2.$$

If the graph is $\mathbb{Z}_p$-barycentric ring magic, then it follows that $a = 0$ which is a contradiction. Thus the graph $K_{2,3}$ cannot be $\mathbb{Z}_p$-barycentric ring magic for any prime p. □

Remark 9.3.13. *If a graph is not $\mathbb{Z}_h$-barycentric magic, then it cannot be $\mathbb{Z}_h$-barycentric ring magic.*

From theorem 3.19 in [26] and remark 9.3.13, we obtain the following result:

Theorem 9.3.14. *$K_{2,3}$ is not $\mathbb{Z}_h$-barycentric ring magic for any h.*

Theorem 9.3.15. *The sun graph $C_n \odot K_1$ is $\mathbb{Z}_p$-barycentric ring magic if and only if $p = 2$.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of the cycle C_n and $v_1, v_2, \cdots, v_n$ be the pendant vertices. Then $deg(u_i) = 3$ and $deg(v_i) = 1$. By lemma 9.2.3, all the edges incident to u_1 is labeled the same element a. Moreover, we get that the edge labels of all those incident to $u_i, i = 2, 3, \cdots, n$ are a itself. Hence we get, $\ell^+(u_i) = 3a$ and $\ell^+(v_i) = a$. If the graph is $\mathbb{Z}_p$-barycentric ring magic we get the congruence, $3a \equiv a(mod\ p)$ which implies $2a \equiv 0(mod\ p)$ which inturn shows that p is even. That is p is exactly equal to 2. From our assumption we also have the congruence,

$$\begin{aligned} \ell^\times(u_i) \equiv \ell^\times(v_i)(mod\ p) &\Rightarrow a^3 \equiv a(mod\ p) \\ &\Rightarrow a(a^2 - 1) \equiv 0(mod\ p). \end{aligned}$$

Since $a \equiv 0(mod\ p)$ is not possible, we have

$$(a^2 - 1) \equiv 0(mod\ p) \Rightarrow a^2 \equiv 1(mod\ p)$$
$$\Rightarrow a = 1, p - 1.$$

Since $p = 2$, $a = 1$. To prove the converse, simply label all the edges by 1. The proof follows. □

Theorem 9.3.16. *Let p be any prime number. Then the wheel graph W_n is $\mathbb{Z}_p$-barycentric ring magic if and only if there exists an $a \in \mathbb{Z}_p \setminus \{0\}$ such that $n \equiv 3(mod\ p)$ and $a^{n-3} \equiv 1(mod\ p)$.*

Proof. Let $u_1, u_2, \cdots, u_n, v$ be the vertices of W_n where v is the central vertex. We have $deg(u_i) = 3, i = 1, 2, \cdots, n$ and $deg(v) = n$. First suppose that W_n is $\mathbb{Z}_p$-barycentric ring magic. Then by lemma 9.2.3, all the edges incident to $u_i, i = 1, 2, \cdots, n$ are labeled by the same element a. Thus we get $\ell^+(u_i) = 3a$, $\ell^+(v) = na$, $\ell^\times(u_i) = a^3$, $\ell^\times(v) = a^n$. By assumption, we obtain the congruence as follows:

$$na \equiv 3a(mod\ p) \Rightarrow (n - 3)a \equiv 0(mod\ p)$$
$$\Rightarrow n \equiv 3(mod\ p).$$

Furthermore,

$$a^n \equiv a^3(mod\ p) \Rightarrow a^3(a^{n-3} - 1) \equiv 0(mod\ p)$$

If $a^3 \equiv 0(mod\ p)$ then $a = 0$ which is a contradiction. Hence $(a^{n-3} - 1) \equiv 0(mod\ p)$ which gives the congruence $a^{n-3} \equiv 1(mod\ p)$, where $a \in \mathbb{Z}_p \setminus \{0\}$.

Conversely suppose that there exists an $a \in \mathbb{Z}_p \setminus \{0\}$ such that $n \equiv 3(mod\ p)$ and $a^{n-3} \equiv 1(mod\ p)$. Then we have,

$$n \equiv 3(mod\ p) \Rightarrow n.1 \equiv 3.1(mod\ p)$$

Obviously we can choose a to be 1 for which the congruence is satisfied. This gives a $\mathbb{Z}_p$-barycentric ring magic labeling of W_n. □

Corollary 9.3.17. *W_n is $\mathbb{Z}_2$ barycentric ring magic if and only if n is odd.*

Proof. Suppose that W_n is $\mathbb{Z}_2$ barycentric ring magic. Then by theorem 9.3.16 we have $n-3 \equiv 0(mod\ 2)$. That is $n-3$ is even which implies n is odd. Conversely suppose that n is odd. The only possible edge label is 1. Then clearly $n \equiv 3(mod\ 2)$ and $a^{n-3} - 1 \equiv 0(mod\ 2)$. □

The following example illustrates theorem 9.3.16.

Example 9.3.18. *Consider the wheel graph W_{10} and the ring $\mathbb{Z}_7$. Here $10 \equiv 3(mod\ 7)$. Let $a = 1$. Then $a^7 = 1^7 \equiv 1(mod\ 7)$. Then W_{10} is $\mathbb{Z}_7$ barycentric ring magic.*

Example 9.3.19. *Consider the wheel graph W_5 and the ring $\mathbb{Z}_3$. Let u_i denote the outer vertices and v denotes the central vertex of W_n. Here $n - 3 = 2$ is not congruent to 0 modulo 3 and $a^2 - 1 = 1^2 - 1 \equiv 0(mod\ 3)$. If we label all the edges by 1, then*

$$\ell^+(u_i) = 3(mod\ 3) = 0$$
$$\ell^+(v) = 5(mod\ 3) = 2$$

Hence W_5 is not $\mathbb{Z}_3$-barycentric ring magic.

Remark 9.3.20. *Example 9.3.19 shows that the condition $n \equiv 3(mod\ p)$ in theorem 9.3.16 is necessary.*

Theorem 9.3.21. *The splitting graph $S(P_3)$ of the path graph P_3 is $\mathbb{Z}_p$-barycentric ring magic if and only if $p = 2$.*

Proof. Let $\{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\}$ be the vertex set of $S(P_3)$. By lemma 9.2.2, all the edges incident to u_1 are labeled by the same element a_1. Similarly all the edges incident to each of the vertices u_3, v_1 and v_3 are labeled by the elements a_2, a_3 and a_4 respectively. If the graph is $\mathbb{Z}_p$-barycentric ring magic, then we get the congruence,

$$2a_1 \equiv 2a_2 \equiv 2a_3 \equiv 2a_4 \equiv a_1 + a_2 + a_3 + a_4(mod\ p). \tag{9.3}$$

We consider the following cases:

i) p is odd.
From congruence 9.3 we get, $a_1 = a_2 = a_3 = a_4 = a$(say). Thus we have $4a \equiv 2a(mod\ p)$ which implies $2a \equiv 0(mod\ p)$ which is a contradiction since p is odd.

ii) $p = 2$.
The only possible edge labeling is 1. This will give $\ell^+(u_i) = \ell^+(v_i) = 0$ and for all vertex u in $S(P_3)$, $deg(u)\ell(u_v v) = \ell^+(u)$. Also $\ell^\times(u_i) = \ell^\times(v_i) = 1$ and $\ell(u_v v)^{deg(u)} = 1$. Thus $S(P_3)$ is $\mathbb{Z}_2$ barycentric ring magic.

The converse is trivial. Hence the proof. □

Theorem 9.3.22. *The Mycielski graph $\mu(P_3)$ is not $\mathbb{Z}_p$-barycentric ring magic for any p.*

Proof. We have $V(\mu(P_3)) = V(S(P_3)) \cup \{u\}$ where u is the newly added vertex. Suppose that $\mu(P_3)$ is $\mathbb{Z}_p$-barycentric ring magic for all p. Since $deg(u) = 3$, by lemma 9.2.3, all the edges incident to u are labeled by the same element $a_1 \in \mathbb{Z}_p \setminus \{0\}$. Then from our assumption it is obvious that the edges incident to both v_1 and v_3 must have the same label. Let the edges incident to u_1 be labeled a_2 and those incident to u_3 be labelled a_3. Then we have,

$$3a_1 \equiv 2a_2 \equiv 2a_3 \equiv 3a_1 + a_2 + a_3 \equiv 2a_1 + a_2 + a_3(mod\ p).$$

Considering the congruence $3a_1 + a_2 + a_3 \equiv 2a_1 + a_2 + a_3(mod\ p)$, it follows that $a_1 \equiv 0(mod\ p)$ which is not possible. □

Theorem 9.3.23. *The planar grid $P_m \times P_n$ is not $\mathbb{Z}_p$-barycentric ring magic for all m, n except $m = n = 2$ and for any p.*

Proof. By lemma 9.2.2, two edges incident to the corner vertices of the planar grid can only be labeled by the same element a. Moreover, by lemma 9.2.3, the edges incident to the adjacent vertices of degree 3 must also be labeled the same element a. Then if $P_m \times P_n$ is $\mathbb{Z}_p$-barycentric ring magic, the following congruence holds.

$$3a \equiv 2a(mod\ p)$$

which is not possible. Furthermore, for $m = n = 2$ the graph $P_2 \times P_2$ is the cycle C_4 itself which is R-barycentric ring magic for all commutative ring R with unity by theorem 9.3.7. □

Consider the Dutch windmill graph $D_3^{(m)}$ or the friendship graph F_m.

Theorem 9.3.24. *The friendship graph F_m is $\mathbb{Z}_2$-barycentric ring magic for all m and $\mathbb{Z}_p$-barycentric ring magic for p odd if and only if there exists an $a \in \mathbb{Z}_p \setminus \{0\}$ such that $m \equiv 1(mod\ p)$ and $a^{2m-2} \equiv 1(mod\ p)$.*

Proof. Let $u_{i,j}, i = 1, 2, \cdots, m, j = 1, 2$ be the vertices of the graph and u be the central vertex. Let the edge $u_{11}u_{12}$ be labeled a_1. Then by lemma 9.2.2, the edges uu_{11} and uu_{12} must also be labeled by the same element a_1. By a similar argument we label the edges $u_{i1}u_{i2}, uu_{i1}, uu_{i2}, i = 2, 3, \cdots, m$ by $a_2, a_3, \cdots, a_m$. Suppose that F_m is $\mathbb{Z}_p$-barycentric ring magic. Then we get the congruences

$$2a_1 \equiv 2a_2 \equiv \cdots \equiv 2a_m \equiv 2(a_1 + a_2 + \cdots + a_m)(mod\ p) \text{ and} \tag{9.4}$$

$$a_1^2 \equiv a_2^2 \equiv \cdots \equiv a_m^2 \equiv a_1^2 a_2^2 a_3^2 \cdots a_m^2(mod\ p) \tag{9.5}$$

We consider the following cases:

Case 1: p is odd.

$2a_1 \equiv 2a_2 \equiv \cdots \equiv 2a_m(mod\ p)$ implies $a_1 = a_2 = \cdots = a_m = a$. Also, $2(\sum_{i=1}^{m} a_i) = 2ma$. Hence we get

$$\begin{aligned} 2ma \equiv 2a(mod\ p) &\Rightarrow (2m-2)a \equiv 0(mod\ p) \\ &\Rightarrow m \equiv 1(mod\ p). \end{aligned}$$

Now consider the congruence 9.5. From the congruence $a_1^2 \equiv a_2^2 \equiv \cdots \equiv a_m^2(mod\ p)$, there arises two cases:

Subcase 1: $a_1 = a_2 = \cdots = a_m = a$

Substituting in (9.5) we get, $a_1^2 a_2^2 a_3^2 \cdots a_m^2 = a^{2m}$. Thus we have

$$\begin{aligned} a^{2m} \equiv a^2(mod\ p) &\Rightarrow a(a^{2m-2} - 1) \equiv 0(mod\ p) \\ &\Rightarrow a^{2m-2} \equiv 1(mod\ p) \end{aligned}$$

Subcase 2: Let $1 \leq k \leq m$ and choose a_i such that

$$a_i = \begin{cases} a, & 1 \leq i \leq k \\ p - a = b, & k+1 \leq i \leq m \end{cases}$$

Therefore, $a_1^2 a_2^2 a_3^2 \cdots a_m^2 \equiv a^{2k} b^{2(m-k)}(mod\ p)$. From the congruence 9.4, we have

$$a^{2k} b^{2(m-k)} \equiv a^2 \equiv b^2 (mod\ p). \tag{9.6}$$

Since $a^2 \equiv b^2(mod\ p)$, we have

$$\begin{aligned} a^{2k} b^{2(m-k)} \equiv (a^2)^k (a^2)^{(m-k)}(mod\ p) &\equiv (a^2)^m (mod\ p) \\ &\equiv a^{2m}(mod\ p) \end{aligned}$$

Substituting in congruence 9.5, we get,

$$\begin{aligned} a^{2m} \equiv a^2(mod\ p) &\Rightarrow a^2(a^{2m-2} - 1) \equiv 0(mod\ p) \\ &\Rightarrow a^{2m-2} \equiv 1(mod\ p). \end{aligned}$$

Case 2: $p = 2$.

$2a_1 \equiv 2a_2 \equiv \cdots \equiv 2a_m(mod\ 2)$ implies that all $a_j's$ can be even distinct. There is no restriction on $a_j's$. If all are equal, we are done. Since we are considering only $\mathbb{Z}_2$-barycentric ring magic graphs, all the edges must have the same label 1. Then we get the following congruences.

$$2m \equiv 2(mod\ 2) \text{ and } 1^m \equiv 1(mod\ 2).$$

which holds for all m.

Conversely assume that $m \equiv 1(mod\ p)$ and $a^{2m-2} \equiv 1(mod\ p)$. Then

$$\begin{aligned} m \equiv 1(mod\ p) &\Rightarrow 2m \equiv 2(mod\ p) \text{ for all } m \\ &\Rightarrow p = 2. \end{aligned}$$

Choose $a = 1$. This choice satisfies the other assumption $1^{2m-2} \equiv 1(mod\ p)$. This gives a $\mathbb{Z}_p$-barycentric ring magic labeling of F_m. □

We prove that the ladder graph L_n and semiladder are not $\mathbb{Z}_p$-barycentric ring magic for any n and p.

Theorem 9.3.25. *The ladder graph L_n is not $\mathbb{Z}_p$-barycentric ring magic for any n and any p.*

Proof. Suppose L_n is $\mathbb{Z}_p$-barycentric ring magic for some n and p. Then by lemmas 9.2.2 and 9.2.3, all the edges must be labeled the same element say a. Then we get the congruence $3a \equiv 2a(mod\ p)$ which is not possible. ☐

Theorem 9.3.26. *L_{n+2} is $\mathbb{Z}_p$-barycentric ring magic if and only if $p = 2$.*

Proof. First we label all the pendant edges by the same element a. Then by lemma 9.2.3, all the other edges of the graph admits the same labeling a. Assuming that L_{n+2} is $\mathbb{Z}_p$-barycentric ring magic, we end up in the congruence, $3a \equiv a(mod\ p)$ which implies that $2a \equiv 0(mod\ p) \Rightarrow p$ is even. By labeling all the edges by 1 we can prove the converse. ☐

Theorem 9.3.27. *Semiladders are not $\mathbb{Z}_p$-barycentric ring magic for any n and any p.*

Proof. Suppose that semiladders are $\mathbb{Z}_p$-barycentric ring magic for some n and p. Then by lemmas 9.2.2 and 9.2.3, the corner vertices must have the same label, say a. Then we have the congruence, $3a \equiv 2a(mod\ p)$ which is not possible. ☐

Theorem 9.3.28. *The graph $C_n(t)$ is $\mathbb{Z}_2$-barycentric ring magic for all n, t and $\mathbb{Z}_p$-barycentric ring magic for p odd if and only if there exists an $a \in \mathbb{Z}_p \setminus \{0\}$ such that $t \equiv 1(mod\ p)$ and $a^{2t-2} \equiv 1(mod\ p)$.*

Proof. The proof is similar to theorem 9.3.24. ☐

Remark 9.3.29. *Theorem 9.3.24 is a special case of theorem 9.3.28.*

Theorem 9.3.30. *Let p be an odd prime. Then the graph $B(n, k)$ is $\mathbb{Z}_p$-barycentric ring magic if and only if the following holds:*

i) $(k-2)a + b \equiv 0(mod\ p)$

ii) $a^{(k-2)}b \equiv 1(mod\ p)$

Proof. Let uv be the common edge and $u_1^j, u_2^j, \cdots, u_{n-2}^j$ be the remaining vertices of the j^{th} cycle C_n. If we label the edge uu_1^1 by a_1, then by lemma 9.2.2, we must label the edges $u_i^1 u_{i+1}^1, i = 1, 2, \cdots, n-2$ by a_1. Similarly we have the edge labels $a_2, a_3, \cdots, a_k$ for the remaining cycles. Let the edge uv be labeled b. If it is $\mathbb{Z}_p$ barycentric ring magic, then we have the congruences

$$2a_1 \equiv 2a_2 \equiv \cdots \equiv 2a_k \equiv a_1 + a_2 + \cdots + a_k + b(mod\ p) \tag{9.7}$$

$$a_1^1 \equiv a_2^2 \equiv \cdots \equiv a_k^2 \equiv a_1 a_2 \cdots a_k b(mod\ p) \tag{9.8}$$

Since we are considering only the case where p is odd, considering $2a_1 \equiv 2a_2 \equiv \cdots \equiv 2a_k (mod\, p)$ in the congruence 9.7 implies that $a_1 = a_2 = \cdots = a_k = a$. Then it follows that $a_1 + a_2 + \cdots + a_k + b \equiv ka + b(mod\, p)$. Combining all this, from congruence 9.7 we get,

$$ka + b \equiv 2a(mod\ p) \Rightarrow (k-2)a + b \equiv 0(mod\ p).$$

Now consider the congruence 9.8. There arises two cases.

Subcase 1: $a_1 = a_2 = \cdots = a_k = a$.
From congruence 9.8, we obtain,

$$\begin{aligned} a^k b \equiv a^2(mod\ p) &\Rightarrow a^2(a^{k-2}b - 1) \equiv 0(mod\ p) \\ &\Rightarrow a^{k-2}b - 1 \equiv 0(mod\ p) \\ &\Rightarrow a^{k-2}b \equiv 1(mod\ p). \end{aligned}$$

Subcase 2:

$$a_i = \begin{cases} a, & 1 \leq i \leq m \\ p - a, & m+1 \leq i \leq k. \end{cases}$$

Then the congruence become,

$$a^m(p-a)^{k-m}b \equiv a^2 \equiv (p-a)^2(mod\ p).$$

We have $(p-a)^{k-m} \equiv a^{k-m}(mod\ p)$. Then

$$\begin{aligned} a^m a^{k-m} b \equiv a^2(mod\ p) &\Rightarrow a^k b \equiv a^2(mod\ p) \\ &\Rightarrow a^2(a^{k-2}b - 1) \equiv 0(mod\ p) \\ &\Rightarrow a^{k-2}b \equiv 1(mod\ p). \end{aligned}$$

Assume the converse. We choose $a = b = 1$. By assumption we have $(k-2)a + b \equiv 0(mod\ p)$.

$$\begin{aligned} (k-2)a + b \equiv 0(mod\ p) &\Rightarrow ka + b \equiv 2a(mod\ p) \\ &\Rightarrow k + 1 \equiv 2(mod\ p). \end{aligned}$$

Moreover, $a^{k-2}b - 1 = 0 \equiv 0(mod\, p)$. This implies $a^k b \equiv a^2(mod\, p)$. This gives a $\mathbb{Z}_p$-barycentric ring magic labeling of $B(n,k)$ where p is an odd prime. □

The following theorem characterizes $\mathbb{Z}_2$-barycentric ring magic n-gon book of k pages $B(n,k)$.

Theorem 9.3.31. *$B(n,k)$ is $\mathbb{Z}_2$-barycentric ring magic if and only if k is odd.*

Proof. Suppose $B(n,k)$ is $\mathbb{Z}_2$-barycentric ring magic. The only possible edge labeling is 1.

Then we have

$$\begin{aligned} k.1+1 \equiv 2(mod\ 2) &\Rightarrow k+1 \equiv 2(mod\ 2) \\ &\Rightarrow k-1 \equiv 0(mod\ 2) \\ &\Rightarrow k \text{ is odd.} \end{aligned}$$

Also the congruence, $1^{k+1} \equiv 1^2(mod\ 2)$ holds for all k. Conversely suppose that k is odd. Then we have $k \equiv 1(mod\ 2)$. That is, $k-1 \equiv 0(mod\ 2)$ which implies $k+1 \equiv 2(mod\ 2)$. Again the congruence $1^{k+1} \equiv 1^2(mod\ 2)$ is vacuosly true. Hence the graph is $\mathbb{Z}_2$-barycentric ring magic for all n. □

Similar characterization theorems hold for book graph $B_n = S_n \times P_2$. In fact, B_n is a special case of $B(n,k)$. We state them as follows:

Theorem 9.3.32. *Let p be an odd prime. B_n is $\mathbb{Z}_p$-barycentric ring magic if and only if the following holds:*

i) $(n-2)a+b \equiv 0(mod\ p)$

ii) $a^{(n-2)}b \equiv 1(mod\ p)$

Proof. The proof is similar to that of theorem 9.3.30. □

Theorem 9.3.33. *B_n is $\mathbb{Z}_2$-barycentric ring magic if and only if n is odd.*

Proof. The proof is similar to that of theorem 9.3.31. □

Theorem 9.3.34. *$C_m@C_n$ is $\mathbb{Z}_2$-barycentric ring magic for all m and n and if it is $\mathbb{Z}_p$-barycentric ring magic for an odd prime p, then $b_i \equiv b_j(mod\ p), i \neq j$ where both i and j are of same parity and b_i's are the edge labels of the cycle C_n.*

Proof. Label the edges of the cycle C_n by $b_1, b_2, \cdots, b_n$. Consider the cycle C_m on the edge labeled b_1. Label one of its edge by a_1. Then by lemma 9.2.2, all the remaining edges of the cycle must have the same label a_1. In a similar manner, we can label all the edges of the remaining cycles by $a_2, a_3, \cdots, a_n$ respectively. First assume that $C_m@C_n$ is $\mathbb{Z}_p$-barycentric ring magic. Then we have the following congruences.

$$\begin{aligned} 2a_1 \equiv 2a_2 \equiv \cdots \equiv 2a_n \equiv a_1+a_2+b_1+b_2 \equiv a_2+a_3+b_2+b_3 \equiv \cdots \equiv a_{n-1}+a_n \\ +b_{n-1}+b_n \equiv a_n+a_1+b_n+b_1(mod\ p) \end{aligned} \tag{9.9}$$

$$a_1^2 \equiv a_2^2 \equiv \cdots \equiv a_n^2 \equiv a_1a_2b_1b_2 \equiv a_2a_3b_2b_3 \equiv \cdots \equiv a_{n-1}a_nb_{n-1}b_n \equiv a_1a_nb_1b_n(mod\ p) \tag{9.10}$$

Case 1: p=2.

The only possible edge label is 1. Then the congruences become $4 \equiv 2(mod\ 2)$ and $1 \equiv 1(mod\ 2)$ which is true. Hence the graph is $\mathbb{Z}_2$-barycentric ring magic.

Case 2: p is odd.

From the congruence 9.9, $2a_1 \equiv 2a_2 \equiv \cdots \equiv 2a_n (mod\ p)$ we get, $a_1 = a_2 = \cdots = a_n = a$. Substituting in 9.9 we get,

$b_1 + b_2 + 2a \equiv b_2 + b_3 + 2a \equiv \cdots \equiv b_{n-1} + b_n + 2a \equiv b_n + b_1 + 2a \equiv 2a(mod\ p)$ Then it follows that

$$\begin{cases} b_1 \equiv b_3 \equiv \cdots b_{n-1}(mod\ p) \\ b_2 \equiv b_4 \equiv \cdots b_n(mod\ p) \end{cases} \quad \text{if } n \text{ is even}$$

$$\begin{cases} b_1 \equiv b_3 \equiv \cdots b_n(mod\ p) \\ b_2 \equiv b_4 \equiv \cdots b_{n-1}(mod\ p) \end{cases} \quad \text{if } n \text{ is odd}$$

$b_i \equiv b_j (mod\ p)$ where both i, j are odd or both are even and $i \neq j$.

Subcase 1: From congruence 9.10, we have $a_1 = a_2 = \cdots = a_n = a$. Moreover,

$$\begin{aligned} b_1b_2a^2 \equiv b_2b_3a^2 \equiv \cdots \equiv b_nb_1a^2 \equiv a^2(mod\ p) &\Rightarrow b_1b_2 \equiv b_2b_3 \equiv \cdots b_nb_1 \equiv 1(mod\ p) \\ &\Rightarrow b_i \equiv b_{i+2}(mod\ p) \text{ for all } i \end{aligned}$$

Thus we get $b_i \equiv b_j(mod\ p)$ where both i, j are odd or both are even and $i \neq j$.

Subcase 2:

$$a_i = \begin{cases} a, & 1 \leq i \leq k \\ p - a, & k+1 \leq i \leq n. \end{cases}$$

Then from congruence 9.10, we get

$$\begin{aligned} a(p-a) \equiv a^2 \equiv a^2b_1b_2 \equiv a^2b_2b_3 \equiv \cdots \equiv a^2b_{k-1}b_k \equiv a(p-a)b_kb_{k+1} \equiv \cdots \\ \equiv a(p-a)b_nb_1(mod\ p) \\ \Rightarrow b_1b_2 \equiv b_2b_3 \equiv \cdots \equiv b_{k-1}b_k \equiv 1(mod\ p) \quad \text{and} \quad b_kb_{k+1} \equiv \cdots \equiv b_nb_1 \equiv 1(mod\ p) \end{aligned}$$

This implies $b_i \equiv b_j(mod\ p)$ where both i, j are odd or both are even and $i \neq j$.

$\square$

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APPENDIX

List of Publications

1. P.T.Vandana, V.Anil Kumar : V_4 *Magic Labelings of Wheel related graphs*, British Journal of Mathematics and Computer Science, 8(3), 189-219, (2015).

2. V.Anil Kumar, P.T.Vandana, V_4*-magic labelings of some Shell related graphs*, British Journal of Mathematics and Computer Science, 9(3), 199-223, (2015).

3. P.T.Vandana, V.Anil Kumar, V_4*-magic labelings of some graphs*, British Journal of Mathematics and Computer Science, 11(5), 1-20,(2015).

4. P.T.Vandana, V.Anil Kumar, *A note on* V_4*-magic labelings of graphs*, Journal of Information & Optimization Sciences, 37(6), 873-880, (2016).

5. P.T.Vandana, V.Anil Kumar, V_4*-magic graphs*, Advances in Computer Networks and Information Technology-Volume II, United Scholars Publications, 140-174, (2016).

6. P.T.Vandana, V.Anil Kumar, *On a class of* V_4*-barycentric magic graphs*, Far East Journal of Mathematical Sciences, 102(1), 67-96, (2017).

7. P.T.Vandana, V.Anil Kumar, V_4*-bimagic graphs*, Advances and Applications in Discrete Mathematics, 18(3), 237-268, (2017).

8. P.T.Vandana, V.Anil Kumar, *Some Special* V_4 *Barycentric Magic Graphs*, Bulletin of Kerala Mathematics Association, 14(2), 235-250, (2017).

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