A STUDY ON PROPERTIES OF L-SLICES, MORPHISM CLASS OF L-SLICES AND GENERALISED LOCALES

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BY

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This is to certify that the thesis entitled ***A Study On Properties of L-Slices, Morphism Class of L-Slices and Generalised Locales***, submitted by part-time research scholar Ms.Mary Elizabeth Antony, Department of Mathematics, Mar Athanasius College, Kothamangalam, to the University of Calicut, in partial fulfilment of requirement for the degree of Doctor of Philosophy in Mathematics, is a bonafide record of research work undertaken by her in the Centre for Research in Mathematical Sciences, St.Joseph's College, Irinjalakuda, under my supervision during the period 2010-2019 and that no part thereof has been presented before for any other degree.

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DECLARATION

I hereby declare that this thesis entitled ***A Study On Properties of L-Slices, Morphism Class of L-Slices and Generalised Locales***, is the record of bonafide research I carried out in the Centre for Research in Mathematical Sciences, St.Joseph's College (Autonomous), Irinjalakuda, under the supervision of Dr.Mangalambal N.R, Research Supervisor, Centre for Research in Mathematical Sciences, St.Joseph's College (Autonomous), Irinjalakuda.

I further declare that this thesis, or any part thereof, has not previously formed the basis for the award of any other degree, diploma, associateship, fellowship or any other similar title of recognition.

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Contents

Notations Used

$\sqsubseteq$: partial order relation on the locale L

$\mathbb{N}$: the set of all Natural numbers

0_L : the bottom element of the locale L

1_L : the top element of the locale L

$\boldsymbol{C}^{op}$: dual or opposite of the category $\boldsymbol{C}$

id : identity slice morphism

$Cl(X)$: the closure of the set X.

$\Omega(X)$: openset lattice of topological space X

$\downarrow(a)$: principal ideal generated by a

$\uparrow(a)$: principal filter generated by a

$O(L)$: the collection of all order preserving maps on the locale L

$(\gamma, J/R)$: quotient slice of L-slice (σ, J) with respect to the congruence R

Σ_a : the collection of all completely prime filters of the locale L containing $a \in L$

$\vee_\gamma$: join with respect to quotient slice $(\gamma, J/R)$

$\mathbf{0}_{Hom}$: Zero morphism of the L-slice $Hom(\sigma, J)$

$\mathbf{0}_{hom}$: Zero morphism of the L-slice $Hom(L, J)$

$Ob\boldsymbol{C}$: class of objects of the category $\boldsymbol{C}$

$Mor\boldsymbol{C}$: class of morphisms of the category $\boldsymbol{C}$

Abbreviations Used

$Ann(J)$: annihilator of the L-slice (σ, J)

$ker f$: kernel of f

$im f$: image of f

L-ideals : ideals of a locale L

(σ, J)-ideals : ideals of the L-slice (σ, J)

$Hom(\sigma, J)$: the collection of all L-slice homomorphisms from (σ, J) to (σ, J)

$Hom(J, K)$: the collection of all L-slice homomorphisms from J to K

Frm : Category of frames and frame homomorphisms

Loc : Category of locales and localic maps

Top : Category of topological spaces and continuous maps

L-slice : Category of L-slices and L-slice homomorphisms

Batch : Category of Batches and Batch morphisms

$Spec(\sigma, C)$: Spectrum of L-component (σ, C)

$Z - Fil$: The collection of all Z-filters

Introduction

Geometry, the study of structural relations is as old as mathematics itself. During the nineteenth century the development of analysis led to the need of understanding the notions of continuity and convergence in a more broader context. The concept of metric spaces was thus developed. But later it was found that some structural properties could not be explained using distances. A major breakthrough came in 1914 through Hausdorff's "Mengenlehre"which paved way for the origin of topology. Hausdorff explained the structure of space using the idea of a neighbourhood.The concept of open sets was favoured by many mathematicians during the 1920s. It was the striking similarity in the definition of topology to that of a lattice which made open sets a favourable concept. The topology on a set is a collection of subsets that are closed under arbitrary unions and finite intersection. This unmistakable similarity between the definition of topology and complete lattices triggered the investigation into the frame work of pointless topology .Frames are complete distributive lattices. In other words,frame theory is the application of lattice theory to topology. Marshal Stone in his famous paper [65] established a representation theorem for Boolean algebras: "Every boolean algebra is isomorphic to the Boolean algebra of open closed sets of totally disconnected compact hausadorff spaces". This theorem made a huge impact in many areas of modern mathematics. It initiated the study of various topological concepts from a lattice theoretic perspective. The study started with Wallmann[70] in 1938 later followed with Mckinsey and Tarski[41], Dowker and Papert[17] among others. This view of pointless topology provides the platform to construct topological spaces from algebraic data.

Ehresmann and Benabon[21] were the first to look at complete lattices with meet distributes over arbitrary joins as generalised topological spaces. They called these lattices as local lattices, while Dowker and Papert[17] called them as frames. The frame theory approach to topology considers the lattice of open sets as the basic notion. Hence, they were referred to as "point free topology". Many of the basic ideas in topology say for example continuity, compactness etc can be defined using open sets alone. With the advent of frames, many topological theorems were generalised to frame theory. Topologists of the period started working with the idea of frames and topology was consistently studied from the lattice theoretic viewpoint. Obvious examples of frames are lattice ΩX of open sets of a topological space X, the complete Boolean algebras and complete chains. A frame that is isomorphic to some ΩX is called spatial. All finite distributive lattices, all complete chains are spatial. But, a complete Boolean algebra is spatial if and only if it is atomic, showing that frames considerably surpass the classical topology. By the late 1980's many of the topological notions like compactness, uniform space, nearness etc. were explored in the context of point free topology. The topologists Dona Papert and C.H.Dowker extended the notion of separation axioms[19] and quotient spaces[17] to a more wider framework of complete lattices. Thus,frame theory became the hotspot for topologists and lattice theorists.From thereon, in works of topology the points of space were seldom mentioned.

One problem that arose was that the frame homomorphisms analogue of continuous maps could not be interpreted as generalised continous maps in the context of frames.This led to the intervention of category theory into frame theory. Isbell in[31] pointed out that frame homomorphisms behave properly in the dual category of frames. J.R.Isbell in "Atomless parts of space "[31] expresses the relationship between frames and spaces in a categorical perspective thus introducing a wide range of categorical tools to work. This motivated him to introduce the separate terminology for dual category of frames as "locales ". Note that frame and locale are synonymous as long as no reference is made to morphisms. In other words, the objects of category **Frm** and category **Loc** are identical. Category theory has played an important role in developing the rich literature in locale theory.

One of the main advantages of frame theory is that it does not require the Axiom of Choice for proving many of theorems like Tychnoff theorem and Stone-Cech compactificationc. The change in the perspective of topology to frame theory gives us more balanced results. For example, in the classical setting coproducts of regular frames does not preserve the Lindeloff property,while in the context of frame theory, coproducts of regular frames preserve the Lindeloff property. Also, frames are more algebraic while locales are topological.The concept of sublocales and subframes emphasises this fact.

When the points are abandoned there arises a natural question whether the geometric information about the spaces are lost. The spaces that are not T_0 are not adequately presentable. That is where the importance of sober spaces arises. Ales Pultr in his book [56] states that "If a complete lattice is isomorphic to the $\Omega(X)$ of a sober space X, then X can be reconstructed from the lattice by purely lattice theoretic methods ".In other words, Sober spaces are fully embedded in the category of locales. Sober spaces highlights the fact that not every space comes from a locale.

Any locale that is isomorphic to the lattice of open sets of a topological space is said to have “enough points “ or it is spatial .The Boolean algebra of all open sets U of the real line with $U = int(Cl(U))$ is an example for non spatial frame.In other words, non spatial frames are not just copies of some already existing topological space. Thus existence of non spatial frames establishes the fact that the category of locales are much larger than that of topological spaces.

Another interesting role of frames is highlighted in the book “Topology via logic” by Steven Vickers. Frames being complete Heyting algebra becomes an order complete model for intuitionistic propositional calculus.The development of lattice theoretic topology favoured the developments in generalised sheaf and topos theory. It is to be noted that the theory of sheaf has a correspondence with predicate formula in the case of propositional logic. Thus a theorem by Marshal stone influenced almost all the branches of mathematics including the representation theory of rings and other generalised algebraic systems.

The connection between Boolean algebra and Boolean rings instigated the development of L-slices.In [59] Sabna K.S. introduced L-slices and their basic properties. L-slices are modelled in line with the concept of modules. Modules can be viewed as the action of a ring over an abelian group. Module theory deals with group actions on vector spaces or equivalently group ring actions which is a generalisation of representation theory. They are also the central notion of commutative algebra. The refoundation of algebraic geometry using locales which are semi rings in place of rings began in[58] .In [58] they discusses the basic properties of L-slices.The factor of L-slices with respect to a subslice is defined. Analogous to the isomorphism theorem in rings, an isomorphism theorem for L-slice is derived. This shows the inevitable relation between algebra and L-slices. The basic theories of algebra are well captured by L-slices. For example,they have shown that finitely generated L-slice of a locale L with n generators is isomorphic to the quotient slice of L-slice $(\sqcap, L^n)$. The benefit of L-slices is that we are equipped with both topological and algebraic tools to study its structure. The relation between the category **L-Slice** of L-slices and **TopWMod** of topological weak modules has been derived. The scope of L-slices is well extended to the branch of Cryptography through Diffie Hellman key exchange protocol.

Now we introduce generalised locales. It is a synonym that we use for quantales. The concept of quantales dates back to 1930s, when M.Ward and Dilworth [16] started working with residuated lattices. The term was coined by C.J.Mulvey [46] from the two words quantum logic and locale. The study on lattices over which an additional binary operation of multiplication or residuation was initiated by the works of Ward and Dilworth[16].They have shown that theory of ideals in rings can be conveniently formulated using the residuated lattices. Locales are the lattice theoretic counter parts of topological spaces which describes commutative C^* algebras. C.J.Mulvey[46] investigated the possibility of a substitute for locales so that they would describe C^* algebras more efficiently. This led to the "quantisation" of the term locale. C.A.Akemann[3] had developed a structure on the lattice of right ideals R(A) of a C*algebra A from which the original structure could be reconstructed. C.J. Mulvey suggested to view R(A) as a lattice with multiplication so that in the commutative case R(A) exhibits the structure of a locale. Thus, quantales were eventually developed as a generalisation of locales. The ideals in commutative algebra could be considered as the paving stone to the development of quantales. Some examples of quantales include frames, ideal lattices of rings and C* algebra. .Niefield and Rosenthal [50] developed the theory of quotients and subobjects and applied this to its spectrum construction. The importance of ideal theory in quantales is as important as the ideal theory in rings. The complete lattice of ideals of a quantale helps in the construcution of various spectrum for the quantale. The relation between locales and quantales has motivated many mathematicians to investigate and analyse the counter parts of localic terms like coherent frames, algebraic frames etc. in the background of quantales. The study of quantales in [39] shows that the theory is developed parallel to that in locales.The definition of subobjects, quotient objects are all exactly the generalisation of what we had in frame theory. Locales have found application in propositional logic while their generalised counterpart quantales have found a remarkable application in analysis of the semantics of linear logic .Through out the development of locale theory mathematicians viewed them as generalised spaces. The striking similarity in the domain of locales and quantales motivates us to consider quantales as “generalised locales”.

The wide possibility of L-slices prompted the study of various properties it exhibits. This motivated the title of this thesis “ A study on properties of L-slices, morphism class of L-slices and generalised locales”.The study in this thesis explores different aspects of L-slices , for a locale L and each chapter deals with different properties. The first chapter introduces a new concept called the Box $\mathfrak{S}$ which leads to the category **Batch**. The second and third chapters are dedicated to the study of L-slices $Hom(\sigma, J)$ and $Hom(L, J)$, for a locale L. The fourth chapter exploits the algebraic property of L-slices to obtain the Zariski topology on L-slices. The fifth chapter deals with quantales which we observe to be generalised locales and give generalised version of L-slices.The final chapter introduces a graph theoretic approach to L-slices. The detailed structure of the thesis is as follows.

Organization of the Thesis

The thesis is divided into seven chapters.The first chapter deals with basic definitions and preliminaries needed for the development of the study.

The second chapter is tilted The Box $\mathfrak{S}$, stack of filters $\mathfrak{S}_x$ and the Category **Batch**. We define regular filter, associated filter based on the filter F_x. Analogous to the notion of sequential continuity, we develop the concept of F-continuity. Some properties of F-continuous slice morphisms are studied. We introduce a particular type of slice called R-A slice, where all regular filters are associated filters. We observe that the F-continuous image of R-A slice is also a R-A slice. The core of the chapter is the concept of Box $\mathfrak{S} = \{(F, x) : F \ \ is \ \ associated \ \ to \ \ x \in (\sigma, J)\}$. We observe the following on a Box $\mathfrak{S}$.

- The projection map π arranges the Box $\mathfrak{S}$ into stacks of filters $\mathfrak{S}_x$ on the germ x. Thus the Box $\mathfrak{S}$ is remodelled into the ordered Box $\mathfrak{S}_{(\sigma,J)}$. Also, $(\lambda, \mathfrak{S}_{(\sigma,J)})$ is an L-slice.

- Each member of the stack $\mathfrak{S}_x$ can be extended to a larger one in the same stack. And for $x \in (\sigma, J)$ and $G \in \mathfrak{S}_x$, $(\mathfrak{S}_x))_G = \{F \in \mathfrak{F} : F \cap G \in \mathfrak{S}_x\}$ is a filter on $\mathfrak{F}$ containing $\mathfrak{S}_x$.

- We define a map called section on $\mathfrak{S}$. The collection of sections $\Gamma((\sigma, J'), \mathfrak{S})$ on $\mathfrak{S}$ is partially ordered as $s \preceq s'$ if and only if $s(x) \leqslant_{\mathfrak{S}} s'(x)$ for all $x \in (\sigma, J')$. We find that $\Gamma((\sigma, J'), \mathfrak{S})$ is a join semilattice with bottom element .

- A Batch is a triplet $(\mathfrak{S}, \pi, (\sigma, J))$ where $\mathfrak{S}$ is a Box over the L-slice (σ, J) and π is the projection of $\mathfrak{S}$ to (σ, J). Also for $X = (\mathfrak{S}, \pi, (\sigma, J))$ and $Y =$

$(\mathfrak{S}', \pi', (\mu, K))$ any two Batches, the morphism between the Batches is defined as a pair (ψ, f).

- We prove that the Batch is a category with class of objects as Batches and the morphism class as the pair (ψ, f).

Chapter 3 studies $Hom(\sigma, J)$ through the ideals of the form $(a : x)_{Hom}$. On $Hom(\sigma, J)$,for each $x \in (\sigma, J)$ the collection $\mathfrak{B}_x = \{(a : x)_{Hom} : a \in L\}$ forms a basis for a topology. For $a \in L$ the collection $J_a = \{(a : x)_{Hom} : x \in (\sigma, J)\}$ forms an L-slice (λ, J_a), where $\lambda : L \times J_a \to J_a$ is defined as $\lambda(b, (a : x)_{Hom}) = (a : \sigma(b, x))_{Hom}$. We observe the following on J_a.

- On (σ, J) if we define a relation as $x \sim_a y$ if and only if $(a : x)_{Hom} = (a : y)_{Hom}$, then $\sim_a$ is a congruence. Consequently, $(\gamma, J/\sim_a)$ is a quotient slice.

- The map $\phi_a : (\gamma, (\sigma, J)/\sim_a) \to (\lambda, J_a)$ defined as $\phi([x]) = (a : x)_{Hom}$ is a surjective slice morphism.

- The map $F_a : (\sigma, J) \to (\lambda, J_a)$ defined as $F_a(x) = (a : x)_{Hom}$ is a slice morphism and $F_a = \phi_a \circ \sim_a$.

The ideals $(a : x)_{Hom}$ also allows a quotienting of the locale L. Fix any $x \in (\sigma, J)$ and consider the corresponding ideal $(a : x)_{Hom}$ of $Hom(\sigma, J)$. The relation R_x on $(\sqcap, L)$ defined as aR_xb if and only if $(a : x)_{Hom} = (b : x)_{Hom}$ is found to be a congruence relation. Thus $(\sqcap_{R_x}, L/R_x)$ is a quotient slice. The next construction leads us to a topology on the L-slice $(\sqcap, L)$. For $f \in Hom(\sigma, J)$ and $x \in (\sigma, J), \mathfrak{B}_L = \{[f : x]_L : f \in Hom(\sigma, J)\}$ forms a basis for topology on the L-slice $(\sqcap, L)$. Hence $\mathfrak{L} = ((\sqcap, L), \mathfrak{B}_L)$ forms a topological space with basis $\mathfrak{B}_L$. Similarly, the subslices of the form $[a : f]_{(\sigma,J)}$ allows $\mathfrak{B}_{(\sigma,J)} = \{[a : f]_{(\sigma,J)} : f \in Hom(\sigma, J)\}$

to form a basis for a topology on (σ, J). In this particular study that we have constructed three topologies on the three different domains involved. And the topologies generated through the ideals on $(\sqcap, L)$ and $Hom(\sigma, J)$ makes the slice morphisms $\psi = \sigma_b$ continuous for every $b \in L$. Similarly the subslices constructed on (σ, J) permits the continuity of the slice morphism σ_x for every $x \in (\sigma, J)$.

The fourth chapter is devoted to the study of the L-slice $Hom(L, J)$. The development of the ring $C(X)$ involves the study of real valued functions on the topological space X . The case under our consideration involves the study of all L-slice morphisms from $(\sqcap, L)$ to (σ, J). The ring $C(X)$ has two important concepts called Zero sets and fixed ideals. We examine the structure and properties of Zero sets and fixed ideals of $Hom(L, J)$.The main results we obtained are as follows:

- The collection of all zero sets $Z(L, J)$ is an L-slice with the action defined as $\lambda : L \times Z(L, J) \to Z(L, J)$ as $\lambda(a, Z(f)) = Z(\delta(a, f))$.

- For $f \in Hom(L, J)$ and an element $x \in (\sigma, J)$ the set $\langle\, f : x \rangle_L = \{r \in L : f(r) \leqslant x\}$ is an ideal of $(\sqcap, L)$. The collection $\mathfrak{B}^{0_J}_{(\sqcap,L)} = \{\langle f : x\rangle_L : f \in Hom(L, J)\}$ the zero sets of slice morphisms from $(\sqcap, L)$ to (σ, J) forms a basis for a topology on $(\sqcap, L)$. Also, if every $f \neq \mathbf{0}_{hom}$ is a unit then the topology generated by $\mathfrak{B}^{0_J}_{(\sqcap,L)}$ is Sierpinski topology.

- We define the concepts of Z-filters, Z-ideals and strong Z-ideals. The interrelation between the concept of Z-filters and Z-ideals are studied. We found that if $\mathcal{F}$ is a Z-filter on L then the family $Z^{\leftarrow}[\mathcal{F}] = \{f \in Hom(L, J) : Z(f) \in \mathcal{F}\}$ is an ideal in $Hom(L, J)$. Also, if I is a strong Z-ideal then $Z[I]$ is a Z-filter on $(\sqcap, L)$.

- Let $\mathfrak{M}$ denote the collection of all fixed ideals M_p. Define $\overline{\wedge} : L \times \mathfrak{M} \to \mathfrak{M}$

as $\overline{\wedge}(a, M_p) = M_{\sqcap(a,p)}$. Then $(\overline{\wedge}, \mathfrak{M})$ is an L-slice. Define a slice morphism $\mu : (\sqcap, L) \to \mathfrak{M}$ as $\mu(a) = M_a$. Let $Z - Fil$ denote the collection of all Z-filters in $Z(L, J)$. The map $\tilde{Z} : \mathfrak{M} \to Z - Fil$ is the natural map that takes each $M_p \in \mathfrak{M}$ to the corresponding Z-Filter $Z[M_p]$. Now, the composition $Z \circ \mu : (\sqcap, L) \to Z - Fil$ takes each element $r \in (\sqcap, L)$ to the Z-Filter $Z[M_r]$. Thus to each $a \in (\sqcap, L)$ we associated a Z-filter in $Z(L, J)$ through $\mathfrak{M}$.

Chapter 5 make use of the algebraic properties of L-slice. We examine the possibility of Zariski topology on L-components. Given a locale L and a L-slice (σ, J), for $m \in (\sigma, J)$ and $r \in L$, we have constructed (σ, J) ideals $[r \to m] = \{n \in (\sigma, J) : \sigma(r, n) \leqslant m\}$. Their properties and characteristics are studied. Similarly for a given L-slice (σ, J) and $n, m \in (\sigma, J)$ we examine the properties of ideals $[r \to m] = \{n \in (\sigma, J) : \sigma(r, n) \leqslant m\}$ on L. We define L-prime element of (σ, C) as an element $p \neq 1_C$, for which every $r \in L$ and $n \in (\sigma, C)$ with $\sigma(r, n) \leqslant p$ implies that either $r \in [1_C \to p]_L$ or $n \leqslant p$. The set of all L -prime elements of (σ, C) is called the spectrum of (σ, C) and is denoted by $Spec(\sigma, C)$. On $Spec(\sigma, C)$ we define the sets $C(n) = \{p \in Spec(\sigma, C) : n \leqslant p\}$.

If the L-prime element of (σ, C) is also a meet irreducible element of C then the we denote the spectrum as $Spec_\wedge(\sigma, C)$. The topology ψ generated by the family of closed sets $\nu = \{C(n) : n \in (\sigma, C)\}$ defined on $Spec_\wedge(\sigma, C)$ is called the Zariski topology on $Spec_\wedge(\sigma, C)$. If we define $C^*(n) = \{p \in Spec(\sigma, C) : [1_C \to n] \subseteq [1_C \to p]\}$ then the collection $\gamma^* = \{C^*(n) : n \in (\sigma, C)\}$ forms a collection of closed sets for the Zariski topology Ω^* on $Spec(\sigma, C)$. We examine the properites exhibited by Ψ and Ω^*. Some of the results obtained are as follows:

- $(Spec_\wedge(\sigma, C), \Psi)$ is a T_0 space.

- If every element of $Spec(\sigma, C)$ is maximal element then the singleton sets will be closed in Ω^* and hence $(Spec(\sigma, C), \Omega^*)$ will be a T_0 space.
- $(Spec(\sigma, C), \Omega^*)$ is irreducible if and only if (σ, C) is without zero divisors.

The sixth chapter emphasies the importance of generalised locales or quantales. In the first section we develop a quotient quantale using a specific ideal. Second section deals with the maps called deductions and their properties. It is well known that a quotient quantale can be constructed through the maps called quantic nucleus. Here we try to do the same through the ideals constructed from the newly defined maps called deductions. The third section introduces the graphs that are associated with quantales. This section motivated us to look into the possibilities of introducing graph theory in the context of L-slices. The last section introduces the generalised L-slice which we call Q-slices. We discuss some of the basic differences in the properties exhibited by L-slices and Q-slices.

The seventh chapter is a graph theoretic approach to L-slices. We introduce two graphs associated with L-slices. We begin with the definition of torsion elements of an L-slice and then define the total graph of L-slice $\Gamma(T(\sigma, J))$. The total graph $\Gamma(T(\sigma, J))$ is complete if and only if the L-slice is not faithful. The subgraph with vertices from $T(\sigma, J)$ is always complete. The chromatic number $\chi(\Gamma(T(\sigma, J)))$ of $\Gamma(T(\sigma, J))$ is such that either always $\Gamma(T(\sigma, J)) = 1$ or $\chi(\Gamma(T(\sigma, J))) = n + 1$, where $n = \mid T(\sigma, J) \mid$. Also if $T(\sigma, J)$ is proper ideal then the diameter and radius of $\Gamma_n(T(\sigma, J))$ will be the same and equal to 1. Whenever $\mid T(\sigma, J) \mid = n \geqslant 3$, then the girth of $\Gamma_n(T(\sigma, J))$ is three and the circumference of $\Gamma_n(T(\sigma, J))$ is n. The second graph is the one that is associated with the weak Zariski topology on L-slice. For a subset T of $Spec(\sigma, C)$ we introduce a graph $G_T(\omega^*)$. The graph exists if and only if f $T = C(\bigwedge T)$ and T is not an irreducible subset of $Spec(\sigma, C)$. The weak Zariski topology graph $G_T(\omega^*)$ is connected and $diam(G_T(\omega^*)) \leqslant 2$. We define a subgraph $G'_T(\omega^*)$ of $G_T(\omega^*)$ and show that it is bipartite.

The study on properties of L-slices, morphism class of L-slices and generalised locales opens up a wide possibility for research in different perspectives. The study can be further taken up by researchers of different interests. We conclude the thesis by emphasizing few open problems that arose during the study.

Chapter 1

Preliminaries

This chapter includes some preliminary concepts on Order theory, Category theory, Frames and Locales ,Quantales and Graph Theory required for the next chapters.

1.1. Order theoretical concepts

Definition 1.1.1. [35] Let L be a set. A partial order on L is a binary relation $\sqsubseteq$ which is

i. reflexive : for all $a \in L$, $a \sqsubseteq a$,

ii. antisymmetric: if $a \sqsubseteq b$ and $b \sqsubseteq a$, then $a = b$, and

iii. transitive: if $a \sqsubseteq b$ and $b \sqsubseteq c$, then $a \sqsubseteq c$.

A partially ordered set (also called poset) is a set equipped with a partial order.

Definition 1.1.2. [12] An element $x \in A \subseteq L$ is called minimal if $a \in A, a \sqsubseteq x$ implies $a = x$. If L has a unique minimal element, then it is called the least element (bottom) of L denoted by 0_L.

Definition 1.1.3. [12] An element $x \in A \subseteq L$ is called maximal if $a \in A, x \sqsubseteq a$ implies $a = x$. If L has a unique maximal element, then it is called the greatest element (top) of L denoted by 1_L.

Definition 1.1.4. [12] An element $x \in L$ is called an upperbound of $A \subseteq L$, if for all $a \in A$, we have $a \sqsubseteq x$. The least element of the set of all upperbounds of A in L, if it exists, is called the least upperbound (supremum)of A. It is denoted by $\bigsqcup A$.

Definition 1.1.5. [12] An element $x \in L$ is called a lowerbound of $A \subseteq L$, if for all $a \in A$, we have $x \sqsubseteq a$. The greatest element of the set of all lowerbounds of A in L, if it exists, is called the greatest lowerbound (infimum) of A. It is denoted by $\bigsqcap A$

Definition 1.1.6. [56] A poset L is called a join-semilattice (resp.meet-semilattice) if there is a supremum $a \sqcup b$ (resp.infimum $a \sqcap b$) for any two $a, b \in L$.

Definition 1.1.7. [35] A partially ordered set L in which for every pair of elements a, b, there exists the supremum $a \sqcup b$ and the infimum $a \sqcap b$ is called a lattice. A partially ordered set L for which every set $A \subseteq L$ has the supremum $\bigsqcup A$ and the infimum $\bigsqcap A$ exist in L is called a complete lattice.

Definition 1.1.8. [35] A lattice L is distributive if $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$ which is equivalent to $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.

Definition 1.1.9. [12] A map $f : L \to M$, where L, M are partially ordered sets, is called monotone(order preserving) if $a \sqsubseteq_L b \Rightarrow f(a) \sqsubseteq_M f(b)$ for all $a, b \in L$. If f is bijective and its inverse f^{-1} is also monotone, then it is called an order isomorphism.

Definition 1.1.10. [35] Let L be a distributive lattice with greatest element 1_L and least element 0_L. The complement a^c of an element $a \in L$ is the one satisfying $a \sqcap a^c = 0_L$ and $a \sqcup a^c = 1_L$.

Definition 1.1.11. [35] A Boolean algebra is a distributive lattice with 0_L and 1_L in which every element has a complement.

Definition 1.1.12. [56] An element $p \neq 1$ in a lattice L is said to be meet-irreducible if for any $a, b \in L$, $a \sqcap b \sqsubseteq p$ implies that either $a \sqsubseteq p$ or $b \sqsubseteq p$.

Definition 1.1.13. [56] An element $p \neq 0$ in a lattice L is join-irreducible if for any $a, b \in L, p \sqsubseteq a \sqcup b$ implies that either $p \sqsubseteq a$ or $p \sqsubseteq b$.

Definition 1.1.14. [56] A lattice A is said to be a Heyting algebra if for each pair of elements (a, b) in A, there exist an element $a \to b$ such that $c \sqsubseteq (a \to b)$ if and only if $c \sqcap a \sqsubseteq b$.

1.2. Categorical Concepts

Definition 1.2.1. [29] A category $\boldsymbol{C}$ consist of:

i. A class $Ob\boldsymbol{C}$ of objects (notation: $A, B, C....$)

ii. A class $Mor\boldsymbol{C}$ of morphisms (notation: $f, g, h...$). Each morphism f has a domain or source A (notation: $dom(f)$) and a codomain or target B (notation: $codom(f)$) which are objects of $\boldsymbol{C}$; this is indicated by writing $f : A \to B$.

iii. A composition law that assign to each pair (f, g) of morphisms satisfying $dom(g) = codom(f)$ a morphism $g \circ f : dom(f) \to codom(g)$, satisfying

(a) $h \circ (g \circ f) = (h \circ g) \circ f$ whenever the compositions are defined.

(b) For each object A of $\boldsymbol{C}$ there is an identiy $id_A : A \to A$ such that $f \circ id_A = f$ and $id_A \circ g = g$ whenever the composition is defined.

Definition 1.2.2. [29] A category $\boldsymbol{B}$ is said to be a subcategory of the category $\boldsymbol{C}$ provided that the following conditions are satisfied.

i. $Ob(\boldsymbol{B}) \subseteq Ob(\boldsymbol{C})$.

ii. $Mor(\boldsymbol{B}) \subseteq Mor(\boldsymbol{C})$.

iii. The domain, codomain and composition functions of $\boldsymbol{B}$ are restriction of the corresponding functions of $\boldsymbol{C}$.

iv. Every $\boldsymbol{B}$-identity is a $\boldsymbol{C}$ -identity.

Definition 1.2.3. [29] If $\boldsymbol{C}$ is a category we can take the same class of objects and morphisms, and interchange the domains and codomains (which leads to inverted composition). Thus $f : A \to B$ is now $f : B \to A$ and we have a composition $f * g = g \circ f$. Thus obtained category is called the dual or opposite of $\boldsymbol{C}$ and denoted by $\boldsymbol{C}^{op}$.

Definition 1.2.4. [29] Let $\boldsymbol{C}$,$\boldsymbol{D}$ be categories. A functor from $\boldsymbol{C}$ to $\boldsymbol{D}$ is a triple ($\boldsymbol{C}$,F,$\boldsymbol{D}$) where F is a function from the class of morphisms of $\boldsymbol{C}$ to the class of morphisms of $\boldsymbol{D}$ (i.e. $F : Mor\,\boldsymbol{C} \to Mor\,\boldsymbol{D}$) satisfying the following conditions.

i. F preserves identities: i.e., if e is a $\boldsymbol{C}$-identity, then $F(e)$ is a $\boldsymbol{D}$- identity.

ii. F preserves composition: $F(f \circ g) = F(f) \circ F(g)$;i.e., whenever
$dom(f) = codom(g)$, then $dom(F(f)) = codom(F(g))$ and the above equality holds.

Definition 1.2.5. [29] A triple ($\boldsymbol{C}$, F, $\boldsymbol{D}$) is called a contravariant functor from $\boldsymbol{C}$ to $\boldsymbol{D}$ if and only if ($\boldsymbol{C}^{op}$, F, $\boldsymbol{D}$) is a functor (or, equivalently, if and only if ($\boldsymbol{C}$, F, $\boldsymbol{D}^{op}$) is a functor).

1.3. Frames and Locales

Definition 1.3.1. [56] A frame is a complete lattice L satisfying the infinite distributivity law $a \sqcap \bigsqcup B = \bigsqcup\{a \sqcap b; b \in B\}$ for all $a \in L$ and $B \subseteq L$.

Definition 1.3.2. [56] A map $f : L \to M$ between frames L, M preserving all finite meets (including the top 1) and all joins (including the bottom 0) is called a frame homomorphism. A bijective frame homomorphism is called a frame isomorphism.

Remark. The category of frames is denoted by **Frm**. The opposite of category **Frm** is the category **Loc** of locales. We can represent the morphism in **Loc** as the infima -preserving $f : L \to M$ such that the corresponding left adjoint $f^* : M \to L$ preserves finite meet. If we do not refer to the morphisms in the category **Loc** of locales and the category **Frm** of frames, then the objects frames and locales are same.

Remark. The category of topological spaces and continuous maps is denoted by **Top**

Definition 1.3.3. [56] The functor Ω : **Top** $\to$ **Frm** maps objects and morphisms as follows

i. A topological spaces $(X, \Omega(X))$ is mapped into frame of open sets $\Omega(X)$

ii. Ω sends morphism $f : X \to Y$ in **Top** to the frame homomorphism
$\Omega(f) : \Omega(Y) \to \Omega(X)$ defined by $\Omega(f)(V) = f^{-1}(V)$.

Theorem 1.3.4. *[56] The functor Ω : **Top** $\to$ **Frm** is a contravariant functor*

Definition 1.3.5. [35] A subset I of a locale L is said to be an ideal if

i. I is a sub-join-semilattice of L; that is $0_L \in I$ and $a \in I, b \in I$ implies $a \sqcup b \in I$;and

ii. I is a lower set; that is $a \in I$ and $b \sqsubseteq a$ imply $b \in I$.

If $a \in L$, the set $\downarrow (a) = \{x \in L; x \sqsubseteq a\}$ is an ideal of L. $\downarrow (a)$ is the smallest ideal containing a and is called the principal ideal generated by a. A proper ideal I is prime if $x \sqcap y \in I$ implies that either $x \in I$ or $y \in I$ [35].

Definition 1.3.6. [56] A subset F of locale L is said to be a filter if

i. F is a sub-meet-semilattice of L; that is $1_L \in F$ and $a \in F$, $b \in F$ imply $a \sqcap b \in F$.

ii. F is an upper set; that is $a \in F$ and $a \sqsubseteq b$ imply $b \in F$.

Definition 1.3.7. [56] A filter F is proper if $F \neq L$, that is if $0_L \notin F$.

A proper filter F in a locale L is prime if $a_1 \sqcup a_2 \in F$ implies that $a_1 \in F$ or $a_2 \in F$.

Definition 1.3.8. [56] A proper filter F in a locale L is a completely prime filter if for any indexing set J and $a_i \in L$, $i \in J$, $\bigsqcup a_i \in F \Rightarrow \exists i \in J$ such that $a_i \in F$. Completely prime filters are denoted by c.p filters.

Example 1.3.9. *[56]* $U(x) = \{V \in \Omega(X); x \in V\}$ *is a completely prime filter in the locale* $\Omega(X)$.

For an element a of a locale L, set $\Sigma_a = \{F \subseteq L; F \neq \phi, F \ \ is \ \ c.p \ \ filters; a \in F\}$. We can easily check that $\Sigma_0 = \phi$, $\Sigma_{\bigsqcup a_i}=\bigcup \Sigma_{a_i}$, $\Sigma_{a \sqcap b} = \Sigma_a \cap \Sigma_b$ and

$\Sigma_1 = \{all \ \ c.p \ \ filters\}$.

The spectrum of a locale is defined as follows.

Sp(L)=($\{all \ \ c.p \ \ filters\}$, $\{\Sigma_a : a \in L\}$). Then $Sp(L)$ is a topological space with the topology $\Omega(Sp(L)) = \{\Sigma_a : a \in L\}$.

Definition 1.3.10. [56] A locale L is said to be spatial if it is isomorphic to $\Omega(X)$ of some topological space X.

Definition 1.3.11. [56] Let L be a frame. An equivalence relation θ on L is said to be a congruence on L if $(a, b) \in \theta \Rightarrow (a \sqcap c, b \sqcap c) \in \theta$ and $(a \sqcup \bigsqcup S, b \sqcup \bigsqcup S) \in \theta$ for all $c \in L, S \subseteq L$.

Definition 1.3.12. [56] A subset of a frame L which is closed under the same finite meets and arbitrary joins in the frame is called a subframe. That is a subframe is itself a frame under the induced order of L.

The concept of sublocale is something different, corresponding to quotient frames.

Definition 1.3.13. [56] Let L be a locale. A subset $S \subseteq L$ is a sublocale of L if

i. S is closed under meets, and

ii. For every $s \in S$ and every $x \in L$, $x \to s \in S$.

A sublocale is always nonempty, since $1 = \bigsqcap \phi \in S$. The least sublocale $\{1\}$ will be denoted by $\mathbf{0}$

Proposition 1.3.14. *[56] Let L be a locale. A subset $S \subseteq L$ is a sublocale if and only if it is a locale in the induced order and the embedding map $j : S \subseteq L$ is a localic map.*

Definition 1.3.15. [56] A nucleus in a locale L is a mapping $v : L \to L$ such that

i. $a \sqsubseteq v(a)$,

ii. $a \sqsubseteq b \Rightarrow v(a) \sqsubseteq v(b)$

iii. $v(v(a)) = v(a)$ and

iv. $v(a \sqcap b) = v(a) \sqcap v(b)$.

Sublocales of a locale L have alternate representations in[56].

i) Sublocales of a locale can also be represented using frame congruence. A sublocale homomorphism $g : L \to M$ induces a frame congruence $E_g = \{(x, y) : g(x) = g(y)\}$ and a frame congruence gives rise to a sublocale homomorphism $x \mapsto Ex : L \to L/E$, where L/E denotes the quotient frame defined by the congruence E, and Ex denotes the E-class.

ii) Sublocales of a locale can also be represented using nucleus. The translation between nuclei and frame congruence resp. sublocale homomorphism is straight forward:

$v \mapsto E_v = \{(x, y) : v(x) = v(y)\}$,

$E \mapsto v_E = (x \mapsto \bigsqcup Ex) : L \to L$;

$v \mapsto v_h = v$ restricted to $L \to v[L]$,

$h \mapsto v_h = (x \mapsto h_* h(x)) : L \to L$

We can relate sublocales and nuclei directly. For a sublocale $S \subseteq L$, set $v_S(a) = j_S^*(a) = \bigsqcap\{s \in S : a \sqsubseteq s\}$ and for a nucleus $v : L \to L$, set $S_v = v[L]$.

1.4. L-Slice

Given a locale L and a join semilattice J with bottom element 0_J, we have introduced a new concept of an action σ of locale L on join semilattice J together with a set of conditions. The pair (σ, J) is called L-slice. L-slice, though algebraic in nature adopts properties of L through the action σ.

Definition 1.4.1. [58] Let L be a locale and J be join semilattice with bottom element 0_J. By the "action of L on J"we mean a function $\sigma : L \times J \to J$ such that the following conditions are satisfied.

i. $\sigma(a, x_1 \vee x_2) = \sigma(a, x_1) \vee \sigma(a, x_2)$ for all $a \in L, x_1, x_2 \in J$.

ii. $\sigma(a, 0_J) = 0_J$ for all $a \in L$.

iii. $\sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x))$ for all $a, b \in L, x \in J$.

iv. $\sigma(1_L, x) = x$ and $\sigma(0_L, x) = 0_J$ for all $x \in J$.

v. $\sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x)$ for $a, b \in L, x \in J$.

If σ is an action of the locale L on a join semilattice J, then we call (σ, J) as L-slice.

Next proposition gives sufficient conditon for a subset $S \subseteq O(L)$ of order preserving maps on L, to be an L-slice.

Proposition 1.4.2. *[58] Let L be a locale, and let S be a set of order preserving maps $L \to L$ such that :*

i. The constant map $\boldsymbol{0} \in S$ *(*$\boldsymbol{0}$ *takes everything to 0).*

ii. If $f, g \in S$*, then* $f \vee g \in S$*.*

iii. For all $a \in L$ *and for all* $f \in S$*, the meet of the constant map* $\boldsymbol{a}$ *and* f *is in* S *(i.e.* $f \sqcap \boldsymbol{a} \in S$*).*

Then the map $\sigma : L \times S \to S$ *defined by* $\sigma(a, f)(x) = f(x) \sqcap a$ *is an action of* L *on* S*.*

Examples 1.4.3. *[58] 1. Let* L *be a locale and* I *be any ideal of* L*. Consider each* $x \in I$ *as constant map* $\boldsymbol{x} : L \to L$*. Then by proposition 1.4.2,* (σ, I) *is an L-slice. In particular* (σ, L) *is an L-slice.*

2. Let the locale L *be a chain with Top and Bottom elements and* J *be any join semilattice with bottom element. Define* $\sigma : L \times J \to J$ *by* $\sigma(a, j) = j \;\; \forall a \neq 0$ *and* $\sigma(0_L, j) = 0_J$*. Then* σ *is an action of* L *on* J *and* (σ, J) *is an L-slice.*

Proposition 1.4.4. *[58] The product of two L-slices of a locale L is an L-slice.*

Definition 1.4.5. [58] Let (σ, J), (μ, K) be L-slices of a locale L. A map $f : (\sigma, J) \to (\mu, K)$ is said to be L-slice homomorphism if

i. $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$ for all $x_1, x_2 \in J$.

ii. $f(\sigma(a, x)) = \mu(a, f(x))$ for all $a \in L$ and all $x \in (\sigma, J)$.

Definition 1.4.6. [58] Let (σ, J) be an L-slice of a locale L. A subjoin semilattice J' of J is said to be L-subslice of J if J' is closed under action by elements of L.

Examples 1.4.7. *[58] 1. Let L be a locale and* $O(L)$ *denotes the collection of all order preserving maps on L. Then* $(\sigma, O(L))$ *is an L-slice, where* $\sigma : L \times O(L) \to O(L)$ *is defined by* $\sigma(a, f) = f_a$*, where* $f_a : L \to L$ *is defined by* $f_a(x) = f(x) \sqcap a$*. Let* $K = \{f \in O(L) : f(x) \sqsubseteq x, \forall x \in L\}$*. Then* (σ, K) *is an L-subslice of the L-slice* $(\sigma, O(L))$*.*

2. Let (σ, J) *be an L-slice and let* $x \in (\sigma, J)$*. Define* $\langle x \rangle = \{\sigma(a, x); a \in L\}$*. Then*

$(\sigma, \langle x \rangle)$ is an L-subslice of (σ, J) and it is the smallest L-subslice of (σ, J) containing x.

Definition 1.4.8. [58] Let (σ, J) be an L-slice of a locale L. An equivalence relation R on (σ, J) is called an L-slice congruence if

i. xRy implies $x \vee zRy \vee z$ for any $x, y, z \in (\sigma, J)$

ii. xRy implies $\sigma(a, x)R\sigma(a, y)$ for all $a \in L, x, y \in (\sigma, J)$.

Proposition 1.4.9. *[58] Let $(\sigma, J), (\mu, K)$ be two L-slices of a locale L and let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Then the relation R on (σ, J) defined by xRy if and only if $f(x) = f(y)$ is a congruence on (σ, J).*

Definition 1.4.10. [58] The L-slice congruence R discussed in proposition 1.4.9 is called natural congruence associated with the L-slice homomorphism $f : (\sigma, J) \to (\mu, K)$.

Let R be a congruence on (σ, J) and let J/R denotes the collection of all equivalence classes with respect to the relation R. Then J/R is a join semilattice with bottom element $[0_J]$, where the partial order $\leqslant$ on J/R is defined by $[x] \leqslant [y]$ if and only if $x \leqslant y$ in (σ, J). In the next proposition, we will show that $(\gamma, J/R)$ is an L-slice where the action $\gamma : L \times J/R \to J/R$ is defined by $\gamma(a, [x]) = [\sigma(a, x)]$.

Proposition 1.4.11. *[58] If R is a congruence relation on (σ, J), then $(\gamma, J/R)$ is an L-slice.*

Definition 1.4.12. [58] Let (σ, J) be an L-slice of a locale L and R be a congruence on (σ, J). Then the L-slice $(\gamma, J/R)$ described in proposition 1.4.11 is called quotient slice of L-slice (σ, J) with respect to the congruence R.

Proposition 1.4.13. *[58] Let R be an L-slice congruence on an L-slice (σ, J) of a locale L and let $(\gamma, J/R)$ be the corresponding quotient slice. Then the map $\pi : (\sigma, J) \to (\gamma, J/R)$ defined by $\pi(x) = [x]$ is an onto L-slice homomorphism.*

Definition 1.4.14. [58] Let (σ, J) be an L-slice of a locale L. For each $a \in L$, the map $\sigma_a : (\sigma, J) \to (\sigma, J)$ defined by $\sigma_a(x) = \sigma(a, x)$ is an L-slice homomorphism.

Definition 1.4.15. [58] A subslice (σ, I) of an L-slice (σ, J) is said to be ideal of (σ, J) if $x \in (\sigma, I)$ and $y \in (\sigma, J)$ are such that $y \leqslant x$, then $y \in (\sigma, I)$.

Definition 1.4.16. [58] An ideal (σ, I) of an L-slice (σ, J) is a prime ideal if it has the following properties:

i. If a and b are any two elements of L such that $\sigma(a \sqcap b, x) \in (\sigma, I)$, then either $\sigma(a, x) \in (\sigma, I)$ or $\sigma(b, x) \in (\sigma, I)$.

ii. (σ, I) is not equal to the whole slice (σ, J).

Definition 1.4.17. [58] Let (σ, J) be an L-slice of a locale L. Then the annihilator of the L-slice (σ, J) is defined by $Ann(J) = \{a \in L : \sigma_a = \mathbf{0}\}$.

Definition 1.4.18. [58] An L-slice (σ, J) of a locale L is said to be faithful if $Ann(J) = \{0\}$.

Example 1.4.19. *[58] The L-slice $(\sqcap, L)$ is faithful.*

Definition 1.4.20. [58] Let (σ, J), (μ, K) be L-slices of a locale L. A map $f : (\sigma, J) \to (\mu, K)$ is said to be L-slice homomorphism if

i. $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$ for all $x_1, x_2 \in (\sigma, J)$.

ii. $f(\sigma(a, x)) = \mu(a, f(x))$ for all $a \in L$ and all $x \in (\sigma, J)$.

Examples 1.4.21. *[58] i. Let (σ, J) be an L-slice and (σ, J') be an L-subslice of (σ, J). Then the inclusion map $i : (\sigma, J') \to (\sigma, J)$ is an L-slice homomorphism.*

ii. Let $I = \downarrow (a), J = \downarrow (b)$ be principal ideals of the locale L. Then (σ, I), (σ, J) are L-slices. Then the map $f : (\sigma, I) \rightarrow (\sigma, J)$ defined by $f(x) = x \sqcap b$ is an L-slice homomorphism.

Proposition 1.4.22. *[58] If $f : (\sigma, J) \rightarrow (\mu, K)$ is a L-slice homomorphism, then $f(0_J) = 0_K$.*

Proposition 1.4.23. *[58] The composition of two L-slice homomorphisms is an L-slice homomorphism.*

Proposition 1.4.24. *[58] Let (σ, J),(μ, K) be L-slices of a locale L and $f : (\sigma, J) \rightarrow (\mu, K)$ be L-slice homomorphism.*

i. Let $kerf = \{x \in J : f(x) = 0_K\}$. Then $(\sigma, kerf)$ is an ideal of (σ, J).

ii. Let $imf = \{y \in K : y = f(x)\ \ for\ \ some\ \ x\ \ \in (\sigma, J)\}$. Then (μ, imf) is an L-subslice of (μ, K).

Definition 1.4.25. [58] Let (σ, J), (μ, K) be L-slices of a locale L. A map $f : (\sigma, J) \rightarrow (\mu, K)$ is said to be an L-slice isomorphism if

i. f is one-one

ii. f is onto

iii. f is an L-slice homomorphism.

Lemma 1.4.26. *[58] Let (σ, J), (μ, K) be two L-slices of a locale L.*

i. The map $\boldsymbol{0}$:$(\sigma, J) \rightarrow (\mu, K)$ defined by $\boldsymbol{0}(x)$=0_K for $x \in (\sigma, J)$ is an L-slice homomorphism.

ii. If $f, g : (\sigma, J) \rightarrow (\mu, K)$ are L-slice homomorphism, then the map $f \vee g : (\sigma, J) \rightarrow (\mu, K)$ defined by $(f \vee g)(x) = f(x) \vee g(x)$ for $x \in (\sigma, J)$ is an L-slice homomorphism.

Proposition 1.4.27. *[58] Let* (σ, J),(μ, K) *be L-slices of a locale L and L- Hom(J,K) denote the collection of all L-slice homomorphisms from* (σ, J) *to* (μ, K). *Then* $(\delta, L - Hom(J, K))$ *is an L-slice, where the action,* $\delta : L \times L - Hom(J, K) \to L - Hom(J, K)$ *is defined by* $\delta(a, f)(x) = \mu(a, f(x))$ *for all* $x \in (\sigma, J)$.

Definition 1.4.28. [58] Let (σ, J) be an L-slice of a locale L. For each $a \in L$, define $\sigma_a : (\sigma, J) \to (\sigma, J)$ by $\sigma_a(x) = \sigma(a, x)$.

Proposition 1.4.29. *[58] Let* (σ, J) *be an L-slice. For each* $a \in L$, $\sigma_a : (\sigma, J) \to (\sigma, J)$ *is an L-slice homomorphism.*

Proposition 1.4.30. *[58] Let* (σ, J) *be an L-slice of a locale L. For each* $x \in J$, $\sigma_x : (\sqcap, L) \to (\sigma, J)$ *is an L-slice homomorphism.*

Proposition 1.4.31. *[58] Let* (σ, J) *be an L-slice of a locale L and let* $P = \{\sigma_x : x \in (\sigma, J)\}$. *Then* (δ, P) *is an L- subslice of* $(\delta, L - Hom(L, J))$.

Proposition 1.4.32. *[58] Let* (σ, J) *be an L-slice of a locale L. For each* $x \in (\sigma, J)$, *let* $F_x = \{a \in L; \sigma(a, x) = x\}$. *Then* F_x *is a filter in L.*

Proposition 1.4.33. *[58] The filter* F_x *is proper for* $x \neq 0_J$.

Proposition 1.4.34. *[58] Let* $x \in (\sigma, J)$ *be join-irreducible element of* (σ, J), *then* F_x *is a prime filter in L.*

Definition 1.4.35. [58] An element $x \in (\sigma, J)$ is said to be compact element of the L-slice (σ, J), if for any collection $\{a_\alpha\}$ of L whenever $\sigma(\sqcup a_\alpha, x) = x$, then there exist a finite sub collection $\{a_1, a_2, ...a_n\}$ of $\{a_\alpha\}$ such that $\sigma(a_1, x) \vee \sigma(a_2, x) \vee ... \vee \sigma(a_n, x) = x$. A slice (σ, J) is compact if each element $x \in (\sigma, J)$ is compact.

Example 1.4.36. *[58] Let (σ, J) be any L-slice. Then 0_J is a compact element.*

Proposition 1.4.37. *[58] Let $x \in (\sigma, J)$ be join-irreducible compact element of (σ, J), then F_x is a completely prime filter.*

Proposition 1.4.38. *[58] Let $F = \{a \in L : \sigma(a, x) = x \ \forall x \in (\sigma, J)\}$. Then $F = \bigcap F_x$ and F is a filter in L*

1.5. Quantales

Definition 1.5.1. [39] The category **SL** of sup-lattices has as its objects complete lattices and if P and Q are complete lattices, a function $f : P \to Q$ is a morphism of sup-lattices iff it preserves arbitrary sups.

Definition 1.5.2. [39] Let P be a poset . An order preserving function $j : P \to P$ is called a closure operator if and only if it satisfies

i) $a \leqslant j(a)$, for all $a \in P$

ii) $j(j(a)) = j(a)$ for all $a \in P$

Let $P_J = \{a \in P : j(a) = a\}$.Then P_j is a complete lattice since it is closed under infimums.

Definition 1.5.3. [39] Let P be a poset . An order preserving map $g : P \to P$ is called a coclosure operator iff it satisfies

i) $g(a) \leqslant a$ for all $a \in P$

ii)$g(g(a)) = g(a)$ for all $a \in P$.

Definition 1.5.4. [39] A quantale is a complete lattice Q with an associative binary operation $*$ satisfying i) $a * (\vee_\alpha b_\alpha) = \vee_\alpha (a * b_\alpha)$ and ii) $(\vee_\alpha b_\alpha) * a = \vee_\alpha (b_\alpha * a)$ for all $a \in Q$ and $\{b_\alpha\} \subseteq Q$.

Since $a * -$ and $- * a$ preserves arbitrary supremums , they have right adjoints we shall denote by $a \rightarrow_r -$ and $a \rightarrow_l -$ respectively. Thus, $a * c \leqslant b$ iff $c \leqslant a \rightarrow_r b$ and $c * a \leqslant b$ iff $c \leqslant a \rightarrow_l b$

Definition 1.5.5. [39] A quantale Q is commutative if and only if $a * b = b * a$ for every $a, b \in Q$.

Definition 1.5.6. [39] Let Q be a quantale and let $a \in Q$. Also, let T denote the top element of the quantale Q.

i) a is right sided iff $a * T \leqslant a$

ii) a is left-sided iff $T * a \leqslant a$

iii) a is two-sided iff a is both right sided and left sided.

iv) a is strictly right (left)sided iff $a * T = a$ $(T * a = a)$

v) a is idempotent iff $a * a = a$

vi) an element $1 \in Q$ is a left unit iff $1 * a = a$ for all $a \in Q$

vii)an element $1 \in Q$ is a right unit iff $a * 1 = a$ for all $a \in Q$

viii) an element $1 \in Q$ is a unit iff it is both a right and left unit.

Definition 1.5.7. [39] Let Q be a quantale

i) Q is right- sided(left- sided) iff every $a \in Q$ is right-sided(left- sided)

ii) Q is two sided iff every every $a \in Q$ is two sided

iii) Q is idempotent iff every $a \in Q$ is idempotent

iv) Q is(left)rightunital iff Q has a (left)right unit 1

v) Q is unital iff Q has a unit 1.

Examples 1.5.8. *[39] i) Any frame is a quantale with $* = \wedge$. It is commutative ,idempotent,unital with unit T(and hence two-sided)* *ii) Sub(R),the set of additive subgroups of R is a quantale with sup $= \Sigma$ and with* $A * B = AB = \{a_1b_1 + a_2b_2 + ...a_nb_n : a_i \in A, b_i \in B\}$

Definition 1.5.9. [15] Let I be a subset of a quantale Q. I is called a left(respectively right)ideal of Q

i) $X \subseteq I$ implies $\bigvee X \in I$

ii) $x \in I$ and $y \leqslant x$ then $y \in I$

iii) $x \in I$ imples $a * x \in I$ (resp $x * a \in I$) for all $a \in Q$

Definition 1.5.10. Let I be a subset of a quantale Q. I is called a left(right) $*$ ideal of Q if

i) $X \subseteq I$ implies $\bigvee X \in I$

ii) $x \in I$ imples $a * x \in I$ (resp $x * a \in I$) for all $a \in Q$

Definition 1.5.11. [43] A nonempty subset F of Q is said to be a filter if it satisfies the following conditions:

i) $0 \notin$ F

ii) If $a \in$ F,$b \in Q$ and $a \leqslant b$ then $b \in$ F

iii) If $a, b \in Q$ then $a * b \in$ F

Definition 1.5.12. [28] A binary relation θ on Q is a congruence on Q if and only if

i) θ is an equivalence relation

ii) If $(x_i, y_i) \in \theta$ for all $i \in I$, then $(\vee_{i\in I}x_i, \vee_{i\in I}y_i) \in \theta$,where I is some indexed set

iii) If $(a, b) \in \theta$, then $(c * a, c * b) \in \theta$ and $(a * c, b * c) \in \theta$ for any $c \in Q$

Definition 1.5.13. [39] Let P and Q be quantales. A function $f : P \to Q$ is a homomorphism of quantales if f preserves arbitrary sups and it also preserves the operation $*$.

Definition 1.5.14. [39] Let Q be a quantale. A quantic nucleus on Q is a closure operator j such that $j(a) * j(b) \leqslant j(a * b)$ for all $a, b \in Q$

Definition 1.5.15. [39] Let Q be a quantle .A quantic conucleus on Q is a coclosure operator g sucht that $g(a) * g(b) \geqslant g(a * b)$

Given a closure operator j on a complete lattice Q, then it can be easily seen that the set $Q_j = \{a \in Q : j(a) = a\}$ is again complete.

Proposition 1.5.16. *[39] If Q is a quantale and $S \subseteq Q$ then $S = Q_j$ for some quantic nucleus j iff S is closed under infs and $a \rightarrow_r s$ and $a \rightarrow_l s$ are in S, whenever $a \in Q$ and $s \in S$.*

Now , such an S is called the quantic quotient of Q. Also, $a *_S b = inf\{s \in S : a * b \leqslant s\}$

Definition 1.5.17. [39] If Q is a quantale , a subset S of Q is a subquantale iff S is closed under sups and $*$.

Theorem 1.5.18. *[39] Let Q be a quantale .If g is a quantic conucleus on Q, then $Q_g = \{a \in Q : g(a) = a\}$ is a subquantale of Q. Moreover,if S is any subquantale of Q, then $S = Q_g$, for some quantic conucleus g.*

Also, subquantales can be described in terms of a quantic conucleus.Thus, quotient quantales and subquantales can be defined using quantic nuclei and quantic conuclei.

1.6. Graph Theory

Definition 1.6.1. [49] A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2 \cdots\}$ called vertices and another set $E = \{e_1, e_2, \cdots\}$ whose elements are called edges such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.The vertices $v_i v_j$ associated with edge e_k are called the end vertices of e_k. Two edges are adjacent if they are incident on a common vertex.

Definition 1.6.2. [49] A graph that has neither self-loops nor parallel edges is called a simple graph.

Definition 1.6.3. [7] The number of edges incident on a vertex v_i is called the degree of the vertex v_i. The minimum(respectively,maximum) of the degrees of the vertices of a graph G is denoted by $\delta(G)$ or δ(respectively, $\Delta(G)$ or Δ)

Definition 1.6.4. [7] A graph in which all vetices are of equal degree is called a regular graph. A vertex having no incident edge is called an isolated vertex. A vertex of degree one is called a pendant vertex . A graph without any edges is called a null graph or an empty graph.

Definition 1.6.5. [7] A graph G is called finite if both $V(G)$ and $E(G)$ are finite. A graph that is not finite is called infinite.The number of vertices of a graph is called the order of G and the number of edges of G is called the size of G.

Definition 1.6.6. [7] A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n. On the other hand,a graph with no edges is called a totally disconnected graph.

Definition 1.6.7. [7] A graph is trivial if its vertex set is a singleton and it contains no edges. A graph is bipartite if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y.The pair (X, Y) is called a bipartition of the bipartite graph. The bipartite graph with bipartition (X, Y) is denoted by $G(X, Y)$.A simple bipartite graph is complete if each vertex of X is adjacent to all vertices of Y. If $G(X,Y)$ is complete with $\mid X \mid = p$ and $\mid Y \mid = q$, then $G(X,Y)$ is denoted by $K_{p,q}$. A complete bipartite graph of the form $K_{1,q}$ is called a star.

Theorem 1.6.8. *[23] A graph is bipartite if and only if it contains no odd cycles.*

Definition 1.6.9. [49] A graph H is called a subgraph of G if all the vertices and all the edges of H are in G and each edge of H has the same end vertices in H as in G.

Definition 1.6.10. [7] A subgraph H of G is said to be an induced subgraph of G if each edge of G having its ends in vertex set of H is also an edge of H. The induced subgraph of G with vertex $S \subseteq V(G)$ is called the subgraph of G induced by S.

Definition 1.6.11. [7] A clique of G is a complete subgraph of G. A clique of G is a maximal clique of G if it is not properly contained in another clique of G. The order of a maximum clique of G is called the clique number of G and is denoted by $\omega(G)$.

Definition 1.6.12. [7] A walk in a graph G is an alternating sequence denoted as $W : v_0 e_1 v_1 e_2 v_2 \cdots e_n v_n$ of vertices and edges beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i.A walk is called a path if all the vertices are distinct.A cycle is a closed path.The length of a walk is the number of edges in it.

Definition 1.6.13. [23] The girth of a graph G,denoted as $g(G)$, is the length of the shortest cycle in G ; the circumference $c(G)$ the length of any longest cycle.

Definition 1.6.14. [7] Let G be a graph.Two vertices u and v of G are said to be connected if there is a $u - v$ path in G. The maximal connected subgraphs of G are called components of G.

Definition 1.6.15. [7] Let G be a connected graph and $d(u,v)$ denotes the length of the shortest $u - v$ path.

i) The diameter of G is defined as $\max\{d(u, v) : u, v \in V(G)\}$ and is denoted by $diam(G)$.

ii) If v is a vertex of G, its eccentricity $e(v)$ is defined by $e(v) = max\{d(u, v) : u \in V(G)\}$.

iii) The radius of G, $r(G)$, is the minimum eccentricity of G, that is , $r(G) = min\{e(v) : v \in V(G)\}$.

Definition 1.6.16. [7] A vertex colouring of G with vertex set V is a map $f : V \to S$, where S is a set of distinct colours;it is proper if adjacent vertices of G recieve distinct colours of S; that is,$uv \in E(G)$, then $f(u) \neq f(v)$.

Definition 1.6.17. [7] The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colours needed for a proper vertex colouring of G. G is k-chromatic ,if $\chi(G) = k$.

Definition 1.6.18. [7] A k-colouring of a graph G is a vertex colouring of G that uses k-colours.

Theorem 1.6.19. *[34] Let G be a non empty graph . Then $\chi(G) = 2$ if and only if G is bipartite.*

Theorem 1.6.20. *[34] Let G be a graph .Then $\chi(G) \geqslant 3$ if and only if G has an odd cycle.*

Definition 1.6.21. [49] A set of vertices in a graph is said to be an independent set of vertices or simply an independent set if no two vertices in the set are adjacent.The number of vertices in the largest independent set of a graph G is called the independence number, $\beta(G)$.

Chapter 2

The Box $\mathfrak{S}$, stack of filters $\mathfrak{S}_x$ and the Category Batch

This chapter presents the novel idea of Batch. The filter F_x on a locale L forms the centre of study.Through this filter we develop the notions of regular filter ,associated filter,F-continuity on L-slices and R-A slice.The idea of associated filter is further developed to construct Box $\mathfrak{S}$ and stack of filters $\mathfrak{S}_x$ on L-slice. We study the sections on Box $\mathfrak{S}$ and their properties are investigated .The Box $\mathfrak{S}$ leads to the concept of Batch and Batch morphisms which are later shown to form a category **Batch** with objects as Batches and morphism class as Batch morphisms . Let L be a locale with top element 1_L and J be a join –semilattice with bottom element 0_J . On the locale L, we have the definition of the L-slice (σ, J) as follows :

Definition 2.0.1. [58] Let L be a locale and J be join semilattice with bottom element 0_J. By the "action of L on J"we mean a function $\sigma : L \times J \rightarrow J$ such that the following conditions are satisfied.

i. $\sigma(a, x_1 \vee x_2) = \sigma(a, x_1) \vee \sigma(a, x_2)$ for all $a \in L, x_1, x_2 \in J$.

ii. $\sigma(a, 0_J) = 0_J$ for all $a \in L$.

iii. $\sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x))$ for all $a, b \in L, x \in J$.

iv. $\sigma(1_L, x) = x$ and $\sigma(0_L, x) = 0_J$ for all $x \in J$.

v. $\sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x)$ for $a, b \in L, x \in J$.

If σ is an action of the locale L on a join semilattice J, then we call (σ, J) as L-slice. For each $x \in (\sigma, J)$, $F_x = \{a \in L : \sigma(a, x) = x\}$ is a filter on L. Any filter F on a locale L is said to be trivial if F= $\{1_L\}$

2.1. Regular Filter F, Idle points with respect to F and $Knot_F$

Definition 2.1.1. A filter F on a locale L is said to regularise $x \in (\sigma, J)$, if $F \cap F_x$ is a nontrivial filter. Then F is called the regular filter of x and x is called the knot point of F.

Definition 2.1.2. For a regular filter F of x, the elements in $F \cap F_x$ are called the idle points of F with respect to the action σ on $x \in (\sigma, J)$. The intersection of any two filters is again a filter . Thus the set $F \cap F_x$ of idle points of F form a filter on L.

Proposition 2.1.3. *Let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism and let F regularise $x \in (\sigma, J)$ then F regularise $f(x) \in (\mu, K)$.*

Proof. Let F regularise $x \in (\sigma, J)$. Then there exist $a \neq 1_L$ in F such that $\sigma(a, x) = x$. Also, $f(\sigma(a, x)) = f(x)$ implies $\mu(a, f(x)) = f(x)$. That is, $a \in F_{f(x)}$ and hence $F \cap F_{f(x)}$ is nontrivial. Thus F regularise $f(x)$. □

Theorem 2.1.4. *Let $f : (\sigma, J) \to (\mu, K)$ be a one-one L-slice homomorphism then F regularise $x \in (\sigma, J)$ if and only if F regularise $f(x) \in (\mu, K)$.*

Proof. From proposition 2.1.3 it follows that if F regularise $x \in (\sigma, J)$ then F regularise $f(x) \in (\mu, K)$. To prove the converse part, suppose that F regularise $f(x)$. We have $F \cap F_{f(x)}$ nontrivial . Let $s \in F_{f(x)} \cap F$

$$\begin{aligned} s \in F_{f(x)} &\Rightarrow \mu(s, f(x)) = f(x) \\ &\Rightarrow f(\sigma(s, x)) = f(x) \\ &\Rightarrow \sigma(s, x) = x \\ &\Rightarrow s \in F \cap F_x \end{aligned}$$

Thus F regularise $x \in (\sigma, J)$. □

Definition 2.1.5. Let F and G be any two regular filters at $x \in (\sigma, J)$. We define $F \sim_x G$, if F and G have the same idle points at $x \in (\sigma, J)$. In other words, $F \sim_x G$ if and only if $F_x \cap F = F_x \cap G$.

Proposition 2.1.6. *Let (σ, J) be an L-slice and $x \in (\sigma, J)$. Then the relation $\sim_x$ on all regular filters at x is an equivalence relation.*

Proof. Since $F \sim_x F$, the relation is reflexive. $F \sim_x G$ if and only if $G \sim_x F$, hence symmetric. Also, if $F \sim_x G$ then $F \cap F_x = G \cap F_x$ and $G \sim_x H$, implies $G \cap F_x = H \cap F_x$. Thus the relation is transitive. □

Remark. For each $x \in (\sigma, J)$, the regular filters at x is partitioned into equivalence classes with respect to the set of same idle points each filter generates.

2.2. $Knot_F$ for a filter F

Definition 2.2.1. For a filter F on the locale L, we define the collection of knot points of filter F as the set $Knot_F = \{x \in (\sigma, J) : F \; regularises \; x\}$.

Note that since the filter $F_{0_J} = L$, $Knot_F$ is nonempty .

Proposition 2.2.2. *If $F \subseteq G$ then $Knot_F \subseteq Knot_G$.*

Proof. If $x \in Knot_F$ then $F \cap F_x$ is nontrivial implies $G \cap F_x$ is also nontrivial. Therefore $x \in Knot_G$. □

Proposition 2.2.3. *If F and G be filters on the locale L then we have*

$Knot_{F \cap G} \subseteq Knot_F \cap Knot_G$.

Proof. If $x \in Knot_{F \cap G}$ then $(F \cap G) \cap F_x$ is nontrivial. Therefore there exists $1_L \neq b \in (F_x \cap F) \cap G$ and hence $F_x \cap F$ is nontrivial. Thus $x \in Knot_F$.

Similarly, $(F_x \cap F) \cap G = F \cap (F_x \cap G)$ is nontrivial implying that $x \in Knot_G$. □

Definition 2.2.4. A subset J' of the L-slice (σ, J) is said to be a semislice if $\sigma(a, x) \in J'$, for $x \in J'$ and any $a \in L$.

Theorem 2.2.5. *For any filter F on the locale L, $Knot_F$ is a semislice of the L-slice (σ, J).*

Proof. For $x \in Knot_F$, $F_x \cap F$ is nontrivial. Let $b \in F_x \cap F$ and $a \in L$, then $\sigma(b, \sigma(a, x)) = \sigma(a, \sigma(b, x)) = \sigma(a, x)$. Thus $b \in F_{\sigma(a,x)}$ and F regularise $\sigma(a, x)$. Therefore $\sigma(a, x) \in Knot_F$ for every $a \in L$. □

2.3. Associated Filter

Definition 2.3.1. Consider a filter F on L.

$a)$ Let $x \in (\sigma, J)$ be such that $F_x \subseteq F$. Then F is called the associated filter of $x \in (\sigma, J)$ and the pair (F, x) is the associated filter element with respect to x.

$b)$Also, if we impose an additional condition that $F_x \neq 1_L$ and $F_x \subseteq F$, then such a filter F is said to be strongly associated to x.

Remark. From now onwards, the tuple (F, x) would suggest that F is an associated filter of the element $x \in (\sigma, J)$ and will be addressed as the associated filter element of x.

Examples 2.3.2. *i) Consider the meet L-slice $(\sqcap, L)$. Then a filter F is said to be associated with x, if $\uparrow x \subseteq F$.*

ii)$F_{0_J} = L$, for $0_J \in (\sigma, J)$.Thus the only associated filter of 0_J is the locale L.

We have the following observations on $(\sqcap, L)$

Observation. *Consider the L-slice $(\sqcap, L)$ and $x \in (\sigma, J)$ with $x \leqslant y$. For any filter F on L , if (F, x), then (F, y).*

Proof. Since $(F, x), \uparrow x \subseteq F$. Also, $x \leqslant y$ implies $\uparrow y \subseteq \uparrow x$. Hence $\uparrow y \subseteq F$. $\square$

Observation. *If (F, x) and (G, y) in L-slice $(\sqcap, L)$ then $(F \cap G, x \vee y)$.*

We now generalise the above results for any L-slice (σ, J).

Lemma 2.3.3. *For any two filters F, G on locale L and $x \in (\sigma, J)$, if (F, x) and (G, x) then $(F \cap G, x)$.*

Proof. Since F and G are associated filters of x, we have $F_x \subseteq F$ and $F_x \subseteq G$. Thus $F_x \subseteq F \cap G$ and hence $(F \cap G, x)$. $\square$

Lemma 2.3.4. *Let F and G be filters on locale L with $F \subseteq G$. If (F, x) then (G, x) for some $x \in (\sigma, J)$.*

Proof. (F, x) implies $F_x \subseteq F$. Since $F \subseteq G$ and $F_x \subseteq F$, we have (G, x). □

Lemma 2.3.5. *Every strongly associated filter of $x \in (\sigma, J)$ regularise x.*

Proof. Let F be a strongly associated filter of $x \in (\sigma, J)$. Then F_x is nontrivial and $F_x \subseteq F$. Hence $F_x = F \cap F_x$ is nontrivial. □

Definition 2.3.6. For $x \in (\sigma, J)$, we define $[F : x]_L$ to be the set of all filters of L that are associated with x.

Theorem 2.3.7. *Let $\mathfrak{P}(L)$ denote the power set locale of a locale L, then $[F : x]_L$ is a filter on $\mathfrak{P}(L)$.*

Proof. Follows from the lemma 2.3.3 and lemma 2.3.4 □

Proposition 2.3.8. *If $\sigma_b : (\sigma, J) \to (\sigma, J)$ is one-one and (F, x) then $(F, \sigma(b, x))$.*

Proof. For $a \in F_{\sigma(b,x)}$, we have $\sigma(a, \sigma(b, x)) = \sigma(b, x)$

$$\begin{aligned}
\sigma(a, \sigma(b, x)) = \sigma(b, x) &\Rightarrow \sigma(a \sqcap b, x) = \sigma(b, x) \\
&\Rightarrow \sigma(b, \sigma(a, x)) = \sigma(b, x) \\
&\Rightarrow \sigma_b(\sigma(a, x)) = \sigma_b(x) \\
&\Rightarrow \sigma(a, x) = x \\
&\Rightarrow a \in F_x
\end{aligned}$$

Thus $F_{\sigma(b,x)} \subseteq F_x \subseteq F$. □

Theorem 2.3.9. *Let $f : (\sigma, J) \to (\mu, K)$ be a one-one slice homomorphism and if a filter F on L is strongly associated to $x \in (\sigma, J)$ then F is strongly associated to $f(x)$.*

Proof. A filter F on L is strongly associated to $x \in (\sigma, J)$ implies F_x is nontrivial and $F_x \subseteq F$. Since f is a slice homomorphism, $F_{f(x)}$ is nontrivial. Now,

$$\begin{aligned} r \in F_{f(x)} &\Rightarrow \mu(r, f(x)) = f(x) \\ &\Rightarrow f(\sigma(r, x)) = f(x) \\ &\Rightarrow \sigma(r, x) = x \\ &\Rightarrow r \in F_x \end{aligned}$$

That is $F_{f(x)} \subseteq F_x \subseteq F$. Thus F is strongly associated to $f(x)$. □

We observe from Theorem 2.3.7 that for each $x \in (\sigma, J)$, the collection of all associated filters form a filter on the power locale $\mathfrak{P}(L)$.

2.4. Continuity of L-slice morphisms with respect to associated filters

This section deals with continuity of L-slice morphisms in terms of associated filters. Classical topology uses the notion of sequences to define the continuity of functions on a topological space. On similar lines , we use associated filters to define the continuity of a slice morphism .

Definition 2.4.1. A slice homomorphism $f : (\sigma, J) \to (\mu, K)$ is said to be semi-continuous at $x \in (\sigma, J)$, if for any filter F associated to x implies F is associated to $f(x)$. A slice homomorphism is said to be semi-continuous on (σ, J), if it is semi-continuous at every $x \in (\sigma, J)$.

Remark. Every slice isomorphism is semi-continuous on (σ, J).

Theorem 2.4.2. *Composition of semi-continuous slice morphism is semi-continuous.*

Proof. Consider any two semi-continuous slice morphisms $f : (\sigma, J) \to (\mu, K)$ and $g : (\mu, K) \to (\delta, M)$. Let (F, x) be an associated filter element. Since f is semi-continous, F associated to $f(x)$. Also, the semi-continuity of g ensures that F is associated to $g(f(x))$. Thus F is associated to $(g \circ f)(x)$. □

Definition 2.4.3. A slice homomorphism $f : (\sigma, J) \to (\mu, K)$ is said to be F-continuous at $x \in (\sigma, J)$ if $F_{f(x)} \subseteq F_x$. A slice morphism is said to be continuous on (σ, J) if it is continuous at every $x \in (\sigma, J)$.

Remark. For the meet slices $(\sqcap, L)$ and $(\sqcap, M)$,continous L-slice homomorphism from $f : (\sqcap, L) \to (\sqcap, M)$ is precisely identity morphisms.

Proposition 2.4.4. *Every F-continuous slice morphism is semi-continuous.*

Proof. Let $f : (\sigma, J) \to (\mu, K)$ be F-continuous at some $x \in (\sigma, J)$. i.e, $F_{f(x)} \subseteq F_x$. Consider an associated filter element (F, x). Then $F_{f(x)} \subseteq F_x \subseteq F$ implies that $(F, f(x))$. Hence f is semi-continous at $x \in (\sigma, J)$. □

Theorem 2.4.5. *If $f : (\sigma, J) \to (\mu, K)$ is F-continuous and F regularise $f(x) \in (\mu, K)$ then F regularise $x \in (\sigma, J)$.*

Proof. F regularise $f(x)$ implies $F \cap F_{f(x)}$ is nontrivial. The F-continuity of f shows that $F_{f(x)} \subseteq F_x$. Hence F regularise x. □

Theorem 2.4.6. *Composition of two F-continous slice morphisms on (σ, J) is F-continous.*

Proof. Consider any two F-continuous slice morphisms $f : (\sigma, J) \to (\mu, K)$ and $g : (\mu, K) \to (\delta, M)$, then $F_{f(x)} \subseteq F_x$ and $F_{g(f(x))} \subseteq F_{f(x)}$, for some $x \in (\sigma, J)$. Thus $g \circ f$ is F-continous at $x \in (\sigma, J)$. □

We have the following results which are obvious.

Theorem 2.4.7. *Let $f : (\sigma, J) \to (\mu, K)$ be bijective L-slice morphism, then $f^{-1} : (\mu, K) \to (\sigma, J)$ is F-continous.*

Theorem 2.4.8. *If $f : (\sigma, J) \to (\mu, K)$ is a F-continuous morphism of L-slices and x is a compact element of (σ, J) then f(x) is a compact element of (μ, K).*

Proof. Let there exist a collection $\{a_\alpha : \alpha \in I,\ for\ some\ indexed\ set\ I\}$ elements of the locale L, such that $\mu(\sqcup a_\alpha, f(x)) = f(x)$. Then $\sqcup a_\alpha \in F_{f(x)} \subseteq F_x$ which implies $\sigma(\sqcup a_\alpha, x) = x$. Since x is a compact element of (σ, J), we can find a finite sub collection $\{a_1, a_2, a_3.....a_n\}$ such that $\sigma(\sqcup a_n, x) = x$. Thus $f(\sigma(\sqcup a_n, x)) = f(x)$ implies $\mu(\sqcup a_n, f(x)) = f(x)$, showing that $f(x)$ is a compact element of (μ, K). □

2.5. R-A slice

Definition 2.5.1. A L-slice (σ, J) is said to be a R-A slice if every regular filter at x is associated to x.

Example 2.5.2. *If (σ, J) is an L-slice having the property that for each $x \in (\sigma, J)$, $\sigma_x : (\sqcap, L) \to (\sigma, J)$ is one-one then (σ, J) is a R-A slice.*

Theorem 2.5.3. *Let $f : (\sigma, J) \to (\mu, K)$ be a slice morphism and (μ, K) be a R-A slice. If F_x is nontrivial for some $x \in (\sigma, J)$ then f is semi-continuous at x.*

Proof. Let F be associated to x. Then F regularise x. Since f is a slice morphism F regularises $f(x)$. Also (μ, K) is a R-A slice would imply that F is associated to $f(x)$. Thus f is semi- continuous at x. □

Theorem 2.5.4. *Let $f : (\sigma, J) \to (\mu, K)$ be a F-continuous L-slice morphism and let (σ, J) be a R- A slice, then (μ, K) is also a R-A slice.*

Proof. Let F regularise $f(x)$, then $F_{f(x)} \cap F$ is nontrivial. Since f is continous at x, we have $F_{f(x)} \subseteq F_x$ which impiles $F_x \cap F$ is nontrivial. Hence F regularise x and F is associated to x. Thus $F_{f(x)} \subseteq F_x \subseteq F$ implies that F associated to $f(x)$. So (μ, K) is also a R-A slice. □

2.6. The Box $\mathfrak{S}$ of associated filter elements and stack of filters $\mathfrak{S}_x$

Definition 2.6.1. *a*) Let $\mathfrak{F}$ denote the collection of filters on the locale L. Then the Box $\mathfrak{S} \subseteq \mathfrak{F} \times (\sigma, J)$ is defined as $\mathfrak{S} = \{(F, x) : F \textit{ is associated to } x\}$. In other words, $\mathfrak{S}$ is the collection of all associated filter elements.
b) We define a projection map on the Box $\mathfrak{S}$ as $\pi : \mathfrak{S} \to (\sigma, J)$ such that $\pi(F, x) = x$. The inverse image $\pi^{-1}(x)$ is the collection of all associated filter elements of x.

As a subobject of the box $\mathfrak{S}$, we define sub-box.

Definition 2.6.2. $\mathfrak{S}'$ is said to be a sub-box of $\mathfrak{S}$ if

i) $\mathfrak{S}' \subseteq \mathfrak{S}$

ii) $\pi(\mathfrak{S}') = (\sigma, J)$

Example 2.6.3. *If we consider $\mathfrak{S}' = \{(F_x, x); x \in (\sigma, J)\}$ then $\mathfrak{S}' \subseteq \mathfrak{S}$ and $\pi(\mathfrak{S}') = (\sigma, J)$*

We investigate the structure of the box $\mathfrak{S}$.

Proposition 2.6.4. *If we define a relation* $\leqslant_{\mathfrak{S}}$ *as* $(F,x) \leqslant_{\mathfrak{S}} (G,y)$ *if* $F \supseteq G$ *and* $x \leqslant y$*, then* $(\mathfrak{S}, \leqslant_{\mathfrak{S}})$ *is a join semilattice.*

Proof. The relation $\leqslant_{\mathfrak{S}}$ is reflexive follows immediately. Suppose $(F,x) \leqslant_{\mathfrak{S}} (G,y)$ and $(G,y) \leqslant_{\mathfrak{S}} (F,x)$, then we have $F \supseteq G$, $G \supseteq F$ and $x \leqslant y, y \leqslant x$. Thus $F = G$ and $x = y$ would imply antisymmetry. Now $(F,x) \leqslant_{\mathfrak{S}} (G,y)$ and $(G,y) \leqslant_{\mathfrak{S}} (H,z)$ would give $F \supseteq G, x \leqslant y$, and $G \supseteq H, y \leqslant z$. Hence $(F,x) \leqslant_{\mathfrak{S}} (H,z)$ proves transitivity. Therefore $(\mathfrak{S}, \leqslant_{\mathfrak{S}})$ is a poset. Also the join can be defined as $(F,x) \sqcup_{\mathfrak{S}} (G,y) = (F \cap G, x \vee y)$. Hence $(\mathfrak{S}, \leqslant_{\mathfrak{S}}, \sqcup_{\mathfrak{S}})$ is a join semilattice. □

2.7. The stack of filters $\mathfrak{S}_x$

Definition 2.7.1. For each $x \in (\sigma, J)$, $\pi^{-1}(x)$ provides us a stack of filters of L over (σ, J) which are associated with x. The stack of filters at x denoted by $\mathfrak{S}_x = \{F \in L : (F,x) \in \pi^{-1}(x)\}$

Observation. *Also by the proposition 2.6.4, the stack of filters at* x *would be a join semilattice with bottom element* (L,x)*.*

Observation. *Algebraically,* $\mathfrak{S}_x$ *has the structure of a commutative idempotent monoid.*

Some Properties of the map π

i) Since there can be more than one filter associated to an element $x \in (\sigma, J)$, generally π is not one-one.

ii) π is a surjective map.

iii) If x is join irreducible compact element, then $\pi/\pi^{-1}(x)$ is injective.

Proof: For any join irreducible and compact element x of (σ, J), the filter F_x is completely prime filter. All completely prime filters are maximal. Hence the only proper filter that is associated to x is F_x.

In the next section we observe that the Box $\mathfrak{S}$ can be transformed into a join semilattice on which an action of the locale L can be defined and thus modelling it into an L-slice.

Definition 2.7.2. The map π arranges the Box $\mathfrak{S}$ into stacks of filters $\mathfrak{S}_x$ at x. The Box can now be viewed as $\mathfrak{S}_{(\sigma,J)} = \{\mathfrak{S}_x : x \in (\sigma, J)\}$.

Lemma 2.7.3. $\mathfrak{S}_{(\sigma,J)}$ *is a join semilattice.*

Proof. We partially order $\mathfrak{S}_{(\sigma,J)}$ as $\mathfrak{S}_x \leqslant'_{\mathfrak{S}} \mathfrak{S}_y$ if and only if $x \leqslant y$, for $x, y \in (\sigma, J)$ Consequently, $\leqslant'_{\mathfrak{S}}$ is a partial order and the join is defined as $\mathfrak{S}_x \sqcup'_{\mathfrak{S}} \mathfrak{S}_y = \mathfrak{S}_{x \vee y}$. Also $\mathfrak{S}_{0_J}$ is the bottom element of $\mathfrak{S}_{(\sigma,J)}$ with respect to $\leqslant'_{\mathfrak{S}}$. □

Thus the Box $\mathfrak{S}$ is remodelled into the ordered Box $\mathfrak{S}_{(\sigma,J)}$. For each $x \in (\sigma, J)$ the ordered Box $\mathfrak{S}_{(\sigma,J)}$ gives the collection of all those associated filters at x.

Theorem 2.7.4. *The map* $\lambda : L \times \mathfrak{S}_{(\sigma,J)} \to \mathfrak{S}_{(\sigma,J)}$ *defined as* $\lambda(a, \mathfrak{S}_x) = \mathfrak{S}_{\sigma(a,x)}$ *is an action on* $\mathfrak{S}_{(\sigma,J)}$ *and* $(\lambda, \mathfrak{S}_{(\sigma,J)})$ *is an L-slice.*

Proof. We examine all the properties of L-slice.

i) $\lambda(a \sqcup b, \mathfrak{S}_x) = \mathfrak{S}_{\sigma(a \sqcup b,x)}$

$= \mathfrak{S}_{\sigma(a,x) \vee \sigma(b,x)}$

$= \mathfrak{S}_{\sigma(a,x)} \sqcup'_{\mathfrak{S}} \mathfrak{S}_{\sigma(b,x)}$

$= \lambda(a, \mathfrak{S}_x) \sqcup'_{\mathfrak{S}} \lambda(b, \mathfrak{S}_x)$

ii) $\lambda(a, \mathfrak{S}_x \sqcup'_{\mathfrak{S}} \mathfrak{S}_x) = \lambda(a, \mathfrak{S}_{x \vee y})$

$= \mathfrak{S}_{\sigma(a,x \vee y)}$

$= \mathfrak{S}_{\sigma(a,x) \vee \sigma(a,y)}$

$= \lambda(a, \mathfrak{S}_x) \sqcup'_{\mathfrak{S}} \lambda(a, \mathfrak{S}_y)$

iii) $\lambda(a, \mathfrak{S}_{0_J}) = \mathfrak{S}_{\sigma(a,0_J)}$

$= \mathfrak{S}_{0_J}$

iv) $\lambda(1_L, \mathfrak{S}_x) = \mathfrak{S}_{\sigma(1_L,x)}$

$= \mathfrak{S}_x$

and

$\lambda(0_L, \mathfrak{S}_x) = \mathfrak{S}_{\sigma(0_L,x)}$

$= \mathfrak{S}_{0_J}$

v) $\lambda(a \sqcap b, \mathfrak{S}_x) = \mathfrak{S}_{\sigma(a \sqcap b,x)}$

$= \mathfrak{S}_{\sigma(a,\sigma(b,x))}$

$= \lambda(a, \mathfrak{S}_{\sigma(b,x)})$

$= \lambda(a, \lambda(b, \mathfrak{S}_x))$

Similarly $\lambda(a \sqcap b, \mathfrak{S}_x) = \lambda(b, \lambda(a, \mathfrak{S}_x))$

Thus $(\lambda, \mathfrak{S}_{(\sigma,J)})$ is an L-slice. $\square$

The next section deals with some constructive properties of the stack of filters $\mathfrak{S}_x$. Note that whenever we study the properties of elements of $\mathfrak{S}_x$, we look at it as an associated filter of x rather than as an associated filter element with respect to x.

2.8. Extending the stack $\mathfrak{S}_x$ to a filter on $\mathfrak{F}$

For $x \in (\sigma, J)$ we observe that any fixed member of the stack of filters $\mathfrak{S}_x$ can be extended to another filter that is associated to x. Fix $F \in \mathfrak{S}_x$ and $a \in L$. Define a set $\langle F|a\rangle = \{b \in L : a \vee b \in F\}$.

Theorem 2.8.1. *$\langle F|a\rangle$ is an associated filter of x and consequently $\langle F|a\rangle \in \mathfrak{S}_x$.*

Proof. First we prove that $\langle F|a\rangle$ is a filter. Let $b_1, b_2 \in \langle F|a\rangle$, then $a \vee b_1 \in F$ and $a \vee b_2 \in F$ implies $(a \vee b_1) \wedge (a \vee b_2) \in F$.
Hence $a \vee (b_1 \wedge b_2) \in F$ would imply $b_1 \wedge b_2 \in \langle F|a\rangle$. Thus $\langle F|a\rangle$ is a meet semilattice. Let $b \in \langle F|a\rangle$ and $b \leqslant c$. Since $b \in \langle F|a\rangle, a \vee b \in F$. Also $b \leqslant c$ implies $a \vee b \leqslant a \vee c$. Because F is a filter and $a \vee b \in F$ would imply $a \vee c \in F$. Thus $c \in \langle F|a\rangle$. Hence $\langle F|a\rangle$ is a upperset of L. Also F being a filter ensures us that $F \subseteq \langle F|a\rangle$ and hence $\langle F|a\rangle \in \mathfrak{S}_x$. $\square$

Thus any member of the stack $\mathfrak{S}_x$ is expanded to a larger one in the same stack. Now we extend the stack $\mathfrak{S}_x$ to a filter on $\mathfrak{F}$. Fix $x \in (\sigma, J)$, $G \in \mathfrak{S}_x$ and define a collection $(\mathfrak{S}_x)_G = \{F \in \mathfrak{S}_x : F \cap G \in \mathfrak{S}_x\}$.

Theorem 2.8.2. *$(\mathfrak{S}_x)_G$ is a filter on F containing $\mathfrak{S}_x$.*

Proof. First we prove that $(\mathfrak{S}_x)_G$ is a filter on F.
$(\mathfrak{S}_x)_G$ is nonempty since $F_x \in (\mathfrak{S}_x)_G$.

$H_1, H_2 \in (\mathfrak{S}_x)_G$ implies $H_1 \cap G \in \mathfrak{S}_x$ and $H_1 \cap G \in \mathfrak{S}_x$. Therefore $(H_1 \cap H_2) \cap G \in \mathfrak{S}_x$ and hence $(\mathfrak{S}_x)_G$ is a meet semilattice.

Suppose $H \in (\mathfrak{S}_x)_G$ and $H \subseteq K$. Then $F_x \subseteq H \cap G \subseteq K \cap G$ implies $K \cap G \in \mathfrak{S}_x$. Therefore $K \in (\mathfrak{S}_x)_G$ and hence $(\mathfrak{S}_x)_G$ is filter on $\mathfrak{F}$.

Also for every $H \in (\mathfrak{S}_x)_G$, $H \cap G \in \mathfrak{S}_x$ implies that $\mathfrak{S}_x \subseteq (\mathfrak{S}_x)_G$. □

We call $(\mathfrak{S}_x)_G$ as the extended stack. We make a few observations on the extended stack $(\mathfrak{S}_x)_G$

i) $(\mathfrak{S}_x)_L = \mathfrak{S}_x = (\mathfrak{S}_x)_{F_x}$

ii) If $H \subseteq G$, then $(\mathfrak{S}_x)_H \subseteq (\mathfrak{S}_x)_G$

iii) If $x \in (\sigma, J)$ and $F_x = \{1_L\}$, then $(\mathfrak{S}_x)_G = \mathfrak{F}$.

Proposition 2.8.3. *For $G_1, G_2 \in \mathfrak{S}_x, (\mathfrak{S}_x)_{G_1} \cap (\mathfrak{S}_x)_{G_2} = (\mathfrak{S}_x)_{G_1 \cap G_2}$.*

Proof. $H \in (\mathfrak{S}_x)_{G_1} \cap (\mathfrak{S}_x)_{G_2}$ gives $H \cap G_1 \in \mathfrak{S}_x$ and $H \cap G_2 \in \mathfrak{S}_x$.

Then $F_x \subseteq (H \cap G_1) \cap (H \cap G_2) = H \cap (G_1 \cap G_2)$. Therefore $H \in (\mathfrak{S}_x)_{G_1 \cap G_2}$.

Now, let $K \in (\mathfrak{S}_x)_{G_1 \cap G_2}$ then $F_x \subseteq K \cap (G_1 \cap G_2) = (K \cap G_1) \cap (K \cap G_2)$.

Hence $F_x \subseteq (K \cap G_1)$ and $F_x \subseteq (K \cap G_2)$. Consequently, $K \in (\mathfrak{S}_x)_{G_1} \cap (\mathfrak{S}_x)_{G_2}$. □

Lemma 2.8.4. *If $\mathfrak{F}$ is the collection of all filters on locale L then $\mathfrak{S}_x$ and $\mathfrak{F}$ are join semilattices .*

Proof. Partially order $\mathfrak{S}_x$ as $F \leqslant_{\mathfrak{S}_x} G$ if and only if $F \supseteq G$. Then $\mathfrak{S}_x$ is a join semilatice with $F \sqcup_{\mathfrak{S}_x} G = F \cap G$. Similarly $\mathfrak{F}$ can be ordered with $\leqslant^*$ such that $F_1 \leqslant^* F_2$ if and only if $F_1 \supseteq F_2$, so that $\mathfrak{F}$ is a join semilattice with $F_1 \sqcup' F_2 = F_1 \cap F_2$. □

Theorem 2.8.5. *The map* $Fil : \mathfrak{S}_x \to \mathfrak{F}$ *defined as* $Fil(G) = (\mathfrak{S}_x)_G$ *is a join semilattice homomorphism.*

Proof. The above proposition shows that $Fil(G_1 \sqcup_{\mathfrak{S}_x} G_2) = Fil(G_1) \sqcup' Fil(G_2)$. Thus Fil is a join semilattice homomorphism. □

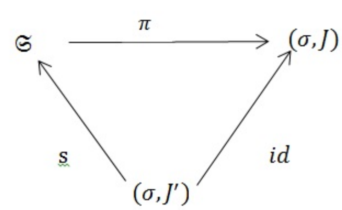
Remark. Consequently the extended stack $(\mathfrak{S}_x)_G$, for a fixed $G \in \mathfrak{S}_x$, can also be viewed as join semilattice homomorphism on $\mathfrak{S}_x$.

The next section deals with maps called section on Box $\mathfrak{S}$.

2.9. Section on the Box $\mathfrak{S}$

Definition 2.9.1. Let (σ, J') be a subslice of (σ, J). The map $s : (\sigma, J') \to \mathfrak{S}$ is said to be a section if

i) The diagram commutes



where id is the inclusion map.

ii. $\tilde{s} : (\sigma, J') \to \mathfrak{S}_{(\sigma,J)}$ defined as $\tilde{s}(x) = \pi^{-1}(\pi \circ s)(x)$ is a slice morphism. Alternatively, $\tilde{s} \in Hom((\sigma, J'), (\lambda, \mathfrak{S}_{(\sigma,J)}))$

Theorem 2.9.2. *Let* (σ, W_1) *and* (σ, W_2) *be two subslices of* (σ, J)*. For a Box* $\mathfrak{S}$ *over* (σ, J) *and* s_1, s_2 *any two sections on* $\mathfrak{S}$ *if we define* $W_0 = \{x \in W_1 \cap W_2 : \tilde{s_1}(x) = \tilde{s_2}(x)\}$*, then* (σ, W_0) *is also a subslice of* (σ, J)*.*

Proof. Let $x_1, x_2 \in W_0$.

$$\begin{aligned}\tilde{s_1}(x_1 \vee x_2) &= \tilde{s_1}(x_1) \vee \tilde{s_1}(x_2) \\ &= \tilde{s_2}(x_1) \vee \tilde{s_2}(x_2) \\ &= \tilde{s_2}(x_1 \vee x_2)\end{aligned}$$

Therefore, $x_1 \vee x_2 \in W_0$.

Suppose, $x \in W_0$ and $a \in L$, then

$$\begin{aligned}\tilde{s_1}(\sigma(a, x)) &= \lambda(a, \tilde{s_1}(x)) \\ &= \lambda(a, \tilde{s_2}(x)) \\ &= \tilde{s_2}(\sigma(a, x))\end{aligned}$$

Therefore $\sigma(a, x) \in W_0$. Hence (σ, W_0) is a subslice. $\square$

We further investigate the structure of collection of all sections on the Box $\mathfrak{S}$.

Definition 2.9.3. Define $\Gamma((\sigma, J'), \mathfrak{S})$ to be the set of all sections on the Box $\mathfrak{S}$. Partially order $\Gamma((\sigma, J'), \mathfrak{S})$ as $s \preceq s'$ if and only if $s(x) \leqslant_{\mathfrak{S}} s'(x), \forall x \in (\sigma, J')$.

Theorem 2.9.4. *$\Gamma((\sigma, J'), \mathfrak{S})$ is a join semilattice with bottom element.*

Proof. If $s, s' \in \Gamma((\sigma, J'), \mathfrak{S})$, then $\tilde{s} : (\sigma, J') \to \mathfrak{S}_{(\sigma,J)}$ and $\tilde{s'} : (\sigma, J') \to \mathfrak{S}_{(\sigma,J)}$ are slice morphisms. The join of two slice morphisms is again a slice morphism.

It remains to prove that $s \tilde{\vee} s' = \tilde{s} \vee \tilde{s'}$.

Suppose $s(x) = (F, x)$ and $s'(x) = (G, x)$, then $s(x) \sqcup_{\mathfrak{S}} s'(x) = (F \cap G, x)$.

$s(x) \sqcup_{\mathfrak{S}} s'(x) = (F \cap G, x)$ implies

$$\begin{aligned} s \tilde{\vee} s'(x) &= \pi^{-1}((\pi \circ (s \vee s')(x)) \\ &= \pi^{-1}((\pi(F \cap G, x)) \\ &= \pi^{-1}(x) \\ &= \mathfrak{S}_x \end{aligned}$$

$\tilde{s}(x) = \mathfrak{S}_x = \tilde{s'}(x)$. Therefore $s \tilde{\vee} s'(x) = \tilde{s}(x) \vee \tilde{s'}(x)$ implies $s \tilde{\vee} s'$ is a section.

Define $\mathbf{0}(x) = (L, x), \forall x \in (\sigma, J')$. Now we prove that $\mathbf{0}$ is a section on $\mathfrak{S}$.

$\tilde{\mathbf{0}}(x) = \pi^{-1}((\pi \circ \mathbf{0})(x)) = \mathfrak{S}_x$

$\tilde{\mathbf{0}}(x \vee y) = \mathfrak{S}_{x \vee y} = \mathfrak{S}_x \sqcup'_{\mathfrak{S}} \mathfrak{S}_y = \tilde{\mathbf{0}}(x) \vee \tilde{\mathbf{0}}(y)$

$\tilde{\mathbf{0}}(\sigma(a, x)) = \mathfrak{S}_{\sigma(a,x)} = \lambda(a, \mathfrak{S}_x) = \lambda(a, \tilde{\mathbf{0}}(x))$. $\tilde{\mathbf{0}}$ is a slice morphism and thus $\mathbf{0}$ is a section on $\mathfrak{S}$. Also, $\mathbf{0}(x) \leqslant_{\mathfrak{S}} s(x), \forall x \in (\sigma, J)$. Therefore $\Gamma((\sigma, J'), \mathfrak{S})$ is a join semilattice with bottom element. □

2.10. The Category Batch

This section introduces a new concept called Batch. Batch is based on the Box $\mathfrak{S}$ and the projection map π. We study some of its properties.

Definition 2.10.1. A Batch is a triplet $(\mathfrak{S}, \pi, (\sigma, J))$, where $\mathfrak{S}$ is a Box over the L-slice (σ, J) , π is the projection of $\mathfrak{S}$ to (σ, J). For each $\pi^{-1}(x)$ we obtain the stack of filters of L associated with $x, \mathfrak{S}_x$.

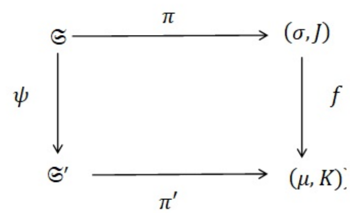
Definition 2.10.2. If $X = (\mathfrak{S}, \pi, (\sigma, J))$ and $Y = (\mathfrak{S}', \pi', (\mu, K))$ are any two Batches, then the morphism between the Batches is defined as a pair (ψ, f) such that

i) $\psi : \mathfrak{S} \to \mathfrak{S}'$ is order preserving

ii) $\psi/\pi^{-1}(x)$ is a bijection from $\pi^{-1}(x)$ to $(\pi')^{-1}(f(x))$

iii) $f : (\sigma, J) \to (\mu, K)$ is a slice morphism

iv) The following diagram commutes.



Definition 2.10.3. Let $\mathfrak{B}_1 = (\mathfrak{S}, \pi, (\sigma, J))$ and $\mathfrak{B}_2 = (\mathfrak{S}', \pi', (\mu, K))$ be two Batches and $(\psi, f) : \mathfrak{B}_1 \to \mathfrak{B}_2$ be the Batch morphism, then we define $Ker\psi = \{(F, x) : \psi(F, x) = (L, 0_K)\}$ and $Kerf = \{x \in (\sigma, J); f(x) = 0_K\}$.

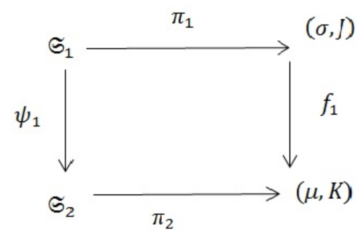
$Im\psi$ denotes the set of image of $\mathfrak{S}$ under ψ and Imf denotes the image of (σ, J) under f.

The composition of two Batch morphisms (ψ_1, f_1) and (ψ_2, f_2) between two batches is defined as $(\psi_1, f_1) \circ (\psi_2, f_2) = (\psi_2 \circ \psi_1, f_2 \circ f_1)$. The next lemma shows that this is again a Batch morphism.

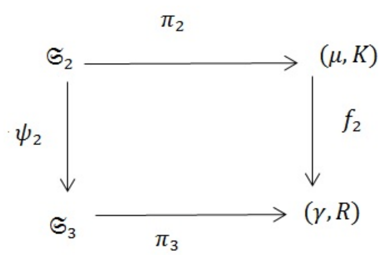
Lemma 2.10.4. *The composition of Batch morphisms is again a Batch morphism.*

Proof. Let $X = (\mathfrak{S}_1, \pi_1, (\sigma, J)), Y = (\mathfrak{S}_2, \pi_2, (\mu, K))$ and $Z = (\mathfrak{S}_3, \pi_3, (\gamma, R))$ be any three Batches. The Batch morphisms from X to Y and Y to Z are (ψ_1, f_1) and (ψ_2, f_2) respectively.

The definition of Batch morphisms provides the following diagrams



such that $\pi_2 \circ \psi_1 = f_1 \circ \pi_1$ and



such that $\pi_3 \circ \psi_2 = f_2 \circ \pi_2$.

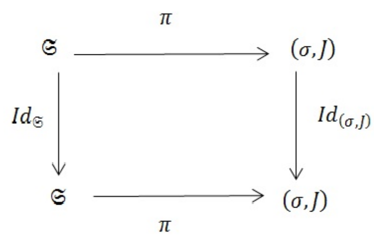
ψ_1 and ψ_2 are order preserving then so is $\psi_2 \circ \psi_1$. Also $\psi_2 \circ \psi_1/\pi_1^{-1}(x)$ is a bijection. Similarly, f_1 and f_2 are slice morphisms implies that $f_2 \circ f_1$ is also a slice morphism. Moreover,

$$\begin{aligned}
\pi_3 \circ (\psi_2 \circ \psi_1) &= (\pi_3 \circ \psi_2) \circ \psi_1 \\
&= (f_2 \circ \pi_2) \circ \psi_1 \\
&= f_2 \circ (\pi_2 \circ \psi_1) \\
&= f_2 \circ (f_1 \circ \pi_1)
\end{aligned}$$

Thus $(\psi_2 \circ \psi_1, f_2 \circ f_1)$is a Batch morphism from X to Y. □

Lemma 2.10.5. *If $Id_{\mathfrak{S}}$ is the identity map on a Box $\mathfrak{S}$ defined as $Id_{\mathfrak{S}}(F, x) = (F, x)$ and $Id_{(\sigma,J)}$ the identity slice morphism, then $(Id_{\mathfrak{S}}, Id_{(\sigma,J)})$ is the identity Batch morphism on $X = (\mathfrak{S}, \pi, (\sigma, J))$.*

Proof. First we prove that $(Id_{\mathfrak{S}}, Id_{(\sigma,J)})$ is a Batch morphism. Since $Id_{\mathfrak{S}}$ is the identity map on $\mathfrak{S}$, it is order preserving. Also, $Id_{\mathfrak{S}}/\pi^{-1}(x) \to \pi^{-1}(Id_{(\sigma,J)}(x))$ is a bijection. Now it remains to prove the commutativity of the following diagram. $(\pi \circ Id_{\mathfrak{S}})(F, x) = \pi(F, x) = x$ and $(Id_{(\sigma,J)} \circ \pi)(F, x) = Id_{(\sigma,J)}(x) = x$.



Therefore the diagram commutes and hence $(Id_{\mathfrak{S}}, Id_{(\sigma,J)})$ is a Batch morphism on X. For any $(\psi, f) : X \to Y$ we have $(Id_{\mathfrak{S}}, Id_{(\sigma,J)}) \circ (\psi, f) = (\psi \circ Id_{\mathfrak{S}}, f \circ Id_{(\sigma,J)}) = (\psi, f)$. Thus $(Id_{\mathfrak{S}}, Id_{(\sigma,J)})$ becomes an identity morphism on X.

□

Theorem 2.10.6. ***Batch** is a category whose objects are Batches, morphisms are Batch morphisms and the composition of morphisms is the composition of Batch morphisms.*

Proof. The above two lemma shows that the collection of Batch morphisms is closed under composition and $(Id_{\mathfrak{S}}, Id_{(\sigma,J)})$ is the identity morphism on a Batch.

We prove that the composition of Batch morphisms is associative.

Let X ,Y and Z be any three Batches. The Batch morphisms between them are $(\psi_1, f_1) : X \to Y$, $(\psi_2, f_2) : Y \to Z$ and $(\psi_3, f_3) : Z \to W$. The composition of order preserving maps as well as slice morphisms are associative . Therefore,

$$\begin{aligned}
(\psi_3, f_3) \circ [(\psi_2, f_2) \circ (\psi_1, f_1)] &= (\psi_3, f_3) \circ [(\psi_2 \circ \psi_1, f_2 \circ f_1)] \\
&= [\psi_3 \circ (\psi_2 \circ \psi_1), f_3 \circ (f_2 \circ f_1)] \\
&= [(\psi_3 \circ \psi_2) \circ \psi_1, (f_3 \circ f_2) \circ f_1] \\
&= (\psi_3 \circ \psi_2, f_3 \circ f_2) \circ (\psi_1, f_1) \\
&= [(\psi_3, f_3) \circ (\psi_2, f_2)] \circ (\psi_1, f_1)
\end{aligned}$$

Thus **Batch** is a category with objects Batches and morphisims Batch morphisms. □

Theorem 2.10.7. *Let* $\mathfrak{B}_1 = (\mathfrak{S}, \pi, (\sigma, J))$ *and* $\mathfrak{B}_2 = (\mathfrak{S}', \pi', (\mu, K))$ *be two Batches and* $(\psi, f) : \mathfrak{B}_1 \to \mathfrak{B}_2$ *be the Batch morphism, then for any* $(F, x) \in \mathfrak{S}, x \in Kerf$ *if and only if* $(F, x) \in Ker\psi$.

Proof.

$$\begin{aligned}
x \in Kerf &\Rightarrow f(x) = 0_K \\
&\Rightarrow f(\pi(F, x)) = 0_K \\
&\Rightarrow (f \circ \pi)(F, x) = 0_K \\
&\Rightarrow (\pi' \circ \psi)[(F, x)] = 0_K \\
&\Rightarrow \pi'(\psi(F, x)) = 0_K \\
&\Rightarrow (\psi(F, x)) = \pi'^{-1}(0_K) \\
&\Rightarrow \psi(F, x) = (L, 0_K)
\end{aligned}$$

Thus, $x \in Kerf$ implies $(F, x) \in Ker\psi$.

For the converse,

$$\begin{aligned}(F, x) \in Ker\psi &\Rightarrow \psi(F, x) = (L, 0_K) \\ &\Rightarrow \pi'(\psi(F, x)) = 0_K \\ &\Rightarrow (\pi' \circ \psi)[(F, x)] = 0_K \\ &\Rightarrow (f \circ \pi)(F, x) = 0_K \\ &\Rightarrow f(x) = 0_K\end{aligned}$$

Consequently, $x \in Kerf$. □

Theorem 2.10.8. *Consider the Batches* $\mathfrak{B}_1 = (\mathfrak{S}, \pi, (\sigma, J))$ *and* $\mathfrak{B}_2 = (\mathfrak{S}', \pi', (\mu, K))$ *and the Batch morphism* $(\psi, f) : \mathfrak{B}_1 \to \mathfrak{B}_2$*. If* f *is injective, then* $|Ker\psi| = 1$*.*

Proof. If f is one-one then $Kerf = \{0_J\}$. Hence the result follows from the above lemma. □

Lemma 2.10.9. *For the Batches* $\mathfrak{B}_1 = (\mathfrak{S}, \pi, (\sigma, J))$ *and* $\mathfrak{B}_2 = (\mathfrak{S}', \pi', (\mu, K))$ *and the Batch morphism* $(\psi, f) : \mathfrak{B}_1 \to \mathfrak{B}_2$*,* $\pi'^{-1}(y) \in Im\psi$ *if and only if* $y \in Imf$*.*

Proof. Suppose that $\pi'^{-1}(y) \in Im\psi$. Then $(G, y) \in \pi'^{-1}(y)$ will imply $(G, y) = \psi(F, x)$, for some $(F, x) \in \mathfrak{S}$ .That is,

$$\begin{aligned}y &= (\pi' \circ \psi)(F, x) \\ &= (f \circ \pi)(F, x) \\ &= f(x)\end{aligned}$$

Thus $y \in Imf$.

For the converse, let $y \in Imf$. Then $y = f(x) = (f \circ \pi)((F, x))$, for some associated filter F of x. Therefore $y = (\pi' \circ \psi)((F, x))$ implies $\psi((F, x)) = \pi'^{-1}(y)$. Thus $\pi'^{-1}(y) \in Im\psi$.

Hence any associated filter of y is an element of $Im\psi$ if and only if $y \in Imf$. $\square$

Theorem 2.10.10. *Consider the Batches* $\mathfrak{B}_1 = (\mathfrak{S}, \pi, (\sigma, J))$ *and* $\mathfrak{B}_2 = (\mathfrak{S}', \pi', (\mu, K))$ *and the Batch morphism* $(\psi, f) : \mathfrak{B}_1 \to \mathfrak{B}_2$*. If* $f : (\sigma, J) \to (\mu, K)$ *is a slice isomorphism, then* $Im\psi$ *is a sub-box of* $\mathfrak{S}$*.*

Proof. Since f is a slice isomorphism, we have $Imf = (\mu, K)$. Also the above lemma states that $y \in (\mu, K) = Imf$ if and only if $y \in \pi(Im\psi)$. Therefore $\pi(Im\psi) = (\mu, K)$. Thus $Im\psi$ is sub-box of $\mathfrak{S}$. $\square$

Chapter 3

A Quotient slice through the ideals of L-slice $Hom(\sigma, J)$

The chapter is mainly concerned with constructive properties of L-slices. We develop ideals of the form $(a : x)_{Hom}$ in $Hom(\sigma, J)$. The L-slice J_a, the system of contractive operators with respect to $a \in L$ is defined .The congruence relation $\sim_a$ on (σ, J) defined as $x \sim_a y$ if and only if $(a : x)_{Hom} = (a : y)_{Hom}$ aids the development of quotient slice of (σ, J) . Also the ideals $(a : x)_{Hom}$ assist in constructing a quotient slice of $(\sqcap, L)$. For each $f \in Hom(\sigma, J)$ and $x \in (\sigma, J)$, through the ideals $[f : x]_L$ of L evolve the topological space $\mathfrak{L} = ((\sqcap, L), \mathfrak{B}_L)$ with basis $\mathfrak{B}_L$. Also the collection of subslices $[a : f]_{(\sigma,J)}$ of (σ, J) guarantees the existence of a topology on (σ, J) with basis $\mathfrak{B}_{(\sigma,J)}$. The last section will deal with the topological continuity of maps σ_b for every $b \in L$ and σ_x for every $x \in (\sigma, J)$. It is already proved that $(Hom(\sigma, J), \preceq)$ with usual ordering $f \preceq g$ if and only if $f(x) \leqslant g(x)$, forms an L-slice. We denote the join of f and g as $f \curlyvee g$ and $\mathbf{0}_{Hom}$ for the bottom element of $Hom(\sigma, J)$. The operator id denotes the identity operator on the L-slice $Hom(\sigma, J)$.

We define two different types of collection of operators on $Hom(\sigma, J)$.

Definition 3.0.1. Consider the L-slice (σ, J). For any $a \in L$ and $x \in (\sigma, J)$ we define the set $[a : x]_{Hom}$ of expansive-slice operators as $[a : x]_{Hom} = \{f \in Hom(\sigma, J) : \sigma(a, x) \leqslant f(x)\}$.

Remark. All increasing operators, that is, all those slice morphisms which have the property that $x \leqslant f(x), \forall x \in (\sigma, J)$ will definitely be in the set $[a : x]_{Hom}$. Hence $[a : x]_{Hom}$ includes, but is not limited to, the increasing operators on (σ, J).

We investigate the properties of $[a : x]_{Hom}$.

Proposition 3.0.2. *$[a : x]_{Hom}$ is nonempty and upward closed set of L-slice $Hom(\sigma, J)$.*

Proof. If $f = id$, then $id \in [a : x]_{Hom}$. Suppose $f \in [a : x]_{Hom}$ and $f \preceq g$, for some $g \in Hom(\sigma, J)$. Since $f \in [a : x]_{Hom}$, we have $\sigma(a, x) \leqslant f(x)$. Also $f \preceq g$ implies $f(x) \leqslant g(x), \forall x \in (\sigma, J)$. Thus $g \in [a : x]_{Hom}$. □

Proposition 3.0.3. *If $a \sqsubseteq b$, then $[b : x]_{Hom} \subseteq [a : x]_{Hom}$.*

Proof. Let $f \in [b : x]_{Hom}$ then $\sigma(b, x) \leqslant f(x)$. But, $a \sqsubseteq b$ implies $\sigma(a, x) \leqslant \sigma(b, x)$. Thus $\sigma(a, x) \leqslant \sigma(b, x) \leqslant f(x)$. Hence $f \in [a : x]_{Hom}$. □

3.1. The class E_a

Fix $a \in L$ and collect all the sets of the form $[a : x]_{Hom}$. The set $E_a = \{[a : x]_{Hom} : x \in (\sigma, J)\}$ denotes a system of expansive operators with respect to $a \in L$. The study on E_a yield the following properties for its elements.

Proposition 3.1.1. $[a : y]_{Hom} \cap [a : x]_{Hom} \subseteq [a : x \vee y]_{Hom}$.

Proof. Let $f \in [a : y]_{Hom} \cap [a : x]_{Hom}$ then $\sigma(a, y) \leqslant f(y)$ and $\sigma(a, x) \leqslant f(x)$. Hence $\sigma(a, y) \vee \sigma(a, x) \leqslant f(y) \vee f(x)$ implies $\sigma(a, x \vee y) \leqslant f(x \vee y)$. Thus the slice morphism $f \in [a : x \vee y]_{Hom}$. □

Proposition 3.1.2. $[a : \sigma(a, x)]_{Hom} \subseteq [a : x]_{Hom}$.

Proof. Let $f \in [a : \sigma(a, x)]_{Hom}$ then $\sigma(a, \sigma(a, x)) \leqslant f(\sigma(a, x))$. Since f is a slice morphism and σ is an action, we obtain the result. □

Proposition 3.1.3. *For any* $b \in L, [a : x]_{Hom} \subseteq [a : \sigma(b, x)]_{Hom}$.

Proof. If $f \in [a : x]_{Hom}$ then proof follows from $\sigma(a, \sigma(b, x)) \leqslant \sigma(a, x) \ \ \forall\, b \in L$. □

3.2. A well behaved class of operators $(a : x)_{Hom}$ in $Hom(\sigma, J)$

Definition 3.2.1. For $f \in Hom(\sigma, J)$ we define a subset $(a : x)_{Hom}$ of $Hom(\sigma, J)$ as $(a : x)_{Hom} = \{f \in Hom(\sigma, J) : \sigma(a, f(x)) \leqslant x\}$ and is called the collection of contractive - slice operators.

An operator in $Hom(\sigma, J)$ is said to be a decreasing operator if it has the property that $f(x) \leqslant x$. Hence $(a : x)_{Hom}$ will definitely contain all decreasing operators. A detailed study of $(a : x)_{Hom}$ led us to the fact that this set is more well behaved than the collection of expansive operators. The structure of the set $[a : x]_{Hom}$ was found to be just an upper closed set, whereas here we get more richer properties. The ideal $(a : x)_{Hom}$ leads to the construction of a quotient slice of (σ, J).

Proposition 3.2.2. $(a : x)_{Hom}$ *is an ideal of* $Hom(\sigma, J)$

Proof. $\mathbf{0}_J : (\sigma, J) \to (\sigma, J)$ defined as $\mathbf{0}_J(x) = \mathbf{0}_J \ \ \forall\, x \in (\sigma, J)$ belongs to $(a : x)_{Hom}$. Hence $(a : x)_{Hom}$ is nonempty. Let $f, g \in (a : x)_{Hom}$, then $\sigma(a, f(x)) \leqslant x$ and $\sigma(a, g(x)) \leqslant x$. Therefore $\sigma(a, f(x) \vee g(x)) = \sigma(a, f(x)) \vee \sigma(a, g(x)) \leqslant x$.
Hence $\sigma(a, f \curlyvee g(x)) \leqslant x$ implies $f \curlyvee g \in (a : x)_{Hom}$. Fix any $f \in (a : x)_{Hom}$. If there exists $h \in Hom(\sigma, J)$ such that $h \preceq f$,then $\sigma(a, h(x)) \leqslant \sigma(a, f(x)) \leqslant x$. Therefore $h \in (a : x)_{Hom}$. Also $\sigma(a, \delta(b, f)(x)) = \sigma(a, \sigma(b, f(x))) = \sigma(b, \sigma(a, f(x))) \leqslant \sigma(b, x) \leqslant x$. That is, $\delta(b, f) \in (a : x)_{Hom}$. Thus $(a : x)_{Hom}$ is an ideal of $Hom(\sigma, J)$. □

The ideal $(a : x)_{Hom}$ exhibits the following properties.

Proposition 3.2.3. *If $a \sqsubseteq b$ then $(b : x)_{Hom} \subseteq (a : x)_{Hom}$.*

Proof. $a \sqsubseteq b$ implies $\sigma(a, x) \leqslant \sigma(b, x)$. Also for $f \in (b : x)_{Hom}$, we have $\sigma(b, f(x)) \leqslant x$. Therefore $\sigma(a, f(x)) \leqslant \sigma(b, f(x)) \leqslant x$ would provide $f \in (a : x)_{Hom}$. □

Proposition 3.2.4. $(a : x)_{Hom} \cap (a : y)_{Hom} \subseteq (a : x \vee y)_{Hom}$.

Proof. Let $f \in (a : x)_{Hom} \cap (a : y)_{Hom}$ implies $\sigma(a, f(x)) \leqslant x$ and $\sigma(a, f(y)) \leqslant y$. Hence $\sigma(a, f(x \vee y)) \leqslant x \vee y$ shows that $f \in (a : x \vee y)_{Hom}$. □

Proposition 3.2.5. $(a : x)_{Hom} \subseteq (a : \sigma(b, x))_{Hom}$*, for $b \in L$.*

Proof. For any $f \in Hom(\sigma, J)$, we have $\sigma(a, f(\sigma(b, x))) = \sigma(a, \sigma(b, f(x))) = \sigma(b, \sigma(a, f(x)))$. If $f \in (a : x)_{Hom}$, then $\sigma(b, \sigma(a, f(x))) \leqslant \sigma(b, x)$. Therefore $\sigma(a, f(\sigma(b, x))) \leqslant \sigma(b, x)$. □

Proposition 3.2.6. $(a : \sigma(b, x))_{Hom} \subseteq (a \sqcap b : x)_{Hom}$ *for any $b \in L$.*

Proof. Let $(a : \sigma(b, x))_{Hom}$. Then we obtain the following

$$\begin{aligned} \sigma(a, f(\sigma(b, x))) \leqslant \sigma(b, x) \leqslant x \ &\Rightarrow\ \sigma(a, \sigma(b, f(x))) \leqslant \sigma(b, x) \leqslant x \\ &\Rightarrow\ \sigma(a \sqcap b, f(x)) \leqslant x \end{aligned}$$

Therefore $f \in (a \sqcap b : x)_{Hom}$. □

Proposition 3.2.7. *If $a, b \in L$ then $(a : x)_{Hom} \cap (b : x)_{Hom} = (a \sqcup b : x)_{Hom}$.*

Proof. If $f \in (a : x)_{Hom} \cap (b : x)_{Hom}$ then $\sigma(a \sqcup b, f(x)) = \sigma(a, f(x)) \vee \sigma(b, f(x)) \leqslant x$. Similarly, if $g \in (a \sqcup b : x)_{Hom}$ then $\sigma(a \sqcup b, g(x)) \leqslant x$.

$$\begin{aligned} \sigma(a \sqcup b, g(x)) \leqslant x &\Rightarrow \sigma(a, g(x)) \vee \sigma(b, g(x)) \leqslant x \\ &\Rightarrow \sigma(a, g(x)) \leqslant x \text{ and } \sigma(b, g(x)) \leqslant x \end{aligned}$$

Thus $g \in (a : x)_{Hom} \cap (b : x)_{Hom}$. Therefore $(a : x)_{Hom} \cap (b : x)_{Hom} = (a \sqcup b : x)_{Hom}$. □

Proposition 3.2.8. $(0_L : x)_{Hom} = Hom(\sigma, J)$.

Proof. $\sigma(0_L, f(x)) = 0_J \leqslant x, \forall x \in (\sigma, J)$. Thus $(0_L : x)_{Hom} = Hom(\sigma, J)$. □

Proposition 3.2.9. *On $Hom(\sigma, J)$, for each $x \in (\sigma, J)$ the collection $\mathfrak{B}_x = \{(a : x)_{Hom} : a \in L\}$ forms a basis for a topology.*

Proof. It follows directly from proposition 3.2.7 and proposition 3.2.8. □

3.3. The L-slice J_a

Definition 3.3.1. The collection $J_a = \{(a : x)_{Hom}; x \in (\sigma, J)\}$ is called the system of contractive operators with respect to $a \in L$.

On J_a we define a binary operation $\uplus$ as $(a : x)_{Hom} \uplus (a : y)_{Hom} = (a : x \vee y)_{Hom}$. The operation $\uplus$ is commutative and idempotent.
Also $(a : x)_{Hom} \uplus (a : 0_J)_{Hom} = (a : x \vee 0_J)_{Hom} = (a : x)_{Hom}$, implies that $(a : 0_J)_{Hom}$ is an identity element with respect to $\uplus$. Algebraically, $(J_a, \uplus)$ is a commutative idempotent monoid.

Lemma 3.3.2. *$(J_a, \uplus)$ is a join semilattice*

Proof. Define a partial ordering $\leqslant_{\uplus}$ such that $(a : x)_{Hom} \leqslant_{\uplus} (a : y)_{Hom}$ if and only if $x \leqslant y$. Then the join would be defined by $(a : x)_{Hom} \vee_{\uplus} (a : y)_{Hom} = (a : x \vee y)_{Hom}$. Consequently $(J_a, \uplus)$ will be a join semilattice with bottom element $(a : 0_J)_{Hom}$. □

Theorem 3.3.3. *(λ, J_a) is an L-slice with action $\lambda : L \times J_a \to J_a$ defined as $\lambda(b, (a : x)_{Hom}) = (a : \sigma(b, x))_{Hom}$*

Proof.

1.$\lambda(b, (a : x)_{Hom} \vee_{\uplus} (a : y)_{Hom}) = \lambda(b, (a : x \vee y)_{Hom})$

$$= (a : \sigma(b, x \vee y))_{Hom}$$
$$= (a : \sigma(b, x))_{Hom} \vee_{\uplus} (a : \sigma(b, y))_{Hom}$$
$$= \lambda(b, (a : x)_{Hom}) \vee_{\uplus} \lambda(b, (a : y)_{Hom})$$

2.$\lambda(b, (a : 0_J)_{Hom}) = (a : \sigma(b, 0_J))_{Hom}$

$$= (a : 0_J)_{Hom}$$

3.$\lambda(b \sqcap c, (a:x)_{Hom}) = (a : \sigma(b \sqcap c, x))_{Hom}$

$$= (a : \sigma(b, \sigma(c, x)))_{Hom}$$

$$= \lambda(b, (a : \sigma(c, x))_{Hom})$$

$$= \lambda(b, \lambda(c, (a : x)_{Hom}))$$

Similarly we get $\lambda(b \sqcap c, (a : x)_{Hom}) = \lambda(c, \lambda(b, (a : x)_{Hom}))$

4.

$$\begin{aligned} \lambda(1_L, (a : x)_{Hom}) &= (a : \sigma(1_L, x))_{Hom} \\ &= (a : x)_{Hom} \end{aligned}$$

and

$$\begin{aligned} \lambda(0_L, (a : x)_{Hom}) &= (a : \sigma(0_L, x))_{Hom} \\ &= (a : 0_J)_{Hom} \end{aligned}$$

5.

$$\begin{aligned} \lambda(b \sqcup c, (a : x)_{Hom}) &= (a : \sigma(b \sqcup c, x))_{Hom} \\ &= (a : \sigma(b, x) \vee \sigma(c, x))_{Hom} \\ &= (a : \sigma(b, x))_{Hom} \vee_{\uplus} (a : \sigma(c, x))_{Hom} \\ &= \lambda(b, (a : x)_{Hom}) \vee_{\uplus} \lambda(c, (a : x)_{Hom}) \end{aligned}$$

Thus (λ, J_a), the system of contractive operators forms an L-slice. $\square$

3.4. A Quotient slice of (σ, J) through the system of contractive operators J_a

On (σ, J) we define a relation as $x \sim_a y$ if and only if $(a : x)_{Hom} = (a : y)_{Hom}$. It can be easily observed that $\sim_a$ is an equivalence relation. Now we prove that $\sim_a$ is a congruence.

Theorem 3.4.1. *$(\gamma, J/\sim_a)$ is a quotient slice.*

Proof. Let $x, y \in (\sigma, J)$ be such that $x \sim_a y$. By definition, $(a : x)_{Hom} = (a : y)_{Hom}$. Also , $(a : x)_{Hom} \vee_{\uplus} (a : z)_{Hom} = (a : y)_{Hom} \vee_{\uplus} (a : z)_{Hom}$ implies that $(a : x \vee z)_{Hom} = (a : y \vee z)_{Hom}$. That is, $x \vee z \sim_a y \vee z$. We have $\lambda(b, (a : x)_{Hom}) = \lambda(b, (a : y)_{Hom})$, for any $b \in L$. Then $(a : \sigma(b, x))_{Hom} = (a : \sigma(b, y))_{Hom}$ would imply $\sigma(b, x) \sim_a \sigma(b, y)$. Therefore $\sim_a$ is a congruence relation. Thus $(\gamma, J/\sim_a)$ becomes a quotient slice with the action defined as $\gamma : L \times (\sigma, J)/\sim_a \rightarrow J/\sim_a$ and $\gamma(a, [x]) = [\sigma(a, x)]$. □

Theorem 3.4.2. *The map $\phi_a : (\gamma, J/\sim_a) \rightarrow (\lambda, J_a)$ defined as $\phi_a([x]) = (a : x)_{Hom}$ is a surjective slice morphism.*

Proof. It is evident from the definition that ϕ_a is surjective. We prove that it is a slice morphism.

$$\begin{aligned} \phi_a([x] \vee [y]) &= \phi_a([x \vee y]) \\ &= (a : x \vee y)_{Hom} \\ &= (a : x)_{Hom} \vee_{\uplus} (a : y)_{Hom} \\ &= \phi_a([x]) \vee_{\uplus} \phi_a([y]) \end{aligned}$$

Thus ϕ_a preserves joins.

$$\begin{aligned}
\phi_a(\gamma(b,[x])) &= \phi_a([\sigma(b,x)]) \\
&= (a:\sigma(b,x))_{Hom} \\
&= \lambda(b,(a:x)_{Hom}) \\
&= \lambda(b,\phi_a([x]))
\end{aligned}$$

ϕ_a preserves action. Hence ϕ_a is a slice morphism. ☐

Theorem 3.4.3. *For each $a \in L$ we can define a slice morphism $\sim_a$: $(\sigma, J) \to (\gamma, J/\sim_a)$ as $\sim_a(x) = [x]$.*

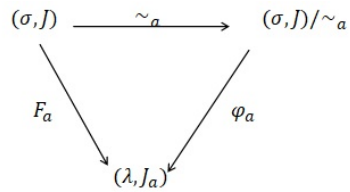
Proof.

$$\begin{aligned}
\sim_a(x \vee y) &= [x \vee y] \\
&= [x] \vee [y] \\
&= \sim_a(x) \vee \sim_a(y)
\end{aligned}$$

$$\begin{aligned}
\sim_a(\sigma(b,x)) &= [\sigma(b,x)] \\
&= \gamma(b,[x]) \\
&= \gamma(b,\sim_a(x))
\end{aligned}$$

Therefore $\sim_a$ is a slice morphism . ☐

The ideals $(a:x)_{Hom}$ yields a quotient slice of (σ, J) and an L-slice on the Hom-slice $Hom(\sigma, J)$ such that there arises a natural slice morphism between (σ, J) and (λ, J_a) which makes the diagram below commute.



Theorem 3.4.4. *The map $F_a : (\sigma, J) \rightarrow (\lambda, J_a)$ defined as $F_a(x) = (a:x)_{Hom}$ is a slice morphism and $F_a = \phi_a \circ \sim_a$.*

Proof. $F_a(x \vee y) = (a : x \vee y)_{Hom} = (a:x)_{Hom} \uplus (a:y)_{Hom} = F_a(x) \uplus F_a(y)$
$F_a(\sigma(b,x)) = (a:\sigma(b,x))_{Hom} = \lambda(b,(a:x)_{Hom}) = \lambda(b, F_a(x))$ Thus F_a is a slice morphism.
Also $\phi_a \circ \sim_a (x) = \phi_a([x]) = (a:x)_{Hom} = F_a(x)$. □

The ideals $(a:x)_{Hom}$ also allows a quotienting of the locale L, viewed as an L-slice $(\sqcap, L)$. Fix any $x \in (\sigma, J)$ and consider the corresponding ideal $(a:x)_{Hom}$ of $Hom(\sigma, J)$. The following lemma gives an equivalence relation on $(\sqcap, L)$

Lemma 3.4.5. *The relation R_x defined on the L-slice $(\sqcap, L)$ as $a \ R_x \ b$ if and only if $(a:x)_{Hom} = (b:x)_{Hom}$ is an equivalence relation.*

Proof. The definition of the relation shows that R_x is reflexive. Also, whenever $a \ R_x \ b$ then $(a:x)_{Hom} = (b:x)_{Hom}$, implies $b \ R_x \ a$ That is, R_x is symmetric. Also, if $a \ R_x \ b$ and $b \ R_x \ c$ then $(a:x)_{Hom} = (b:x)_{Hom}$ and $(b:x)_{Hom} = (c:x)_{Hom}$. Thus we obtain the transitivity property of R_x. Hence R_x is an equivalence relation on $(\sqcap, L)$. □

Proposition 3.4.6. *R_x defines a congruence relation on $(\sqcap, L)$.*

Proof. Let aR_xb and $c \in L$. It follows from proposition 3.2.7 that $(a \sqcup c : x)_{Hom} = (a : x)_{Hom} \cap (c : x)_{Hom} = (b : x)_{Hom} \cap (c : x)_{Hom} = (b \sqcup c : x)_{Hom}$. Hence $a \sqcup c \ R_x \ b \sqcup c$.

Now it remains to show that R_x is compatible with the action $\sqcap$ on $(\sqcap, L)$.

If $f \in (a \sqcap c : x)_{Hom}$ then

$$\begin{aligned}
\sigma(a \sqcap c, f(x)) \leqslant x &\Rightarrow \sigma(a, \sigma(c, f(x))) \leqslant x \\
&\Rightarrow \sigma(a, \delta(c, f)(x)) \leqslant x \\
&\Rightarrow \delta(c, f) \in (a : x)_{Hom} = (b : x)_{Hom} \\
&\Rightarrow \sigma(b, \delta(c, f)(x)) \leqslant x \\
&\Rightarrow \sigma(b, \sigma(c, f(x))) \leqslant x \\
&\Rightarrow \sigma(b \sqcap c, f(x)) \leqslant x \\
&\Rightarrow f \in (b \sqcap c : x)_{Hom}
\end{aligned}$$

Thus we obtain $(a \sqcap c : x)_{Hom} \subseteq (b \sqcap c : x)_{Hom}$. Similarly we can prove that $(b \sqcap c : x)_{Hom} \subseteq (a \sqcap c : x)_{Hom}$. Consequently, we observe that $(a \sqcap c : x)_{Hom} = (b \sqcap c : x)_{Hom}$. Hence $(\sqcap(a, c) : x)_{Hom} = (\sqcap(b, c) : x)_{Hom}$ implies that R_x is a congruence relation on $(\sqcap, L)$. □

Theorem 3.4.7. *$(\sqcap_{R_x}, L/R_x)$ is a quotient slice with action $\sqcap_{R_x} : L \times L/R_x \to L/R_x$ defined as $\sqcap_{R_x}(a, [b]) = [a \sqcap b]$.*

Proof. Follows from the above proposition. □

Remark. The congruence relation can also be viewed as a map between the two L-slices $(\sqcap, L)$ and $(\sqcap_{R_x}, L/R_x)$ defined as $R_x(a) = [a]$. We propose the following theorem.

Theorem 3.4.8. $R_x : (\sqcap, L) \to (\sqcap_{R_x}, L/R_x)$ *is an onto slice morphism.*

Proof. Proof follows directly from Theorem 3.4.7. □

The ideals of $Hom(\sigma, J)$ are utilised to construct quotient slices on $(\sqcap, L)$ and (σ, J). Further, we prove that there exists a special slice morphism between the two quotient slices.

Theorem 3.4.9. *The map* $\phi : (\sqcap_{R_x},, L/R_x) \to (\gamma, J/\sim_a)$ *defined as* $\phi([b]) = [\sigma(b, x)]$ *is a slice morphism.*

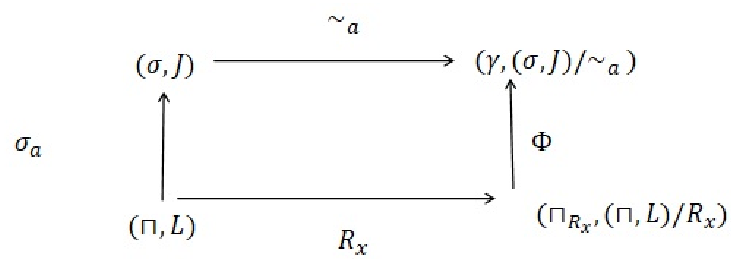
Proof.

$$\begin{aligned} \phi([a] \sqcup [[b]]) = \phi([a \sqcup b]) &= [\sigma(a \sqcup b, x)] = [\sigma(a, x) \vee \sigma(b, x)] \\ &= [\sigma(a, x)] \vee [\sigma(b, x)] = \phi([a]) \vee \phi([b]) \end{aligned}$$

$$\begin{aligned} \phi(\sqcap_{R_x}(c, [a])) = \phi([c \sqcup a]) &= [\sigma(c \sqcup a, x)] = [\sigma(c, \sigma(a, x))] \\ &= \gamma(c, [\sigma(a, x)]) = \gamma(c, \phi([a])) \end{aligned}$$

Hence ϕ is a slice morphism. □

Remark. Summarising the above results : For a fixed $a \in L$ and $x \in (\sigma, J)$, we have developed two L-slices and their respective quotient slices. The diagram with L-slices and the corresponding maps between them commutes.



3.5. A Prime Ideal on L through the Hom-slice $Hom(\sigma, J)$

For each L-slice morphism f on (σ, J), we define a prime ideal on L. We study some of its properties and define a basis for a topology on (σ, J).

Definition 3.5.1. Let f be a slice morphism in $Hom(\sigma, J)$ and $x \in (\sigma, J)$. Define the set $[f : x]_L = \{a \in L : \sigma(a, f(x)) \leqslant x\}$. Since $0_L \in [f : x]_L$ shows that the set is nonempty.

Theorem 3.5.2. *The set $[f : x]_L$ is an ideal of L.*

Proof. If $a, b \in [f : x]_L$, then $\sigma(a, f(x)) \leqslant x$ and $\sigma(b, f(x)) \leqslant x$ will imply that $\sigma(a \sqcup b, f(x)) \leqslant x$. Therefore $a \sqcup b \in [f : x]_L$ proving that it is a join semilattice. Also, if $b \leqslant a$ and $a \in [f : x]_L$, then $\sigma(b, f(x)) \leqslant \sigma(a, f(x)) \leqslant x$. Hence $b \in [f : x]_L$ and $[f : x]_L$ is a lower set. Any nonempty set which is a lower set and is a join semilattice is an ideal of the locale L. □

The above theorem shows that $[f:x]_L$ is also an ideal of the meet slice $(\sqcap, L)$. The next theorem establishes the possibility of $[f:x]_L$ to be a prime ideal.

Theorem 3.5.3. *If x is a prime element of (σ, J), then $[f:x]_L$ is a prime ideal.*

Proof. Whenever $a \sqcap b \in [f:x]_L$, we have $\sigma(a \sqcap b, f(x)) \leqslant x$. Since x is prime, $\sigma(a \sqcap b, f(x)) \leqslant x$ implies either $\sigma(a, f(x)) \leqslant x$ or $\sigma(b, f(x)) \leqslant x$. That is, either $a \in [f:x]_L$ or $b \in [f:x]_L$. Hence $[f:x]_L$ is a prime ideal of L. □

We study the properties of the ideal $[f:x]_L$ of L.

Proposition 3.5.4. *For $f, g \in Hom(\sigma, J)$, if $f \preceq g$, then $[g:x]_L \subseteq [f:x]_L$ for any $x \in (\sigma, J)$.*

Proof. If $f \preceq g$ then $f(x) \leqslant g(x)$ will give $\sigma(a, f(x)) \leqslant \sigma(a, g(x))$ $\forall\, a \in L$. If $b \in [g:x]_L$ then $\sigma(b, g(x)) \leqslant x$ would imply $\sigma(b, f(x)) \leqslant x$. Thus $b \in [f:x]_L$ and $[g:x]_L \subseteq [f:x]_L$. □

Proposition 3.5.5. *For $x, y \in (\sigma, J)$ and a fixed slice morphism $f \in Hom(\sigma, J)$, $[f:x]_L \cap [f:y]_L \subseteq [f:x \vee y]_L$.*

Proof. Let $a \in [f:x]_L \cap [f:y]_L$ then $\sigma(a, f(x \vee y)) = \sigma(a, f(x) \vee f(y)) = \sigma(a, f(x)) \vee \sigma(a, f(y)) \leqslant x \vee y$. Therefore $[f:x]_L \cap [f:y]_L \subseteq [f:x \vee y]_L$. □

Proposition 3.5.6. *For the slice morphisms $f, g \in Hom(\sigma, J)$ and a fixed $x \in (\sigma, J)$, $[f:x]_L \cap [g:x]_L = [f \curlyvee g:x]_L$*

Proof. $a \in [f:x]_L \cap [g:x]_L$ implies $\sigma(a, f(x)) \leqslant x$ and $\sigma(a, g(x)) \leqslant x$. Therefore $\sigma(a, (f \curlyvee g)(x)) = \sigma(a, f(x) \vee g(x)) = \sigma(a, f(x)) \vee \sigma(a, g(x)) \leqslant x$ will imply that $a \in [f \curlyvee g:x]_L$. Thus $[f:x]_L \cap [g:x]_L \subseteq [f \curlyvee g:x]_L$.

Now for the reverse inequality, let us consider $b \in [f \curlyvee g : x]_L$. Then $\sigma(b, (f \curlyvee g)(x)) \leqslant x$ implies that $\sigma(b, f(x)) \vee \sigma(b, g(x)) \leqslant x$. That is, $\sigma(b, f(x)) \leqslant x$ and $\sigma(b, g(x)) \leqslant x$. Therefore $b \in [f : x]_L \cap [g : x]_L$. □

Proposition 3.5.7. *For any* $x \in (\sigma, J), [\boldsymbol{0}_{Hom} : x]_L = L$

Proof. The property follows from the definition of $\mathbf{0}_{Hom}$. □

Fix $x \in (\sigma, J)$. For each slice morphism $f \in Hom(\sigma, J)$ consider the collection of ideals $\mathfrak{B}_L = \{[f : x]_L : f \in Hom(\sigma, J)\}$.

Theorem 3.5.8. $\mathfrak{L} = ((\sqcap, L), \mathfrak{B}_L)$ *forms a topological space with basis* $\mathfrak{B}_L$.

Proof. The propositions 3.5.6 and 3.5.7 shows that the collection $\mathfrak{B}_L$ forms a basis for topology on the L-slice $(\sqcap, L)$. □

Remark. Through the L-slice morphisms in $Hom(\sigma, J)$, we have constructed ideals in $Hom(\sigma, J)$ and on the L-slice $(\sqcap, L)$.

Analogous to the above definitions, we try to develop and study the structure of sets $[a : f]_{(\sigma,J)}$ in (σ, J).

3.6. The Subslice $[a : f]_{(\sigma,J)}$ of (σ, J)

The structure of sets $(a : x)_{Hom}$ defined on $Hom(\sigma, J)$ and $[f : x]_L$ defined on $(\sqcap, L)$ were that of ideals in the respective domains. In contrast, here we obtain a weaker structure which is of a subslice.

Definition 3.6.1. We define the set $[a : f]_{(\sigma,J)}$ as a subset of (σ, J) . For $a \in L$ and $f \in Hom(\sigma, J), [a : f]_{(\sigma,J)} = \{x \in (\sigma, J) : \sigma(a, f(x)) \leqslant x\}$.Since $0_J \in [a : f]_{(\sigma,J)}$, it is nonempty.

Theorem 3.6.2. *The set* $[a : f]_{(\sigma,J)}$ *is a subslice of* (σ, J).

Proof. Let $x, y \in [a : f]_{(\sigma,J)}$ then $\sigma(a, f(x)) \leqslant x$ and $\sigma(a, f(y)) \leqslant y$. We have $\sigma(a, f(x \vee y)) = \sigma(a, f(x)) \vee \sigma(a, f(y)) \leqslant x \vee y$. Thus $[a : f]_{(\sigma,J)}$ is a join semilattice. Let $b \in L$ and $x \in [a : f]_{(\sigma,J)}$. Now $\sigma(a, f(\sigma(b, x))) = \sigma(a, \sigma(b, f(x))) = \sigma(b, \sigma(a, f(x))) \leqslant \sigma(b, x) \leqslant x$. Hence $\sigma(b, x) \in [a : f]_{(\sigma,J)}$. □

Theorem 3.6.3. *For* $a \in L$*, the collection* $\mathfrak{B}_{(\sigma,J)} = \{[a : f]_{(\sigma,J)} : f \in Hom(\sigma, J)\}$ *forms a basis for a topology on* (σ, J).

Proof. First we prove that $[a : f]_{(\sigma,J)} \cap [a : g]_{(\sigma,J)} = [a : f \curlyvee g]_{(\sigma,J)}$. Consider $x \in [a : f]_{(\sigma,J)} \cap [a : g]_{(\sigma,J)}$, then $\sigma(a, f \curlyvee g(x)) = \sigma(a, f(x)) \vee \sigma(a, g(x)) \leqslant x$. Therefore $x \in [a : f \curlyvee g]_{(\sigma,J)}$ and $[a : f]_{(\sigma,J)} \cap [a : g]_{(\sigma,J)} \subseteq [a : f \curlyvee g]_{(\sigma,J)}$. For the reverse inequality, let us consider $y \in [a : f \curlyvee g]_{(\sigma,J)}$. The relation $\sigma(a, (f \curlyvee g)(y)) \leqslant y$ implies that $\sigma(a, f(y)) \vee \sigma(a, g(y)) \leqslant y$. Therefore $\sigma(a, f(y)) \leqslant y$ and $\sigma(a, g(y)) \leqslant y$. Hence $y \in [a : f]_{(\sigma,J)} \cap [a : g]_{(\sigma,J)}$. Also, $[a : \mathbf{0}_{Hom}]_{(\sigma,J)} = (\sigma, J)$. Thus the collection $\mathfrak{B}_{(\sigma,J)} = \{[a : f]_{(\sigma,J)} : f \in Hom(\sigma, J)\}$ forms a basis for a topology on (σ, J). □

Remark. The collection of ideals $\mathfrak{B}_x$ yields a topology on the L-slice $Hom(\sigma, J)$. Similarly, through the collection of subslices $\mathfrak{B}_{(\sigma,J)}$ on (σ, J) and collection of ideals $\mathfrak{B}_L$ on $(\sqcap, L)$ we obtain three topologies on the three different domains involved. Once topology is defined on a structure we can talk about the continuity of morphisms defined on the structure. In this regard, we propose the following two theorems.

Theorem 3.6.4. *The map* $\psi : (\sqcap, L) \to Hom(\sigma, J)$ *defined as* $\psi(b) = \sigma_b$ *is a continuous slice morphism.*

Proof. $\psi(b \sqcup c) = \sigma_{b \sqcup c}$. But, $\sigma_{b \sqcup c}(x) = \sigma(b \vee c, x) = \sigma(b, x) \vee \sigma(c, x) = \sigma_b(x) \vee \sigma_c(x)$. Therefore $\psi(b \sqcup c) = \sigma_{b \sqcup c} = \psi(b) \curlyvee \psi(c)$. By definition $\psi(\sqcap(b, c)) = \sigma_{b \sqcap c}$.

Also, we have the following equations :

$$\begin{aligned}
\sigma_{b \sqcap c}(x) &= \sigma(b \sqcap c, x) \\
&= \sigma(b, \sigma(c, x)) \\
&= \sigma(b, \sigma_c(x)) \\
&= \delta(b, \sigma_c)(x) \\
&= \delta(b, \psi(c))(x)
\end{aligned}$$

. Thus ψ is a slice morphism. To prove continuity, let us consider an open set $(a : x)_{Hom}$ in $Hom(\sigma, J)$

$$\begin{aligned}
\psi^{-1}(a : x)_{Hom} &= \{b \in L : \psi(b) \in (a : x)_{Hom}\} \\
&= \{b \in L : \sigma_b \in (a : x)_{Hom}\} \\
&= \{b \in L : \sigma(a, \sigma_b(x)) \leqslant x\} \\
&= \{b \in L : \sigma(a, \sigma(b, x)) \leqslant x\} \\
&= \{b \in L : \sigma(b, \sigma_a(x)) \leqslant x\} \\
&= [\sigma_a : x]_L \in \mathfrak{B}_L
\end{aligned}$$

□

Similarly, the continuity of the slice morphism σ_x defined on $(\sqcap, L)$ is established in the following theorem

Theorem 3.6.5. *The slice morphism* $\sigma_x : (\sqcap, L) \to (\sigma, J)$ *defined as* $\sigma_x(a) = \sigma(a, x)$ *is continuous for every* $x \in (\sigma, J)$.

Proof. To prove continuity of σ_x , consider the open set $[a : f]_{(\sigma,J)}$ of (σ, J).

$$\begin{aligned}
\sigma_x^{-1}[a : f]_{(\sigma,J)} &= \{b \in L : \sigma_x(b) \in [a : f]_{(\sigma,J)}\} \\
&= \{b \in L : \sigma(b, x) \in [a : f]_{(\sigma,J)}\} \\
&= \{b \in L : \sigma(a, f(\sigma(b, x))) \leqslant \sigma(b, x) \leqslant x\} \\
&= \{b \in L : \sigma(a, \sigma(b, f(x))) \leqslant x\} \\
&= \{b \in L : \sigma(b, \delta(a, f)(x) \leqslant x\} \\
&= [\delta(a, f) : x]_L \in \mathfrak{B}_L
\end{aligned}$$

□

Thus the topologies generated through the ideals on $(\sqcap, L)$ and $Hom(\sigma, J)$ makes the slice morphisms $\psi = \sigma_b$ continuous for every $b \in L$. Similarly, the subslices which are constructed on (σ, J) permits the continuity of the slice morphism σ_x for every $x \in (\sigma, J)$.

Chapter 4

The Zero sets and Fixed Ideals of $Hom(L, J)$

The set $C(X)$ of all real valued continuous functions on a toplogical space X is extensively studied by Gillman and Jerison in [25] . On similar lines we have tried to study the special morphism class $Hom(L, J)$. The collection $Hom(J, K)$ of all L-slice morphisms between the L-slice (σ, J) and (μ, K) is an L-slice $(\delta, Hom(J, K))$. Also we know that any locale L can be viewed as the meet L-slice $(\sqcap, L)$. Consider the L-slice $(\delta, Hom(L, J))$ of all slice morphisms from $(\sqcap, L)$ to (σ, J). $Hom(L, J)$ is nonempty through the existence of the slice morphism $\sigma_x : (\sqcap, L) \rightarrow (\sigma, J)$ defined as $\sigma_x(a) = \sigma(a, x)$. Thus there is a possibility of extending the study of L-slices through the morphism class $Hom(L, J)$. In this chapter we try to develop theories analogous to that in the ring $C(X)$. Hence we utilise the same terminologies as in [25] so as to observe the interrelations between the two theories. Also throughout this chapter whenever we mention $x \in (\sigma, J)$ is nonzero it will imply that $x \neq 0_J$.

4.1. The sets $Z(L, J)$ and $Coz(L, J)$

On studying the properties of $Hom(L, J)$, we observe the properties of subsets of the form $\{a \in (\sqcap, L) : f(a) = 0_J\}$. We begin with the definition of zero divisors on the locale L.

Definition 4.1.1. A locale L is said to have no zero divisors on (σ, J) if $\sigma(a, x) = 0_J$ implies either $a = 0_L$ or $x = 0_J$. A locale L is then called a σ-domain.

Example 4.1.2. *If L is a chain then L is a $\sqcap$-domain.*

Definition 4.1.3. The zero set of a slice morphism $f \in Hom(L, J)$ is defined as $Z(f) = \{a \in (\sqcap, L) : f(a) = 0_J\}$. The collection of all zero sets is denoted as $Z(L, J)$.

Definition 4.1.4. The cozero set is the dual of zero set and is defined as the set $Coz(f) = \{a \in (\sqcap, L) : f(a) \neq 0_J\}$. The collection of all cozero sets is denoted as $Coz(L, J)$.

Definition 4.1.5. A slice morphism $f \in Hom(L, J)$ is said to be a multiple of $g \in Hom(L, J)$ if $f = \delta(r, g)$ for some $r \in L$

We investigate a few basic properties of the zero set.

Proposition 4.1.6. *If the slice morphism $f \in Hom(L, J)$ is a multiple of the slice morphism $g \in Hom(L, J)$ then $Z(g) \subseteq Z(f)$.*

Proof. If $a \in Z(g)$ then $g(a) = 0_J$. Also $f(a) = \delta(r, g)(a) = \sigma(r, g(a)) = 0_J$. Thus $Z(g) \subseteq Z(f)$. □

Proposition 4.1.7. *If $f \leqslant g$ then $Z(g) \subseteq Z(f)$*

Proof. If $a \in Z(g)$ then $f(a) \leqslant g(a) = 0_J$ implies $f(a) = 0_J$. □

The next proposition gives a better understanding of the structure of $Z(f)$ for $f \in Hom(L, J)$

Proposition 4.1.8. *$Z(f)$ is an ideal of the meet L-slice $(\sqcap, L)$.*

Proof. For $a, b \in Z(f)$ then $f(a \sqcup b) = f(a) \vee f(b) = 0_J$ implies $a \sqcup b \in Z(f)$.
If $c \in Z(f)$ and $r \in L$ then $f(\sqcap(r, c)) = \sigma(r, f(c)) = \sigma(r, 0_J) = 0_J$. Thus $\sqcap(r, c) \in Z(f)$. Also if $a \in Z(f)$ and $d \leqslant a$ then $f(d) \leqslant f(a) = 0_J$ implies $f(d) = 0_J$. Thus $d \in Z(f)$. Hence $Z(f)$ is an ideal of the meet L-slice $(\sqcap, L)$.
Thus $Z(L, J)$ is a collection of ideals of $(\sqcap, L)$. $\square$

Remark. If $\sigma_a : J \to J$ is one-one for every $a \in L$ then for $b \in Z(f), a \to b \in Z(f)$.

We investigate the properties of $Coz(f)$, for $f \in Hom(L, J)$ and we have the following observations.

1. If $f \neq \mathbf{0}_{hom}$, then there exists $a \in (\sqcap, L)$ such that $f(a) \neq 0_J$. So, $Coz(f)$ is non-empty if $f \neq \mathbf{0}_{hom}$.

2. If $a, b \in Coz(f)$ then $f(a) \neq 0_J, f(b) \neq 0_J$ will imply $f(a \sqcup b) = f(a) \vee f(b) \neq 0_J$. Thus $a \sqcup b \in Coz(f)$.

3. For $c \in Coz(f)$ and $a \in L, \sqcap(a, c) = a \sqcap c$ need not belong to $Coz(f)$.

4. Let $c \in Coz(f)$ and $a \sqsubseteq b$ for some $b \in L$. Since f is a slice morphism and $f(a) \neq 0_J$ implies $f(a) \leqslant f(b)$ and $f(b) \neq 0_J$. Therefore $b \in Coz(f)$.

Thus from the above observations $Coz(f)$ is an upperset and $Coz(L, J)$ is a collection of upper sets of $(\sqcap, L)$.

4.2. The subslice $Hom^*(L, J)$

We begin this section by introducing bounded slice morphisms.

Definition 4.2.1. A function $f : (\sqcap, L) \to (\sigma, J)$ is said to be bounded if there exists some $y, z \in (\sigma, J)$ such that $y \leqslant f(a) \leqslant z \ \forall \ a \in L$.

Example 4.2.2. *Fix* $x \in (\sigma, J)$*, then* $\sigma_x : (\sqcap, L) \to (\sigma, J)$ *defined as* $\sigma_x(a) = \sigma(a, x)$ *is a bounded slice morphism with bounds* 0_J *and* x*.*

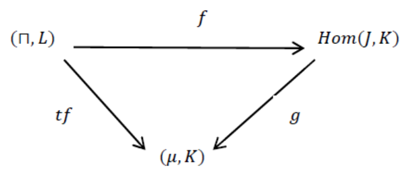
The collection of all bounded functions are denoted as $Hom^*(L, J)$.

Theorem 4.2.3. $Hom^*(L, J)$*is a subslice of* $Hom(L, J)$

Proof. For $f, g \in Hom^*(L, J)$, we show that $f \curlyvee g \in Hom^*(L, J)$. Let y, z be bounds for f and m, n be that of g. Then $y \vee m \leqslant f(a) \vee g(a) \leqslant z \vee n \ \forall \ a \in L$. Thus $f \vee g$ is a bounded slice morphism with bounds $y \vee m$ and $z \vee n$. Also if f is bounded, we can show that $\delta(a, f)$ is bounded. Let x_1 and x_2 be the bounds of f. Then $x_1 \leqslant f(b) \leqslant x_2, \ \forall \ b \in L$ implies $\sigma(a, x_1) \leqslant \sigma(a, f(b)) \leqslant \sigma(a, x_2)$. Therefore the relations $\sigma(a, x_1) \leqslant \delta(a, f)(b) \leqslant \sigma(a, x_2) \ \forall \ b \in L$ shows that $\delta(a, f)$ is bounded. Thus $Hom^*(L, J)$ is a subslice of $Hom(L, J)$. □

The theorem below gives the condition when a bounded function in a Hom-slice $Hom(L, J)$ becomes a bounded function of another Hom-slice. Consider two L-slices $Hom(L, J)$ and $Hom(L, K)$. Let t be a slice morphism between these two L-slices .

Theorem 4.2.4. *Every slice morphism* $t : Hom(L, J) \to Hom(L, K)$ *takes bounded functions to bounded functions if* $|Hom(J, K)| > 1$ *and the diagram below commutes.*



Note that tf denotes $t(f)$ and $|Hom(J,K)|$ is the cardinality of the set $Hom(J,K)$

Proof. Suppose $f \in Hom^*(L,J)$ then there exists $x, y \in (\sigma, J)$ such that $x \leqslant f(a) \leqslant y$ $\forall\ a \in L$. Subsequently, $g(x) \leqslant g(f(a)) \leqslant g(y)$ implies $g(x) \leqslant tf(a) \leqslant g(y)$. Therefore tf is bounded. □

4.3. The L-slice $Z(L,J)$

Let $Z(L,J) = \{Z(f) : f \in Hom(L,J)\}$. For each $f \in Hom(L,J)$ we have already shown that $Z(f)$ is an ideal of $(\sqcap, L)$. Now we look at the structure of the collection of all zero sets.

Theorem 4.3.1. *$Z(L,J)$ is a join semilattice.*

Proof. Partially order $Z(L,J)$ as $Z(f) \leqslant' Z(g)$ if and only if $Z(g) \subseteq Z(f)$.
Consider $a \in Z(f) \cap Z(g)$. Accordingly, $f(a) = 0_J$ and $g(a) = 0_J$ will imply that $(f \curlyvee g)(a) = f(a) \vee g(a) = 0_J$. Thus $a \in Z(f \curlyvee g)$ and $Z(f) \cap Z(g) \subseteq Z(f \curlyvee g)$. Simliarly for $b \in Z(f \curlyvee g)$ we have $(f \curlyvee g)(b) = 0_J$. Also $f(b) \vee g(b) = 0_J$ implies $f(b) = 0_J$ and $g(b) = 0_J$. Therefore $b \in Z(f) \cap Z(g)$ implies $Z(f \curlyvee g) = Z(f) \cap Z(g)$. Hence $Z(L,J)$ is closed under finite intersection. Thus $Z(L,J)$ is a join semilattice with join defined as the intersection of zero sets. Also $Z(\mathbf{0}_{hom}) = (\sqcap, L)$. Consequently, we obtain that $(Z(L,J), \leqslant')$ is a join semilatttice with bottom element $Z(\mathbf{0}_{hom})$. □

Theorem 4.3.2. *The map $\lambda : L \times Z(L,J) \to Z(L,J)$ defined as $\lambda(a, Z(f)) = Z(\delta(a,f))$ is an action on $Z(L,J)$ and $(\lambda, Z(L,J))$ is an L-slice.*

Proof. We prove all the axioms for λ to be an action on $Z(L,J)$.

i) $\lambda(a, Z(f) \cap Z(g)) = \lambda(a, Z(f \vee g))$

$= Z(\delta(a, f \vee g))$

$= Z(\delta(a,f) \vee \delta(a,g))$

$= Z(\delta(a,f)) \cap Z(\delta(a,g))$

$= \lambda(a, Z(f)) \cap \lambda(a, Z(g))$.

ii) $\lambda(a, Z(\mathbf{0}_{hom})) = Z(\delta(a, \mathbf{0}_{hom}))$

$= Z(\mathbf{0}_{hom}), \forall a \in L$.

iii) $\lambda(a \sqcap b, Z(f)) = Z(\delta(a \sqcap b, f))$.

And $\lambda(a, \lambda(b, Z(f))) = \lambda(a, Z(\delta(b,f))$

$= Z(\delta(a, \delta(b,f)))$

$= Z(\delta(a \sqcap b, f))$.

Similarly $\lambda(b, \lambda(a, Z(f))) = Z(\delta(a \sqcap b, f))$.

Thus $\lambda(a \sqcap b, Z(f)) = \lambda(a, \lambda(b, Z(f))) = \lambda(b, \lambda(a, Z(f)))$.

iv) $\lambda(1_L, Z(f)) = Z(\delta(1_L, f)) = Z(f)$ and

$\lambda(0_L, Z(f)) = Z(\delta(0_L, f)) = Z(\mathbf{0}_{hom})$ for all $f \in Hom(L,J)$.

v) $\lambda(a, Z(f)) \cap \lambda(b, Z(f)) = Z(\delta(a,f)) \cap Z(\delta(b,f))$

$= Z(\delta(a,f) \curlyvee \delta(b,f))$

$= Z(\delta(a \sqcup b, f))$

$= \lambda(a \sqcup b, Z(f))$.

Thus $(\lambda, Z(L,J))$ is a L-slice. □

Remark. We may also view $Z(L, J)$ as the image set of an onto slice morphism $\mathcal{Z}$. The map $\mathcal{Z} : (\delta, Hom(L, J)) \to (\lambda, Z(L, J))$ is defined as $\mathcal{Z}(f) = Z(f)$ .It is a slice morphism because $\mathcal{Z}(f \curlyvee g) = Z(f \curlyvee g) = Z(f) \cap Z(g) = \mathcal{Z}(f) \cap \mathcal{Z}(g)$ and $\mathcal{Z}(\delta(a, f)) = \lambda(a, Z(f)) = \lambda(a, \mathcal{Z}(f))$. Also the kernel of $\mathcal{Z}$ is $\{\mathbf{0}_{hom}\}$.

Definition 4.3.3. A slice morphism $f \in Hom(L, J)$ is called a unit of $Hom(L, J)$ if $Z(f) = \{0_L\}$

Example 4.3.4. *Fix a nonzero $x \in (\sigma, J)$. Define $\sigma_x : (\sqcap, L) \to (\sigma, J)$ as $\sigma_x(0_L) = 0_J$ and $\sigma_x(a) = x$, for every $a \neq 0_L$. Then σ_x is a slice morphism which is a unit of $Hom(L, J)$.*

Definition 4.3.5. The set of all units of $Hom(L, J)$ is called *Units*.

Lemma 4.3.6. *Units is a join semilattice.*

Proof. If $f, g \in$ *Units* then $Z(f) \cap Z(g) = Z(f \curlyvee g) = \{0_L\}$. Thus $f \curlyvee g$ is a unit. □

Lemma 4.3.7. *If g is a unit and $g \preceq f$ then $f \in$ Units.*

Proof. $g \preceq f$ implies $Z(f) \subseteq Z(g)$. Since g is a unit $Z(g) = \{0_L\}$ will imply that $Z(f) = \{0_L\}$. Thus f is a unit. □

Remark. Units is an upper set.

Next we try to endow a topology on the locale L. Analogous to the topology we have developed in $Hom(\sigma, J)$, we try to construct a topology on $(\sqcap, L)$ through the zero sets of the L-slice $Hom(L, J)$.For each slice morphism f in $Hom(L, J)$ we obtain a collection of ideals on the locale L and study the topology so obtained.

4.4. The ideal $\langle f : x \rangle_L$ and the Sierpinski topology

Definition 4.4.1. Let f be a slice morphism such that $f \in Hom(L, J)$ and $x \in (\sigma, J)$ then the set $\langle f : x \rangle_L = \{r \in L : f(r) \leqslant x\}$.

Note that the zero set of f is always a subset of $\langle f : x \rangle_L$.

Theorem 4.4.2. *$\langle f : x \rangle_L$ is an ideal of $(\sqcap, L)$.*

Proof. $0_L \in \langle f : x \rangle_L$ for every f and for every x. Hence it is nonempty. Let $r, s \in \langle f : x \rangle_L$. Then $f(r) \leqslant x$ and $f(s) \leqslant x$ implies $f(r \sqcup s) \leqslant x$. Thus $r \sqcup s \in \langle f : x \rangle_L$. If $r \in \langle f : x \rangle_L$ and $b \in (\sqcap, L)$ then $f(\sqcap(b, r)) = \sigma(b, f(r)) \leqslant \sigma(b, x) \leqslant x$.Therefore, $\sqcap(b, r) \in \langle f : x \rangle_L$. Also if $a \in \langle f : x \rangle_L$ and $b \sqsubseteq a$ then clearly $b \in \langle f : x \rangle_L$. Hence $\langle f : x \rangle_L$ is an ideal of $(\sqcap, L)$. □

Properties of the ideal $\langle f : x \rangle_L$

Property 1. *If f is surjective then for every $y \in (\sigma, J)$ there exists $a \in (\sqcap, L)$ such that $f(a) = y$. Then $\downarrow a \subseteq \langle f : y \rangle_L$.*

Property 2. *If $f \preceq g$ then $\langle f : x \rangle_L \supseteq \langle g : x \rangle_L$.*

Property 3. *If $x \leqslant y$ then $\langle f : x \rangle_L \subseteq \langle f : y \rangle_L$.*

Property 4. $\langle \boldsymbol{0}_{hom} : x \rangle_L = L = \langle \sigma_x : x \rangle_L$.

Property 5. $\langle f : 0_J \rangle_L = Z(f)$.

Proof. If $r \in \langle f : 0_J \rangle_L$ then $f(r) = 0_J$ implies $r \in Z(f)$. Also $Z(f)$ is always a subset of $\langle f : x \rangle_L$ for every $x \in (\sigma, J)$. Thus $\langle f : 0_J \rangle_L = Z(f)$. □

Property 6. $\langle f : x\rangle_L \cap \langle f : y\rangle_L \subseteq \langle f : x \vee y\rangle_L$.

Proof. If $r \in \langle f : x\rangle_L \cap \langle f : y\rangle_L$ then $f(r) \leqslant x$ and $f(r) \leqslant y$ will imply $f(r) \leqslant x \vee y$. Thus $\langle f : x\rangle_L \cap \langle f : y\rangle_L \subseteq \langle f : x \vee y\rangle_L$. □

Property 7. $\langle f : x\rangle_L \cap \langle g : x\rangle_L = \langle f \curlyvee g : x\rangle_L$.

Proof. If $r \in \langle f : x\rangle_L \cap \langle g : x\rangle_L$ then $f(r) \leqslant x$ and $g(r) \leqslant x$ shows that $f \curlyvee g(r) \leqslant x$. Therefore $\langle f : x\rangle_L \cap \langle g : x\rangle_L \subseteq \langle f \curlyvee g : x\rangle_L$. Also if $s \in \langle f \curlyvee g : x\rangle_L$ then $f \curlyvee g(s) \leqslant x$ implies $f(s) \leqslant x$ and $g(s) \leqslant x$. Thus $\langle f : x\rangle_L \cap \langle g : x\rangle_L = \langle f \curlyvee g : x\rangle_L$. □

Definition 4.4.3. Consider the collection $\mathfrak{B}^{x}_{(\sqcap,L)} = \{\langle f : x\rangle_L : f \in Hom(L, J)\}$. The property 4 and property 7 shows that the collection $\mathfrak{B}^{x}_{(\sqcap,L)}$ forms a basis for a topology on $(\sqcap, L)$.

Also each open set of the topology so generated contains a zero set.In particular, for the collection $\mathfrak{B}^{0_J}_{(\sqcap,L)} = \{\langle f : 0_J\rangle_L : f \in Hom(L, J)\}$ the zero sets of slice morphisms from $(\sqcap, L)$ to (σ, J) forms a basis for a topology on $(\sqcap, L)$.

Proposition 4.4.4. *If every $f \neq \mathbf{0}_{hom}$ is a unit then the topology generated by $\mathfrak{B}^{0_J}_{(\sqcap,L)}$ is Sierpinski topology.*

Proof. The property 4 shows that $\langle \mathbf{0}_{hom} : x\rangle_L = L$. If $x = 0_J$ then $\langle \mathbf{0}_{hom} : 0_J\rangle_L = L$. Also , since every slice morphism $f \in Hom(L, J)$ is a unit we have $Z(f) = \{0_L\}$. The definition of $\mathfrak{B}^{0_J}_{(\sqcap,L)}$ shows that the topology so generated by $\mathfrak{B}^{0_J}_{(\sqcap,L)}$ will be $\tau_{0_J} = \{L, \{0_L\}, \phi\}$. □

4.5. Ideals and Z-filter of $Hom(L, J)$

$Hom(L, J)$ is an L-slice and an ideal of $Hom(L, J)$ can be defined accordingly.

Definition 4.5.1. An ideal I in $Hom(L, J)$ is a set such that it satisfies the following conditions.

i) $f, g \in I$ implies $f \vee g \in I$

ii) If $f \in I$ then $\delta(r, f) \in I$

iii) If $f \in I$ and $h \preceq f$ then $h \in I$.

Note that an ideal will always mean a proper ideal that is, $I \neq \{\mathbf{0}_{hom}\}$

Definition 4.5.2. An ideal I is said to be prime ideal if $\delta(r \sqcap s, f) \in I$ implies either $\delta(r, f) \in I$ or $\delta(s, f) \in I$.

Remark. In the theory of locales, filters are defined as a generalisation of the concept of filters in a topological space. Filters are not defined in the usual setting of an L-slice. When the locale L is viewed as an L-slice there arises the problem of defining a filter in terms of the action involved. Here we define the notion of Z-filter in terms of the zero sets of the L-slice $Hom(L, J)$.

Definition 4.5.3. A nonempty subfamily $\mathcal{F}$ of $Z(L, J)$ is called Z-filter on $(\sqcap, L)$ provided

i) $\{0_L\} \notin \mathcal{F}$

ii)$Z(f), Z(g) \in \mathcal{F}$ implies $Z(f) \cap Z(g) \in \mathcal{F}$.

iii)$Z(f) \in \mathcal{F}$ and $Z(f) \subseteq Z(g)$ then $Z(g) \in \mathcal{F}$.

Remark. $Z(f)$ is nonempty for every $f \in Hom(L, J)$. Also $Z(\mathbf{0}_{hom}) = L$ implies every Z-filter contains L.

Note that a filter F on L will denote the usual order theoretic filter on L. We have the following theorem on Z-filters.

Theorem 4.5.4. *The intersection of any filter F on L with $Z(L, J)$ is a Z-filter on $(\sqcap, L)$.*

Proof. We have already shown that L belongs to every Z-filter. Thus $F \cap Z(L, J)$ is nonempty. Consider the set $Z_F = \{F \cap Z(f) : Z(f) \in Z(Hom(L, J))\}$. By the above observation each set belonging to Z_F is nonempty. Let $F \cap Z(f), F \cap Z(g) \in Z_F$. Then $(F \cap Z(f)) \cap (F \cap Z(g)) = F \cap (Z(f) \cap Z(g)) = F \cap (Z(f \curlyvee g))$. Therefore $(F \cap Z(f)) \cap (F \cap Z(g)) \in Z_F$. □

Analogous to Z-filters we introduce two new terminologies Z-ideals and strong Z-ideal.

Definition 4.5.5. An ideal I of $Hom(L, J)$ is said to be a Z-ideal if $Z(f) \in Z[I]$ then $f \in I$.

Definition 4.5.6. An ideal I of $Hom(L, J)$ is said to be a strong Z-ideal if $f \in I$ and $Z(f) \subseteq Z(g)$ then $g \preceq f$. Consequently, $g \in I$.

We now study the relationship between strong Z-ideals and Z-filters.

Theorem 4.5.7. *Let $\mathcal{F}$ be a Z-filter on L. The sub-family $Z^{\leftarrow}[\mathcal{F}]$ of $Hom(L, J)$ defined as $Z^{\leftarrow}[\mathcal{F}] = \{f \in Hom(L, J) : Z(f) \in \mathcal{F}\}$ is an ideal in $Hom(L, J)$.*

Proof. Let $f, g \in Z^{\leftarrow}[\mathcal{F}]$ then $Z(f) \in \mathcal{F}, Z(g) \in \mathcal{F}$ implies $Z(f) \cap Z(g) = Z(f \curlyvee g) \in \mathcal{F}$. Therefore $f \curlyvee g \in Z^{\leftarrow}[\mathcal{F}]$. Hence $Z^{\leftarrow}[\mathcal{F}]$ is a join semilattice. Let $f \in Z^{\leftarrow}[\mathcal{F}]$. We know that $\delta(r, f) \preceq f$ implies $Z(f) \subseteq Z(\delta(r, f))$. Since $\mathcal{F}$ is Z-filter $Z(\delta(r, f)) \in \mathcal{F}$ and thus $\delta(r, f) \in Z^{\leftarrow}[\mathcal{F}]$. Also whenever $g \preceq f, Z(f) \subseteq Z(g)$ and $\mathcal{F}$ being a Z-filter guarantees that $g \in Z^{\leftarrow}[\mathcal{F}]$. Hence $Z^{\leftarrow}[\mathcal{F}]$ is an ideal of $Hom(L, J)$. □

Remark. The converse of the theorem need not be true. That is, if I is an ideal of $Hom(L, J)$ then $Z[I] = \{Z(f) : f \in I\}$ is not necessarily a Z-filter. $Z[I]$ will be a set which has the finite intersection property.

Theorem 4.5.8. *If I is a strong Z-ideal of $Hom(L, J)$ then $Z[I]$ is a Z-filter on $(\sqcap, L)$.*

Proof. $Z[I] \neq \{0_L\}$. For $Z(f), Z(g) \in Z[I]$, $Z(f \vee g) = Z(f) \cap Z(g) \in Z[I]$. Also if $Z(f) \in Z[I]$ and $Z(g) \in Z(L, J)$ such that $Z(f) \subseteq Z(g)$ then I being strong Z-ideal shows that $g \in I$. Therefore $Z(g) \in Z[I]$. Thus $Z[I]$ is a Z-filter. □

Remark. If for some Z-filter $\mathcal{F}$, $Z^{\leftarrow}[\mathcal{F}]$ is a strong Z-ideal,then $Z[Z^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$.

We examine the change in the structure of $Z^{\leftarrow}[\mathcal{F}]$ when $\mathcal{F}$ is given an additional condition. Let us consider $Z^{\leftarrow}[\mathcal{F}]$ as the image of the map $Z^{\leftarrow}$. Let us denote the collection of all Z-filters as $\mathbf{Z}$ and the collection of all ideals in $Hom(L, J)$ as $\mathbf{I}$. The map $Z^{\leftarrow}$ is a map from $\mathbf{Z}$ to $\mathbf{I}$.

Definition 4.5.9. A maximal Z-filter is called Z-ultra filter on $(\sqcap, L)$.

Theorem 4.5.10. *If $\mathcal{U}$ is an Z-ultra filter on $(\sqcap, L)$, then $Z^{\leftarrow}[\mathcal{U}]$ is a maximal ideal in $Hom(L, J)$.*

Proof. We know that $Z^{\leftarrow}[\mathcal{U}]$ is an ideal in $Hom(L, J)$. It remains to show that the map $Z^{\leftarrow} : \mathbf{Z} \to \mathbf{I}$ preserves inclusion. Let $\mathcal{F}$ and $\mathcal{G}$ be two Z-filters such that $\mathcal{F} \subseteq \mathcal{G}$. We show that $Z^{\leftarrow}(\mathcal{F}) \subseteq Z^{\leftarrow}(\mathcal{G})$. The set $Z^{\leftarrow}[\mathcal{F}]$ is an ideal of $Hom(L, J)$. If $f \in Z^{\leftarrow}[\mathcal{F}]$ then $Z(f) \in \mathcal{F}$ will imply $Z(f) \in \mathcal{G}$. Therefore $Z^{\leftarrow}(\mathcal{F}) \subseteq Z^{\leftarrow}(\mathcal{G})$ and thus $Z^{\leftarrow}$ preseves inclusion. Hence if $\mathcal{U}$ is an Z-ultra filter then $Z^{\leftarrow}[\mathcal{U}]$ is a maximal ideal. □

The next theorem gives a characterisation for Z-ultra filter.

Theorem 4.5.11. *Let $\mathcal{U}$ be an Z-ultra filter on $(\sqcap, L)$. If a zero set Z has nontrivial intersection with every member of the ultra filter $\mathcal{U}$ then $Z \in \mathcal{U}$.*

Proof. Since $\mathcal{U} \cup \{Z\}$ has the finite intersection property, $\mathcal{U} \cup \{Z\}$ generates a Z-filter. Now this Z-filter contains $\mathcal{U}$ and $\mathcal{U}$ being maximal the Z-filter so generated must be $\mathcal{U}$ itself. Therefore, $Z \in \mathcal{U}$. □

Example 4.5.12. *Let $a \in (\sqcap, L)$ and $I = \{f \in Hom(L, J) :\downarrow a \subseteq Z(f)\}$, then I is an Z-ideal.*

Proof. It can be easily shown that I is an ideal. To prove that it is Z-ideal, let us consider $f \in I$ and any $g \in Hom(L, J)$ such that $Z(f) = Z(g)$. Then $\downarrow a \subseteq Z(f) = Z(g)$ implies $g \in I$. □

Lemma 4.5.13. *Let I and J be any two Z-ideals, then $Z[I \cap J] = Z[I] \cap Z[J]$.*

Proof. Since I and J are ideals, so is $I \cap J$. Also $Z[I \cap J] = \{Z(f) : f \in I \cap J\}$. Then $Z(f) \in Z[I]$ and $Z(f) \in Z[J]$. Hence $Z[I \cap J] \subseteq Z[I] \cap Z[J]$.
Let $Z(g) \in Z[I] \cap Z[J]$. Then I and J being Z-ideals imply that $g \in I$ and $g \in J$. Therefore $Z(g) \in Z[I \cap J]$. Thus $Z[I \cap J] = Z[I] \cap Z[J]$. □

Theorem 4.5.14. *The intersection of any two Z-ideals is a Z-ideal.*

Proof. Let I and J be any two Z-ideals. Let $Z(f) \in Z[I \cap J] = Z[I] \cap Z[J]$. Since I and J are Z-ideals $Z(f) \in Z[I]$ and $Z(f) \in Z[J]$ shows that $f \in I$ and $f \in J$. Hence $f \in I \cap J$. □

Proposition 4.5.15. *Let S be a subset of a locale L. The family of all functions in $Hom(L, J)$ such that $f(S) = \{0_J\}$ is a Z-ideal.*

Proof. $I = \{f \in Hom(L, J) : f(s) = 0_J, \forall s \in S\}$. For any $f \in I$, we have $S \subseteq Z(f)$. If $f, g \in I$ then $(f \vee g)(s) = f(s) \vee g(s) = 0_J, \forall s \in S$. Therefore $f \vee g \in I$. Also if $f \in I$ and $h \leqslant f$ then $S \subseteq Z(f) \subseteq Z(h)$ implies $h \in I$. For $f \in I, S \subseteq Z(f) \subseteq Z(\delta(r, f))$ implies $\delta(r, f) \in I$. Thus I is an ideal of $Hom(L, J)$. Consider $g \in Hom(L, J)$ such that $Z(f) = Z(g)$, for some $f \in I$. Then $S \subseteq Z(f) = Z(g)$.Thus $g \in I$ and consequently I is a Z-ideal . □

Theorem 4.5.16. *Let I be any ideal in $Hom(L, J)$ that contains a prime ideal. If there exists a $g \in Hom(L, J)$ such that $\delta(r \sqcap s, g) = 0_{Hom}$ then either $\delta(r, g) \in I$ or $\delta(s, g) \in I$.*

Proof. Let P be the prime ideal contained in I . Let g be such that there exists $r, s \in (\sqcap, L)$ with $(r \sqcap s, g) = 0_{Hom}$. Since P is a prime ideal $\delta(r \sqcap s, g) = 0_{Hom} \in P$ implies either $\delta(r, g) \in P$ or $\delta(s, g) \in P$. Hence $\delta(r, g) \in I$ or $\delta(s, g) \in I$. □

4.6. Fixed Ideals of $Hom(L, J)$

For any ideal I of $Hom(L, J)$define $\bigcap Z[I] = \{Z(f) : f \in I\}$. For any slice morphism $f \in Hom(L, J), Z(f)$ always contains $\{0_L\}$ and hence nonempty. Whenever we say that a set involving zero sets is nontrivial, it would imply that the set contains an element other than 0_L.In the first chapter we had introduced regular filters on the locale L. In a similar manner we give a definition of ideals on $Hom(L, J)$.

Definition 4.6.1. An ideal I is said to be fixed if $\bigcap Z[I]$ is nontrivial. Also, if $\bigcap Z[I] = \{0_L\}$ then I is said to be free.

Obviously the ideal $I = \{0_{Hom}\}$ is fixed. Also any ideal I is a free ideal if and only if for every $a \in (\sqcap, L)$ there is a slice morphism in I such that it does not vanish at a.

Note that in [25] Gillmann and Jerrison defines fixed ideal as that ideal for which $\bigcap Z[I]$ is nonempty. But in our case this arises naturally for every ideal in the L-slice $Hom(L, J)$. Every zero set contains 0_L and hence any intersection of zero sets is nonempty. Similar is the case of free ideal. Thus the definitions in the background of L-slices differ and we explore the various results associated with it.

Proposition 4.6.2. *If Z(f) is nontrivial for some $f \in Hom(L, J)$ then the principal ideal $\downarrow f$ is fixed.*

Proof. For every $g \in \downarrow f, Z(f) \subseteq Z(g)$. Let $a(\neq 0_L) \in Z(f)$ then $a \in Z(g)$ for every $g \in \downarrow f$.Thus $a \in \bigcap Z[\downarrow f]$ and consequently $\downarrow f$ is fixed. □

Remark. Suppose that the ideal I is free. If there exists $f \in I$ such that $Z(f)$ is nontrivial then by the above proposition $\downarrow f$ is fixed. Thus every free ideal contains a fixed ideal. However, the converse is not true. If I is a fixed ideal then $\bigcap Z[I]$ is nontrivial and hence it will never contain a free ideal.

Example 4.6.3. *Consider a nonempty set H of the L-slice $(\sqcap, L)$. Also let $H \neq \{0_L\}$. We have already shown that $I = \{f \in Hom(L, J) : f[H] = \{0_J\}\}$ is an Z-ideal in $Hom(L, J)$. Obviously, I is a fixed ideal.*

Proposition 4.6.4. *The intersection of any two fixed ideals is also a fixed ideal.*

Proof. Let I and J be any two fixed ideals. We have that $I \cap J$ is also an ideal. Let $\bigcap Z[I] = A$, where A is nonempty and $A \subseteq (\sqcap, L)$. For every $f \in I$ and $a \in A, f(a) = 0_J$. Similarly for the fixed ideal J there exists a nonempty set $B \subseteq (\sqcap, L)$ such that$g(b) = 0_J$ for every $g \in J$ and $b \in B$. If $h \in I \cap J$ then $h(a) = 0_J$ and $h(b) = 0_J$ for every $a \in A$ and $b \in B$ .Thus $A \cup B \subseteq Z(h)$ for every $h \in I \cap J$.Therefore $I \cap J$ is a fixed ideal. □

Note that the above proposition is not true in the case of free ideals. But we have the following proposition concerning maximal ideals and free ideals.

Proposition 4.6.5. *Let J be any free ideal then the maximal ideal containing J is also a free ideal.*

Proof. Let M be the maximal ideal containing$J$. Since J is a free ideal we have $\bigcap Z[J] = \{0_L\}$. Suppose that $a \neq 0_L \in \bigcap Z[M]$. Then $f(a) = 0_J, \forall f \in M$ and $J \subseteq M$ shows that $g(a) = 0_J, \forall g \in J$.But J is a free ideal .Thus we arrive at a contradiction. Hence M is also a free ideal. □

4.7. The fixed ideal M_p of $Hom(L, J)$

Definition 4.7.1. For each p in the L-slice $(\sqcap, L)$, we define a subset of $Hom(L, J)$ as $M_p = \{f \in Hom(L, J) : f(p) = 0_J\}$. In other words, $f \in M_p$ if and only if $p \in Z(f)$.

Remark. The set M_p is the set of all those functions that vanish at p.Thus whenever p belongs to the zero set of a slice morphism f it will belong to the set M_p. Hence the zero sets and the sets M_p are related to each other.

We investigate the structure and properties of the set M_p. The previous section on zero sets have shown that the zero set of a slice morphism is an ideal. Similarly the next theorem shows that for each $p \in (\sqcap, L)$ the set M_p is an ideal of $Hom(L, J)$

Theorem 4.7.2. *The set M_p is an ideal of $Hom(L, J)$ for every $p \in (\sqcap, L)$,*

Proof. If $f, g \in M_p$ then $f \vee g(p) = f(p) \vee g(p) = 0_J$. Therefore $f \vee g \in M_p$.
Also if $h \leqslant f$ then $h(p) = 0_J$ and $\delta(a, f)(p) = \sigma(a, f(p)) = 0_J$ shows that M_p is an ideal of $Hom(L, J)$. □

Proposition 4.7.3. *If L is a σ - domain then M_p is a prime ideal for every $p \in (\sqcap, L)$.*

Proof. If $\delta(a \sqcap b, f) \in M_p$ then $\delta(a \sqcap b, f)(p) = \sigma(a \sqcap b, f(p)) = \sigma(a, \sigma(b, f(p)) = 0_J$. Since L is a σ - domain, either $a = 0_L$ or $\sigma(b, f(p)) = 0_J$. That is, either we have $\sigma(a, f(p)) = 0_J$ or $\sigma(b, f(p)) = 0_J$.Therefore either $\delta(a, f) \in M_p$ or $\delta(b, f) \in M_p$. $\square$

Proposition 4.7.4. *The ideal M_p exhibits the following properties*

i)If $p \sqsubseteq q$ then $M_q \subseteq M_p$

ii) $M_p \cap M_q = M_{p \sqcup q}$

iii) $M_{0_L} = Hom(L, J)$

iv)For $p \neq 0_L, M_p$ is a fixed ideal.

Proof. i) Let $f \in M_q$. Since f is a slice morphism $f(p) \leqslant f(q) = 0_J$ implies $f(p) = 0_J$. Therefore$f \in M_p$.

ii) If $f \in M_p \cap M_q$, then $f(p \sqcup q) = f(p) \vee f(q) = 0_J$. Therefore $f \in M_{p \sqcup q}$ and $M_p \cap M_q \subseteq M_{p \sqcup q}$. Let $g \in M_{p \sqcup q}$. Then $g(p \sqcup q) = g(p) \vee g(q) = 0_J$ implies that $g(p) = 0_J$ and $g(q) = 0_J$.Thus $g \in M_p \cap M_q$. Consequently, $M_p \cap M_q = M_{p \sqcup q}$.

iii) $f(0_L) = 0_J$ for every $f \in Hom(L, J)$. Hence , $M_{0_L} = Hom(L, J)$.

iv)Let $p \neq 0_L$. From the definition of the ideal M_p it follows $p \in \bigcap Z[M_p]$.Thus M_p is fixed. $\square$

The similarity in the properties of fixed ideals M_p of $Hom(L, J)$ and the zero sets $(\sqcap, L)$ are illustrated by i) and ii) of the above proposition.

The next theorem gives more insight into the relationship between the zero sets and fixed ideal M_p.

Theorem 4.7.5. *i)If $M_p \subseteq M_q$, then $\mid \bigcap Z[M_p] \mid \geqslant 3$*

ii) M_p is a strong Z-ideal.

iii) $Z[M_p]$ is Z-filter.

Proof. i) $M_p \subseteq M_q$ shows that to every $f \in M_p$, the corresponding zero set $Z(f)$ contains the set $\{0_L, p, q\}$. Thus $| \bigcap Z[M_p] | \geqslant 3$.

ii) Let $f \in M_p$ and $Z(f) \subseteq Z(g)$. Therefore $p \in Z(g)$ and hence $g \in M_p$.Thus M_p is strong Z-ideal.

iii) Follows from theorem 4.5.8 □

In the previous section we have developed topology which contains zero sets as open sets. Here on similar lines, we can develop a topology for which the fixed ideals M_p will form a basis .

Theorem 4.7.6. *The collection $\mathfrak{M} = \{M_p : p \in (\sqcap, L)\}$ forms a basis for a topology on $Hom(L, J)$.*

Proof. The proposition 4.7.4 guarantees the existence of such a basis and we hence the theorem. □

Remark. Consider the open set $\langle f : 0_J \rangle_L$ belonging to the basis $\mathfrak{B}^{0_J}_{(\sqcap,L)}$. For any r in $(\sqcap, L)$, $r \in \langle f : 0_J \rangle_L$ implies that $f \in M_r$, the open set in $\mathfrak{M}$.

We show that $\mathfrak{M}$ is a join semilattice and eventually construct an L-slice on $\mathfrak{M}$.

Lemma 4.7.7. $\mathfrak{M}$ *is a join semilattice.*

Proof. Partially order the collection $\mathfrak{M}$ as $M_p \leqq M_q$ if and only if $M_p \supseteq M_q$. The join is then defined as $M_p \veebar M_q = M_p \cap M_q = M_{p \sqcup q}$. Also, $M_{0_L} = Hom(L, J)$. Thus $\mathfrak{M}$ is a join semilattice with bottom element M_{0_L}. □

Theorem 4.7.8. *The map* $\overline{\wedge} : L \times \mathfrak{M} \to \mathfrak{M}$ *defined as* $\overline{\wedge}(a, M_p) = M_{\sqcap(a,p)}$ *is an action on* $\mathfrak{M}$ *and consequently* $(\overline{\wedge}, \mathfrak{M})$ *is an L-slice.*

Proof.

1. $\overline{\wedge}(a, M_p \veebar M_q) = \overline{\wedge}(a, M_{p\sqcup q})$

$$= M_{\sqcap(a,p\sqcup q)}$$

$$= M_{\sqcap(a,p)\ \sqcup\ \sqcap(a,q)}$$

$$= M_{\sqcap(a,p)} \veebar M_{\sqcap(a,q)}$$

$$= \overline{\wedge}(a, M_p) \veebar \overline{\wedge}(a, M_q), \text{ for all } a \in L \text{ and } M_p, M_q \in \mathfrak{M}.$$

2. $\overline{\wedge}(a, M_{0_L}) = M_{\sqcap(a,0_L)} = M_{0_L}$ for all $a \in L$.

3. For all $a, b \in L$ and $M_p \in \mathfrak{M}$, $\overline{\wedge}(a \sqcap b, M_p) = M_{\sqcap(a\sqcap b,p)}$

$$= M_{\sqcap(a,\sqcap(b,p))}$$

$$= \overline{\wedge}(a, M_{\sqcap(b,p)})$$

$$= \overline{\wedge}(a, \overline{\wedge}(b, M_p)).$$

Also, $\overline{\wedge}(a \sqcap b, M_p) = \overline{\wedge}(b, \overline{\wedge}(a, M_p))$.

4. $\overline{\wedge}(1_L, M_p) = M_{\sqcap(1_L,p)} = M_p$ and $\overline{\wedge}(0_L, M_p) = M_{\sqcap(0_L,p)} = M_{0_L}$, for all $M_p \in \mathfrak{M}$

5. $\overline{\wedge}(a \sqcup b, M_p) = M_{\sqcap(a\sqcup b,p)} = M_{\sqcap(a,p)\ \sqcup\ \sqcap(b,p)}$

$$= M_{\sqcap(a,p)} \veebar M_{\sqcap(b,p)}$$

$$= \overline{\wedge}(a, M_p) \veebar \overline{\wedge}(b, M_p), \text{ for all } a, b \in L \text{ and } M_p \in \mathfrak{M}.$$

Thus $\overline{\wedge}$ is an action of locale L on $\mathfrak{M}$ and $(\overline{\wedge}, \mathfrak{M})$ is an L-slice. □

Now we may associate to each $a \in (\sqcap, L)$ an Z-filter in $Z(L, J)$ through a slice morphism.

Theorem 4.7.9. *The map* $\mu : (\sqcap, L) \to (\overline{\curlywedge}, \mathfrak{M})$ *defined as* $\mu(a) = M_a$ *is a slice morphism.*

Proof. $\mu(a \sqcup b) = M_{a \sqcup b} = M_a \veebar M_b = \mu(a) \veebar \mu(b)$ for every $a, b \in (\sqcap, L)$.Thus μ preserves join. Also $\mu(\sqcap(a, r)) = M_{\sqcap(a,r)} = \overline{\curlywedge}(a, M_r) = \overline{\curlywedge}(a, \mu(r))$, for all $r \in (\sqcap, L)$. Therefore it preserves action. Hence μ is a slice morphism. □

Let $Z - Fil$ denote the collection of all Z-filters on the locale L. Define a map on $\mathfrak{M}$ as $\tilde{Z} : \mathfrak{M} \to Z - Fil$ is the natural map that takes each $M_p \in \mathfrak{M}$ to the corresponding Z-Filter , $Z[M_p]$. The composition $Z \circ \mu : (\sqcap, L) \to Z - Fil$ takes each element $r \in (\sqcap, L)$ to the Z-Filter $Z[M_r]$.

Chapter 5

Zariski topology on L-slices

Modules are the action of a ring on a group. The postulates for Modules and L-slices are somewhat similar. The algebraic properties of L-slices prompts us to elaborate the study of L-slices in the direction of Modules. In [59] L-slices and TopW-Modules are found to be related. This resulted in observing the L-slice (σ, J) as a module.

We have tried to extend the idea of Zariski topology on modules to L-slices. Given a locale L and a L-slice (σ, J), for $m \in (\sigma, J)$ and $r \in L$, we have constructed (σ, J) ideals $[r \to m]_{(\sigma,J)} = \{n \in (\sigma, J) : \sigma(r, n) \leqslant m\}$. Their properties and characteristics are studied. Similarly for a given L-slice (σ, J) and $n, m \in (\sigma, J)$, we examine the properties of L-ideals $[n \to m]_L = \{r \in (\sigma, J) : \sigma(r, n) \leqslant m\}$. The notion of L-prime elements on (σ, J) and their properties are discussed. The collection of L-prime elements is defined as $Spec(\sigma, J)$ and the possibility of existence of Zariski topology on it is examined.

5.1. Implicative Ideals of L

In this section we define sets on the locale L which are found to be ideals called implicative ideals.

Let L be a locale and (σ, J) be an L - slice with bottom element 0_J. For $n, l \in (\sigma, J)$, define a set $[l \to n]_L = \{r \in L : \sigma(r, l) \leqslant n\}$.

Proposition 5.1.1. *For $n, l \in (\sigma, J)$, $[l \to n]_L$ is an ideal of L.*

Proof. We know that $\sigma(0_L, l) \leqslant n \ \forall \ l \in (\sigma, J)$. Hence $[l \to n]_L$ is nonempty.

Let $a, b \in [l \to n]_L$. Then $\sigma(a, l) \leqslant n$ and $\sigma(b, l) \leqslant n$ implies $\sigma(a \vee b, l) \leqslant n$. Therefore $a \vee b \in [l \to n]_L$. If $c \in L$ and $c \leqslant a$ then $\sigma(c, l) \leqslant \sigma(a, l) \leqslant n$ implies that $c \in [l \to n]_L$. Hence $[l \to n]_L$ is an ideal of L. □

Definition 5.1.2. For $n, l \in (\sigma, J)$, the ideal $[l \to n]_L$ is called the implicative ideal of the locale L.

Proposition 5.1.3. *i) If $n, l \in (\sigma, J)$ and $n \leqslant l$ then $[x \to n]_L \subseteq [x \to l]_L \ \ \forall \ x \in (\sigma, J)$*

ii) If $n, l, k \in (\sigma, J)$ and $n \leqslant l$ then $[l \to k]_L \subseteq [n \to k]_L$.

Proof. i) If $r \in [x \to n]_L$ then $\sigma(r, x) \leqslant n \leqslant l$ implies $r \in [x \to l]_L$. Therefore $[x \to n]_L \subseteq [x \to l]_L \ \ \forall \ x \in (\sigma, J)$.

ii) $s \in [l \to k]_L \Rightarrow \sigma(s, l) \leqslant k$. Since $n \leqslant l, \sigma(s, n) \leqslant \sigma(s, l) \leqslant k$. Therefore $s \in [n \to k]_L$. Thus $[l \to k]_L \subseteq [n \to k]_L$. □

Proposition 5.1.4. *For $n, l \in (\sigma, J)$, $[l \to k]_L \cap [n \to k]_L = [l \vee n \to k]_L$.*

Proof. Let $r \in [l \to k]_L \cap [n \to k]_L$.

$$\begin{aligned} r \in [l \to k]_L \cap [n \to k]_L &\Leftrightarrow \sigma(r,l) \leqslant k \text{ and } \sigma(r,n) \leqslant k \\ &\Leftrightarrow \sigma(r,l) \vee \sigma(r,n) \leqslant k \\ &\Leftrightarrow \sigma(r, l \vee n) \leqslant k \\ &\Leftrightarrow r \in [l \vee n \to k]_L \end{aligned}$$

Therefore $[l \to k]_L \cap [n \to k]_L = [l \vee n \to k]_L$. □

Similarly we obtain the following proposition.

Proposition 5.1.5. *For $n, l, k \in (\sigma, J)$, $[k \to l]_L \cap [k \to n]_L = [k \to l \vee n]_L$.*

5.2. Implicative Ideals of (σ, J)

Let $n \in (\sigma, J)$ and $r \in L$ then we define a set $[r \to n]_{(\sigma,J)} = \{l \in (\sigma, J) : \sigma(r,l) \leqslant n\}$.

Proposition 5.2.1. *For $r \in L$ and $n \in (\sigma, J)$, $[r \to n]_{(\sigma,J)}$ is an ideal of (σ, J).*

Proof. $0_J \in [r \to n]_{(\sigma,J)}$ and hence it is nonempty.Let $l, m \in [r \to n]_{(\sigma,J)}$ then $\sigma(r,l) \leqslant n$ and $\sigma(r,m) \leqslant n$ implies $\sigma(r, l \vee m) \leqslant n$. Hence $l \vee m \in [r \to n]_{(\sigma,J)}$. Also, if $x \leqslant l$ for some $x \in (\sigma, J)$ then $\sigma(r,x) \leqslant \sigma(r,l) \leqslant n$. Thus $x \in [r \to n]_{(\sigma,J)}$. If $x \in [r \to n]_{(\sigma,J)}$ and $a \in L$ then $\sigma(r, \sigma(a,x)) = \sigma(a, \sigma(r,x)) \leqslant \sigma(r,x) \leqslant n$. Therefore $[r \to n]_{(\sigma,J)}$ is an ideal of (σ, J). □

Definition 5.2.2. The ideal $[r \to n]_{(\sigma,J)}$ is called the implicative ideal of (σ, J).

The following properties of implicative ideals of (σ, J) can be easily verified.

Proposition 5.2.3. *i) For $r, s \in L$, if $r \leqslant s$, then $[r \to n]_{(\sigma,J)} \supseteq [s \to n]_{(\sigma,J)}$*

ii) For $m, n \in (\sigma, J)$, $n \leqslant m$ implies $[r \to n]_{(\sigma,J)} \subseteq [r \to m]_{(\sigma,J)}$

iii) $[r \to n]_{(\sigma,J)} \cup [r \to m]_{(\sigma,J)} \subseteq [r \to n \vee m]_{(\sigma,J)}$

Lemma 5.2.4. *Let $n \in (\sigma, J)$ and $r, s \in L$*

i) $[0_L \to n]_{(\sigma,J)} = (\sigma, J)$

ii) $[1_L \to n]_{(\sigma,J)} = \downarrow n$

iii) $[r \to n]_{(\sigma,J)} \cap [s \to n]_{(\sigma,J)} = [r \sqcup s \to n]_{(\sigma,J)}$, where $r, s \in L$.

Proof. i, ii follows from definitions. To prove iii , let $l \in [r \to n]_{(\sigma,J)} \cap [s \to n]_{(\sigma,J)}$ then $\sigma(r, l) \leqslant n$ and $\sigma(s, l) \leqslant n$ implies $\sigma(r \sqcup s, l) \leqslant n$. Hence $l \in [r \sqcup s \to n]_{(\sigma,J)}$ and $[r \to n]_{(\sigma,J)} \cap [s \to n]_{(\sigma,J)} \subseteq [r \sqcup s \to n]_{(\sigma,J)}$. Consider $m \in [r \sqcup s \to n]_{(\sigma,J)}$. Then

$$\begin{aligned} m \in [r \sqcup s \to n]_{(\sigma,J)} &\Rightarrow \sigma(r \sqcup s, m) \leqslant n \\ &\Rightarrow \sigma(r, m) \vee \sigma(s, m) \leqslant n \\ &\Rightarrow \sigma(r, m) \leqslant n \text{ and } \sigma(s, m) \leqslant n \\ &\Rightarrow m \in [r \to n]_{(\sigma,J)} \cap [s \to n]_{(\sigma,J)} \end{aligned}$$

Thus $[r \to n]_{(\sigma,J)} \cap [s \to n]_{(\sigma,J)} \supseteq [r \sqcup s \to n]_{(\sigma,J)}$.

Hence $[r \to n]_{(\sigma,J)} \cap [s \to n]_{(\sigma,J)} = [r \sqcup s \to n]_{(\sigma,J)}$.

□

The following theorem is obvious consequence of the above lemma.

Theorem 5.2.5. *The collection $\{[r \to n]_{(\sigma,J)} : n \in (\sigma, J), r \in L\}$ forms a basis for a topology on the L-slice (σ, J).*

5.3. Ideal Element of (σ, C)

In this section we introduce the concept of L -component (σ, C). We construct a L-slice over a sup lattice. A complete L -slice is called a L -component, that is, it is the action of a locale on a sup lattice. Further, we define the action of an ideal on $x \in (\sigma, C)$ and ideal elements of (σ, C).

Definition 5.3.1. Let L be a locale and C be a sup lattice with bottom element 0_C and top element 1_C. The L -component (σ, C) is the action of L on C which is defined as a map $\sigma : L \times C \to C$ such that it satisfies the following conditions in addition to the definition of L-slice

i) $\sigma(a, \bigvee x_i) = \bigvee \sigma(a, x_i)$ for $\{x_i\}_{i\in I} \in (\sigma, C)$, for some indexed set I

ii) $\sigma(\bigsqcup_i a_i, x) = \bigvee_i \sigma(a_i, x)$ for $\{a_i\}_{i\in I} \in L$, for some indexed set I.

Example 5.3.2. $(\sqcap, L)$ *is an L-component.*

Definition 5.3.3. Let I be an ideal of the locale L. For $x \in (\sigma, C)$ we define the action of I on x as $\sigma(I, x) = \bigvee\{\vee_{i=1}^{k}\sigma(a_i, x) : k \in N, a_i \in I\}$ and is denoted as Ix.

Definition 5.3.4. Let I be an ideal of the locale L and $n \in (\sigma, C)$. The element $[n : I] = \bigvee\{x \in (\sigma, C) : Ix \leqslant n\}$ of (σ, C) is called an ideal element of (σ, C).

Properties of ideal elements

Proposition 5.3.5. *Let (σ, C) be a L- component and I be an ideal of a locale L then for every $n \in (\sigma, C)$*

i) $n \leqslant [n : I]$

ii) $I[n : I] \leqslant n$.

Proof. i) We have

$$\begin{aligned}
\sigma(a_i, n) \leqslant n \ \ \forall\, a_i \in I \ &\Rightarrow\ \vee_{i=1}^{k}\sigma(a_i, n) \leqslant n \\
&\Rightarrow\ \bigvee\{\vee_{i=1}^{k}\sigma(a_i, x) : k \in N, a_i \in I\} \leqslant n \\
&\Rightarrow\ In \leqslant n
\end{aligned}$$

Therefore $n \leqslant [n : I]$.

ii) Let $[n : I] = m$ and $\{x \in (\sigma, C) : Ix \leqslant n\} = X$ so that we have $\bigvee_{x \in X} x = m$.
For any $a \in I, \sigma(a, m) = \sigma(a, \bigvee_{x \in X} x) = \bigvee_{x \in X} \sigma(a, x)$. Therefore $\sigma(a, x) \leqslant Ix \leqslant n$ implies $\bigvee_{x \in X} \sigma(a, x) \leqslant n$ so that $\sigma(a, m) \leqslant n$. Thus for every $a_i \in I$,

$$\begin{aligned}
\sigma(a_i, m) \leqslant n \ &\Rightarrow\ \vee_{i=1}^{k}\sigma(a_i, m) \leqslant n \\
&\Rightarrow\ \bigvee\{\vee_{i=1}^{k}\sigma(a_i, m) : k \in \mathbb{N}\} \leqslant n \\
&\Rightarrow\ Im \leqslant n \\
&\Rightarrow\ I[n : I] \leqslant n.
\end{aligned}$$

□

Proposition 5.3.6. *Let I, J be ideals of a locale L and $x \in (\sigma, C)$. If $I \subseteq J$ then $Ix \leqslant Jx$.*

Proof. $Ix = \bigvee\{\vee_{i=1}^{k}\sigma(a_i, x) : a_i \in I, k \in \mathbb{N}\} \leqslant \bigvee\{\vee_{i=1}^{k}\sigma(b_i, x) : b_i \in J, k \in N\} = Jx$. Thus $Ix \leqslant Jx$ for all $x \in (\sigma, C)$. □

Proposition 5.3.7. *For* $x, n \in (\sigma, C), [x \to n]_L \subseteq [x \to [x \to n]_L x]_L$.

Proof. Let $r \in [x \to n]_L$. Then $\sigma(r, x) \leqslant n$ implies $\sigma(r, x) \leqslant [x \to n]_L x$. Hence $r \in [x \to [x \to n]_L x]_L$. Therefore $[x \to n]_L \subseteq [x \to [x \to n]_L x]_L$. □

Proposition 5.3.8. *For* $x, n \in (\sigma, C), [x \to n]_L x \leqslant n$.

Proof. $[x \to n]_L x = \bigvee\{\vee_{i=1}^k \sigma(a_i, x) : a_i \in [x \to n]_L, k \in \mathbb{N}\} = \bigvee\{\vee_{i=1}^k \sigma(a_i, x) : \sigma(a_i, x) \leqslant n, k \in \mathbb{N}\}$. Hence the proof. □

5.4. L-Prime Elements and *Spec*(σ, C)

Definition 5.4.1. An element $p \neq 1_C$ of (σ, C) is said to be L-prime element if for every $r \in L$ and $n \in (\sigma, C), \sigma(r, n) \leqslant p$ implies that either $r \in [1_C \to p]_L$ or $n \leqslant p$.

Example 5.4.2. *If we consider the L-slice* $(\sqcap, L)$ *then the L-prime elements are precisely the meet irreducible elements of* L.

Properties of L-prime elements

Theorem 5.4.3. *If* p *be a L-prime element and* $x \in (\sigma, C)$ *then* $[x \to p]_L$ *is a prime ideal of* L.

Proof. Let $r \sqcap s \in [x \to p]$, then $\sigma(r \sqcap s, x) \leqslant p$ implies $\sigma(r, \sigma(s, x)) \leqslant p$. The L-prime element p shows that either $\sigma(s, x) \leqslant p$ or $r \in [1_C \to p]_L$. That is, either $x \leqslant p$ or $s \in [x \to p]_L$ or $r \in [1_C \to p]_L$. From proposition 5.1.3 that $[1_C \to p]_L \subseteq [x \to p]_L$. Thus $s \in [x \to p]_L$ or $r \in [x \to p]_L$. Hence $[x \to p]_L$ is a prime ideal. □

Corollary 5.4.4. *If* p *is a L-prime element then* $[1_C \to p]_L$ *is a prime ideal of a locale* L.

Proof. Follows from the above theorem. □

Theorem 5.4.5. *If p and q are any two L-prime elements then so is $p \wedge q$.*

Proof. Let $\sigma(r,n) \leqslant p \wedge q$, for some $n \in (\sigma, C)$ and $r \in L$. Then $\sigma(r,n) \leqslant p$ and $\sigma(r,n) \leqslant q$. Since p and q are L-prime elements, we have the following statements: i) either $n \leqslant p$ or $r \in [1_C \to p]_L$ and ii) either $n \leqslant q$ or $r \in [1_C \to q]_L$. From these two statements the theorem follows. □

Definition 5.4.6. The set of all L -prime elements of (σ, C) is called the spectrum of (σ, C) and is denoted by $Spec(\sigma, C)$.

5.5. Zariski topology on $Spec_\wedge(\sigma, C)$

Definition 5.5.1. For $n \in (\sigma, C)$ we define $C(n) = \{p \in Spec(\sigma, C) : n \leqslant p\}$.

Proposition 5.5.2. *For $n \in (\sigma, C)$ and $p \in Spec(\sigma, C)$ we have the following :*

i) $C(0_C) = Spec(\sigma, C)$

ii) $C(1_C) = \phi$

iii) $\bigcap_{i \in I} C(n_i) = C(\bigvee_{i \in I} n_i)$*,for some indexed set I*

iv) $C(n) \cup C(l) \subseteq C(n \wedge l)$.

Proof. i) $C(0_C) = \{p \in Spec(\sigma, C) : 0_C \leqslant p\}$. Since 0_C is the bottom element, $0_C \leqslant p$ for every $p \in Spec(\sigma, C)$. Hence $C(0_C) = Spec(\sigma, C)$.

ii) $C(1_C) = \{p \in Spec(\sigma, C) : 1_C \leqslant p\}$. Since 1_C is the top element, no $p \in Spec(\sigma, C)$ belongs to $C(1_C)$. Hence $C(1_C)$ is empty.

iii) $p \in \bigcap_{i \in I} C(n_i)$ implies $n_i \leqslant p$, for every i in some indexed set I. Then we have $\bigvee_{i \in I} n_i \leqslant p$. Therefore $p \in C(\bigvee_{i \in I} n_i)$. Thus $\bigcap_{i \in I} C(n_i) \subseteq C(\bigvee_{i \in I} n_i)$.

Suppose $p \in C(\bigvee_{i\in I} n_i)$.

$$\begin{aligned} \bigvee_{i\in I} n_i \leqslant p &\Rightarrow n_i \leqslant p \;\forall\; i \in I \\ &\Rightarrow p \in C(n_i) \;\forall\; i \in I \end{aligned}$$

Hence $C(\bigvee_{i\in I} n_i) \subseteq \bigcap_{i\in I} C(n_i)$ and consequently $\bigcap_{i\in I} C(n_i) = C(\bigvee_{i\in I} n_i)$.

iv) Let $p \in C(n) \cup C(l)$.

$$\begin{aligned} p \in C(n) \cup C(l) &\Rightarrow p \in C(n) \text{ or } p \in C(l) \\ &\Rightarrow n \leqslant p \text{ or } l \leqslant p \\ &\Rightarrow n \wedge l \leqslant p \\ &\Rightarrow p \in C(n \wedge l) \end{aligned}$$

Hence $C(n) \cup C(l) \subseteq C(n \wedge l)$. □

The above proposition leads us to the following theorem.

Theorem 5.5.3. *On Spec*$(\sigma, C), \Lambda = \{C(n) : n \in (\sigma, C)\}$ *forms a basis for some topology* Ω.

Definition 5.5.4. *Spec*$_\wedge(\sigma, C)$ is the set of all $p \in (\sigma, C)$ such that p is meet irreducible as well as an L-prime element of (σ, C)

Proposition 5.5.5. *On Spec*$_\wedge(\sigma, C)$, $C(n) \cup C(l) = C(n \wedge l)$.

Proof. We have $C(n) \cup C(l) \subseteq C(n \wedge l)$. If $p \in C(n \wedge l)$ then $n \wedge l \leqslant p$. The L-prime element p being meet irreducible, either $n \leqslant p$ or $l \leqslant p$. That is, either $p \in C(n)$ or $p \in C(l)$. Hence $p \in C(n) \cup C(l)$ and $C(n) \cup C(l) = C(n \wedge l)$. □

Proposition 5.5.6. *The collection $\nu = \{C(n) : n \in (\sigma, C)\}$ defined on $Spec_{\wedge}(\sigma, C)$ forms a family of closed sets for some topology on $Spec_{\wedge}(\sigma, C)$.*

Proof. Follows from Proposition 5.5.2 and proposition 5.5.5. □

Definition 5.5.7. The topology Ψ generated by the family of closed sets ν is called the Zariski topology on $Spec_{\wedge}(\sigma, C)$.

5.6. Zariski Topology of L-Component

In this section we define a new type of sets $C^*(n)$ and have tried to define a Zariski topology on $Spec(\sigma, C)$.

Definition 5.6.1. For $n \in (\sigma, C)$, we define $C^*(n) = \{p \in Spec(\sigma, C) : [1_C \to n]_L \subseteq [1_C \to p]_L\}$.

Proposition 5.6.2. *Let (σ, C) be a L-component. The set $C^*(n)$ has the following properties*

i) $C^*(0_C) = Spec(\sigma, C)$

ii) $C^*(1_C) = \phi$

iii) $\bigcap_{i \in I} C^*(n_i) = C^*(\bigvee_{i \in I}[1_C \to n_i]1_C)$, for some indexed set I

iv) $C^*(n) \cup C^*(l) = C^*(n \wedge l)$.

Proof. i) $C^*(0_C) = \{p \in Spec(\sigma, C) : [1_C \to 0_C]_L \subseteq [1_C \to p]_L\}$. It is obvious that $[1_C \to 0_C]_L \subseteq [1_C \to p]_L, \forall p \in (\sigma, C)$. Hence $C^*(0_C) = Spec(\sigma, C)$.

ii) $C^*(1_C) = \{p \in Spec(\sigma, C) : [1_C \to 1_C]_L \subseteq [1_C \to p]_L\}$. And $[1_C \to 1_C]_L = \{r \in L; \sigma(r, 1_C) \leqslant 1_C\} = L$. Since $p \neq 1_C, C^*(1_C) = \phi$.

iii) Let $p \in \bigcap_{i \in I} C^*(n_i)$. From Propositions 5.3.6 and 5.3.8 $[1_C \to n_i]_L 1_C \leqslant p, \forall i \in I$ implies that $\bigvee_{i \in I}[1_C \to n_i]_L 1_C \leqslant p$. Thus $p \in C^*(\bigvee_{i \in I}[1_C \to n_i]_L 1_C)$.

Consider $p \in C^*(\bigvee_{i \in I}[1_C \to n_i]_L 1_C)$. For any j in the indexed set I

$[1_C \to n_j]_L \subseteq [1_C \to [1_C \to n_j]_L 1_C]_L \subseteq [1_C \to \bigvee_{J \in I}[1_C \to n_j]_L 1_C]_L \subseteq [1_C \to p]_L$.

Therefore $p \in C^*(n_j), \forall j \in I$.

iv) Let $n, l \in (\sigma, C)$. For $p \in C^*(n \wedge l)$, $[1_C \to n \wedge l]_L \subseteq [1_C \to p]_L$. That is, $[1_C \to n]_L \cap [1_C \to l]_L \subseteq [1_C \to p]_L$. Since $[1_C \to p]_L$ is a prime ideal, either $[1_C \to n]_L \subseteq [1_C \to p]_L$ or $[1_C \to l]_L \subseteq [1_C \to p]_L$. Therefore $p \in C^*(n) \cup C^*(l)$.

If $p \in C^*(n) \cup C^*(l)$, then $[1_C \to n]_L \subseteq [1_C \to p]_L$ or $[1_C \to l]_L \subseteq [1_C \to p]_L$. Hence $[1_C \to n \wedge l]_L \subseteq [1_C \to p]_L$ which implies $p \in C^*(n \wedge l)$. □

Theorem 5.6.3. *The collection* $\gamma^* = \{C^*(n) : n \in (\sigma, C)\}$ *forms a collection of closed sets for some topology* Ω^* *on Spec*(σ, C).

Proof. Follows from the above proposition. □

Definition 5.6.4. The topology Ω^* on *Spec*(σ, C) with γ^* as the collection of closed sets is called the Zariski topology on *Spec*(σ, C).

Theorem 5.6.5. *If every L-prime element of* (σ, C) *is meet irreducible in* C*, then* $\nu \subseteq \gamma^*$.

Proof. Let $C(n) \in \nu$ then $p \in C(n)$ implies $[1_C \to n] \subseteq [1_C \to p]$. Hence $p \in C^*(n)$ and $\nu \subseteq \gamma^*$. □

Remark. On *Spec*(σ, C) we have defined two topologies, one with respect to the closed sets $\{C^*(n) : n \in (\sigma, C)\}$ and another with $\{C(n) : n \in (\sigma, C)\}$ as basis for open sets.Also,for $A \in \Omega^*$ and $B \in \Lambda$, $A \cap B^c \neq \phi$.

5.7. Properties of Zariski Topology Ω^* and Ψ

Definition 5.7.1. An element $m \in Spec(\sigma, C)$ is said to be σ- maximal if it satisfies the conditions

i) $m \leqslant n$ implies $m = n$ for $n \in Spec(\sigma, C)$

ii) $[1_C \to m]_L$ is a maximal ideal of the locale L.

Theorem 5.7.2. *If $x \in Spec(\sigma, C)$ is a σ- maximal element then $\{x\}$ is closed.*

Proof. Suppose x is a σ- maximal element of (σ, C). By definition $[1_C \to x]_L$ is a maximal ideal of the locale L. Thus $C^*(x) = \{x\}$ and hence it is closed. □

Theorem 5.7.3. *For $q \in Spec(\sigma, C), Cl(\{q\}) = C^*(q)$.*

Proof. We have to show that the smallest closed set containing $\{q\}$ is $C^*(q)$. Obviously, $q \in C^*(q)$. Now let $q \in C^*(n)$ for some $n \in (\sigma, C)$. We prove $C^*(q) \subseteq C^*(n)$. If $y \in C^*(q)$ then $[1_C \to q]_L \subseteq [1_C \to y]_L$. Also, $[1_C \to n]_L \subseteq [1_C \to q]_L$. Therefore $y \in C^*(n)$. Thus $Cl(\{q\}) = C^*(q)$. □

Theorem 5.7.4. *If $p \leqslant q$ for some $p, q \in Spec(\sigma, C)$ then $q \in Cl(\{p\})$.*

Proof. $p \leqslant q$ implies $[1_C \to p]_L \subseteq [1_C \to q]_L$. Thus $q \in C^*(p)$ implies $q \in Cl(\{p\})$. □

Remark. If every element of $Spec(\sigma, C)$ is σ-maximal, then the singleton sets will be closed in Ω^* and hence $(Spec(\sigma, C), \Omega^*)$ will be a T_1 space.

Theorem 5.7.5. *On $(Spec_\wedge(\sigma, C), \Psi)$ we have the following*

i) $Cl(\{p\}) = C(p)$, for $p \in Spec_\wedge(\sigma, C)$

ii) $\{p\}$ is closed in $Spec_\wedge(\sigma, C)$ if and only if p is maximal in $Spec_\wedge(\sigma, C)$

iii) $q \in Cl(\{p\})$ if and only if $p \leqslant q$, for $p, q \in Spec_\wedge(\sigma, C)$

iv) $(Spec_\wedge(\sigma, C), \Psi)$ is a T_0 space.

Proof. i) Clearly $p \in C(p)$. We show that $C(p)$ is the smallest closed set containing p. If $p \in C(n)$ for some $n \in (\sigma, C)$ and $q \in C(p)$ then we have $p \leqslant q$ and $n \leqslant p$. Clearly, $q \in C(n)$. Thus $C(p) \subseteq C(n)$.

ii) Suppose $\{p\}$ is closed in $Spec_{\wedge}(\sigma, C)$ then $\{p\} = C(n)$ for some $n \in (\sigma, C)$. That is, p is the only element such that $n \leqslant p$. If p is not maximal in $Spec_{\wedge}(\sigma, C)$ then for any $q \neq p$, with $p \leqslant q$, implies $q \in C(n)$. But $\{p\} = C(n)$. Hence we arrive at a contradiction. Therefore p is a maximal element of $Spec_{\wedge}(\sigma, C)$.

Now suppose p is maximal then we have to show that $Cl(\{p\}) = \{p\}$. We know that $Cl(\{p\}) = C(p)$.If $p \neq q \in C(p)$ then $p \leqslant q$ is a contradiction to the fact that p is maximal. Thus $\{p\}$ is closed in $Spec_{\wedge}(\sigma, C)$.

iii) Follows from definition.

iv) Let p and q be two distinct points then either $p \nleqslant q$ or $q \nleqslant p$. Without loss of generality, suppose the latter. Then $p \notin Cl(\{q\})$ and hence $p \in Cl(\{q\})^c$ which definitely deos not contain q. Thus $(Spec_{\wedge}(\sigma, C), \Psi)$ is a T_0 space. □

Remark. $(Spec_{\wedge}(\sigma, C), \Psi)$ is a T_0 space while $(Spec(\sigma, C), \Omega^*)$ is not.

5.8. A Study on $(Spec(\sigma, C), \Omega^*)$

In this section we study a few properties of Ω^*.

Definition 5.8.1. An L-Component (σ, C) is said to be without zero divisors if for $p \neq 0_C$ and $q \neq 0_C$ implies $p \wedge q \neq 0_C$.

Examples 5.8.2. *i) If C is a chain then the L-Component (σ, C) is without zero divisors.*

ii) If C is atomic then L-Component (σ, C) is without zero divisors.

Lemma 5.8.3. *Consider $Spec(\sigma, C)$ of the L-slice (σ, C). Let $X^*(n)$ denote the complement of $C^*(n)$ in $(Spec(\sigma, C), \Omega^*)$. Then we have the following*

i) $X^*(1_C) = Spec(\sigma, C)$

ii) $X^*(n) = \varnothing$ if and only if $n = 0_C$

iii) $X^*(n) \cap X^*(m) = X^*(n \wedge m)$

Proof. The proof of i) and ii) follows easily from definitions. To prove iii), suppose $p \in X^*(n) \cap X^*(m)$. Then $[1_C \to n]_L \nsubseteq [1_C \to p]_L$ and $[1_C \to m]_L \nsubseteq [1_C \to p]_L$. Since $[1_C \to p]_L$ is a prime ideal $[1_C \to n]_L \cap [1_C \to m]_L \nsubseteq [1_C \to p]_L$. That is, $[1_C \to n \wedge m]_L \nsubseteq [1_C \to p]_L$. Therefore $p \in X^*(n \wedge m)$ and hence we get $X^*(n) \cap X^*(m) \subseteq X^*(n \wedge m)$. Similarly we can prove the reverse inclusion. □

Theorem 5.8.4. *$(Spec(\sigma, C), \Omega^*)$ is irreducible if and only if (σ, C) is without zero divisors.*

Proof. $(Spec(\sigma, C), \Omega^*)$ is irreducible if and only if the intersection of any pair of nonempty open sets is nonempty. If $X^*(n)$ and $X^*(m)$ are two non empty open sets then by the above lemma $X^*(n \wedge m)$ is nonempty. That is, $X^*(n \wedge m) \neq \phi$ if and only if whenever $n \neq 0_C$ and $m \neq 0_C$ implies $n \wedge m \neq 0_C$. Thus $X^*(n \wedge m) \neq \phi$ if and only if (σ, C) is without zero divisors. □

Theorem 5.8.5. *Let $f : (\sigma, C) \to (\mu, K)$ be the L-component isomorphism between the L-components (σ, C) and (μ, K) . If $q \in Spec(\mu, K)$ then $f^{-1}(q) \in Spec(\sigma, C)$.*

Proof. Let $f^{-1}(q) = p$. Also, if $\sigma(r, n) \leqslant p$ then $f(\sigma(r, n)) \leqslant f(p)$ will imply that $\mu(r, f(n)) \leqslant f(p)$. Since $f(p) = q \in Spec(\mu, K)$, we have that either $f(n) \leqslant f(p)$ or $\mu(r, 1_K) \leqslant f(p)$. Thus either $n \leqslant p$ or $\sigma(r, 1_C) \leqslant p$. Therefore we have that $f^{-1}(q) = p \in Spec(\sigma, C)$. □

Theorem 5.8.6. *If $f : (\sigma, C) \to (\mu, K)$ be a L-component isomorphism then f induces a map $f_* : Spec(\mu, K) \to Spec(\sigma, C)$ such that f_* is continous.*

Proof. We know that

$$\begin{aligned} q \in f_*^{-1}(X^*(n)) &\Leftrightarrow f_*(q) \in X^*(n) \\ &\Leftrightarrow [1_C \to n] \nsubseteq [1_C \to f_*(q)] \\ &\Leftrightarrow [1_K \to f(n)] \nsubseteq [1_K \to f(f_*(q))] \\ &\Leftrightarrow f(f_*(q)) \in X^*(f(n)) \\ &\Leftrightarrow q \in X^*(f(n)). \end{aligned}$$

□

Chapter 6

Generalised locales and the Q-Slice D(Q)

This chapter deals with the abstract notion of locales called quantales. Algebraically quantales can be considered as semirings. Topologically speaking they are the abstract notion of generalised spaces which are in turn named locales. Locales or frames are complete lattices where meet distributes over infinite joins. While defining quantales C.J. Mulvey introduced an associative binary operation * on a complete lattice such that * distributes over infinite joins.The similarity in the definitions of locales and quantales justifies the review of the already established definition of quantales as generalised locales. Thus quantales can be rightfully called as generalised locales. Since the notion of quantales are already established and much developed, we prefer the terminology quantales to that of generalised locales. This chapter is divided into four sections. In the first section we develop a quotienting of quantale using a specific ideal. Second section deals with the maps called deductions and their properties.It is well known that a quotient quantale can be constructed through the maps called quantic nucleus. Here we try to do the analogue through the ideals constructed from the newly defined maps called deductions . The third section introduces the graphs that are associated with quantales. This section motivated us to look into the possibilities of introducing graph theory in the context of L-slices.The last section introduces the generalised L-slice which we call Q-slices. We discuss the differences in the basic properties exhibited by L-slices and Q-slices.

6.1. Ideals of a Generalised locales or Quantales and their properties

In [59] authors have developed a quotient frame using the ideals of a locale. We investigate the possibility of existence of such a quotient quantale in the context of generalised locales or rather quantales.

Let $(Q, \bigvee, \wedge, *)$ be a quantale with top element T and bottom element 0. Let I be any Q-ideal in Q. For each $a \in Q$ define $I_a = \{x \in Q : a * x \in I\}$.

Proposition 6.1.1. *I_a is a Q-ideal of a commutative quantale Q.*

Proof. Since $0 = a * 0$ I_a is nonempty. Let J be any indexed set and $\{x_i\}_{i \in J} \in I_a$. Then $a * x_i \in I, \forall i \in J$. Since I is an ideal, $\bigvee_{i \in J} a * x_i \in I$. Also $\bigvee_{i \in J} a * x_i = a * \bigvee_{i \in J} x_i \in I$. Thus $\bigvee_{i \in J} x_i \in I_a$. Let $x \in I_a$ and $y \in Q$ then $a * x \in I$ implies $y * (a * x) \in I$. Since Q is commutative $y * (a * x) = a * (y * x) \in I$.Therefore $y * x \in I_a$. Similarly $x * y \in I_a$. If $y \in I_a$ and $x \in Q$ with $x \leqslant y$ then $a * x \leqslant a * y$ implies that $a * x$ belongs to the ideal I. Thus $x \in I_a$. □

Note that if $a \in I$ then $I_a = Q$.

Proposition 6.1.2. *Let Q be a quantale and I be any Q-ideal then*

i) If $a, b \in Q$ with $a \leqslant b$ then $I_a \supseteq I_b$

ii) $I \subseteq I_a \;\; \forall a \in Q$

Proof. i) Let $a, b \in Q$ such that $a \leqslant b$. For $x \in I_b, b * x \in I$ and $a * x \leqslant b * x$ implies that $a * x \in I$. Therefore $x \in I_a$.

ii)follows directly from the definition of I_a. □

Theorem 6.1.3. *For any Q-ideal I and $a \in Q$, $I_a \cap I_b = I_{a \vee b}$.*

Proof. We have

$$\begin{aligned} x \in I_a \cap I_b &\Leftrightarrow x \in I_a \text{ and } x \in I_b \\ &\Leftrightarrow a * x \in I \text{ and } b * x \in I \\ &\Leftrightarrow (a * x) \vee (b * x) \in I \\ &\Leftrightarrow x \in I_{a \vee b} \end{aligned}$$

Therefore, $I_a \cap I_b = I_{a \vee b}$. □

Theorem 6.1.4. *Let Q be a commutative and idempotent quantale. If I is a prime Q-ideal then $I_a \cup I_b = I_{a*b}$.*

Proof. Let $x \in I_a \cup I_b$.

$$\begin{aligned} x \in I_a \text{ or } x \in I_b &\Rightarrow a * x \in I \text{ or } b * x \in I \\ &\Rightarrow (a * x) * (b * x) \in I \\ &\Rightarrow (a * b) * x \in I \\ &\Rightarrow x \in I_{a*b} \end{aligned}$$

Therefore $I_a \cup I_b \subseteq I_{a*b}$

For the reverse inclusion, consider $y \in I_{a*b}$.Then we have $(a * b) * y \in I$. Also $y * (a * b * y) = (a * y) * (b * y)$ belongs to the ideal I. Since I is a prime ideal, either $a * y \in I$ or $b * y \in I$. That is, either $x \in I_a$ or $y \in I_b$. Therefore $y \in I_a \cup I_b$. Thus $I_{a*b} \subseteq I_a \cup I_b$ and consequently $I_a \cup I_b = I_{a*b}$. □

Theorem 6.1.5. *If I is a prime Q-ideal of a quantale Q then for each $a \in Q, I_a$ is a prime Q-ideal.*

Proof. Let I be a prime Q-ideal and let $x * y \in I_a$. Then $a * (x * y) \in I$ would imply that either $a * x \in I$ or $y \in I$. That is, either $x \in I_a$ or $y \in I$. Since $I \subseteq I_a$ we have that either $x \in I_a$ or $y \in I_a$. Thus I_a is a prime Q-ideal. □

Definition 6.1.6. An element $a \in Q$ is said to be quasi prime to a Q-ideal I, if $a \notin I$ and $a * x \in I$ implies $x \in I$.

Proposition 6.1.7. *If $a \in Q$ is quasi prime to a Q-ideal I of Q then $I_a = I$.*

Proof. If $y \in I_a$ then $a * y \in I$ implies $y \in I$. Hence $I_a \subseteq I$. The proof follows from Proposition 6.1.2. □

Remark. If I is a maximal Q-ideal then $I_a = I$.

Lemma 6.1.8. *Let I is a Q-ideal of Q. For any $a, b, c \in Q$ and $S \subseteq Q$, if $I_a = I_b$ then*

i) $I_{a*c} = I_{b*c}$

ii) $I_{a \vee \bigvee S} = I_{b \vee \bigvee S}$

Proof. i. Let $a, b, c \in L$ and $I_a = I_b$.

$$\begin{aligned} x \in I_{a*c} &\text{ if and only if } (a*c)*x = a*(c*x) \in I \\ &\text{ if and only if } c*x \in I_a = I_b \\ &\text{ if and only if } b*(c*x) = (b*c)*x \in I \\ &\text{ if and only if } x \in I_{b*c}. \end{aligned}$$

Therefore $I_a = I_b$ implies $I_{a*c} = I_{b*c}$.

ii. Let $I_a = I_b$ and $S \subseteq L$.

$$\begin{aligned} x \in I_{a\vee\bigvee S} &\text{ if and only if } (a \vee \bigvee S)*x = \bigvee(a \vee s)*x \in I \\ &\text{ if and only if } (a*x) \vee (s*x) \in I \text{ for all } s \in S \\ &\text{ if and only if } (a*x) \in I \text{ and } (s*x) \in I \text{ for all } s \in S \\ &\text{ if and only if } b*x \in I \text{ and } (s*x) \in I \text{ for all } s \in S \\ &\text{ if and only if } b*x \in I \text{ and } \bigvee_{s\in S}(s*x) \in I \\ &\text{ if and only if } b*x \vee \bigvee_{s\in S}(s*x) \in I \\ &\text{ if and only if } (b \vee \bigvee_{s\in S})*x \in I \\ &\text{ if and only if } x \in I_{(b\vee\bigvee_{s\in S})}. \end{aligned}$$

Hence $I_a = I_b$ implies $I_{a\vee\bigvee S} = I_{b\vee\bigvee S}$. □

Remark. Q being commutative will give us $I_{c*a} = I_{c*b}$.

Definition 6.1.9. Fix an ideal I in Q and define a relation θ_I on Q as $a\ \theta_I\ b$ if and only if $I_a = I_b$.

The following theorems are an immediate consequence of the above lemma.

Theorem 6.1.10. *For an ideal I in a commutative quantale Q , θ_I is a congruence relation on Q.*

Theorem 6.1.11. *The quotient Q/θ_I is a quantale.*

6.2. Deductions on Quantales

In this section we introduce the map deductions on quantales and study some of its basic properties.

Definition 6.2.1. Consider the quantale $(Q, \bigvee, \wedge, *)$. A map $d : Q \to Q$ on a quantale Q is called a deduction on Q if it satisfies the following conditions.

i) $d(\bigvee_{i \in I} b_i) = \bigvee_{i \in I} d(b_i)$, where I is some indexed set.

ii) $d(a * b) = a * d(b)$, this property is called translation in second variable.

If Q is commutative we impose the additional condition that $d(a*b) = a*d(b) = d(a)*b$.

Example 6.2.2. *Let $(Q, \bigvee, \wedge, *)$ be a quantale. Define $d_a : Q \to Q$ for $a \in Q$ such that $d_a(x) = x * a$.*

Lemma 6.2.3. *Let d be a deduction on a quantale $(Q, \vee, \wedge, *)$. For any $x, y \in Q$ we have*

i) $d(0) = 0$

ii) $x \leqslant y$ implies $d(x) \leqslant d(y)$

Proof. i) $d(0) = d(0 * 0) = 0 * d(0) = 0$

ii)$d(y) = d(x \vee y) = d(x) \vee d(y)$. Thus $d(x) \leqslant d(y)$. □

Lemma 6.2.4. *If Q is right unital with e as the right unit and $e \leqslant d(e)$ then*

i) $a \leqslant d(a)$

ii) $T = d(T)$, where T is the top element of Q.

Proof. i) Since $e \leqslant d(e)$, we have $a * e \leqslant a * d(e) = d(a * e) = d(a)$. That is, $a \leqslant d(a)$.

ii) $T \leqslant d(T)$. Since T is the top element, $d(T) = T$. □

Proposition 6.2.5. *If Q is right unital with right unit e and $d(e) = e$ then the only deduction map is the identity.*

Proof. Let $d : Q \to Q$ be a deduction. $d(a) = d(a * e) = a * d(e) = a * e = a \ \forall \ a \in Q$ Hence d is the identity map on Q. □

Proposition 6.2.6. *The deduction map $d : Q \to Q$ is a Čech closure operator, if Q is right unital with right unit e and $e \leqslant d(e)$.*

Proof. Follows from definition and lemma 6.2.3 and 6.2.4. □

Proposition 6.2.7. *If Q is right unital with right unit e and $e \leqslant d(e)$ for a deduction d on Q then $d(a * b) \leqslant d(a) * d(b), \forall a, b \in Q$.*

Proof. We have

$$\begin{aligned} d(a) * d(b) &= (d(a) \vee a) * d(b) \\ &= (d(a) * d(b)) \vee (a * d(b)) \\ &= (d(a) * d(b)) \vee d(a * b) \end{aligned}$$

Hence the proof. □

Definition 6.2.8. A quantic dual on a quantale Q is a preclosure operator d such that $d(a*b) \leqslant d(a)*d(b) \ \forall \ a, b \in Q$. A quantic dual is said to be strict if $d(a*b) = d(a)*d(b)$.

The above proposition will give the following theorem.

Theorem 6.2.9. *Let Q be a right unital quantale with right unit e and $e \leqslant d(e)$. Then any deduction d on Q is a quantic dual on Q.*

Theorem 6.2.10. *The kernel $K_d = \{a \in Q : d(a) = 0\}$ of the deduction d on Q is a left-sided Q-ideal.*

Proof. If $\{a_i\}_{i\in I} \in K_d$ then $d(a_i) = 0, \forall i \in I$.
$d(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} d(a_i) = 0$. Therefore $\bigvee_{i\in I} a_i \in K_d$.
If $b \in Q$ and $a \in K_d$ then $d(b * a) = b * d(a) = b * 0 = 0$. Therefore K_d is a left-sided Q-ideal. □

6.3. The Sub-Quantale Q_d

Definition 6.3.1. For a quantale Q and a deduction d on it we define the stationary points of d to be the set $Q_d = \{a \in Q : d(a) = a\}$.

Theorem 6.3.2. *$(Q_d, \bigvee, \wedge, *)$ is a subquantale of $(Q, \bigvee, \wedge, *)$.*

Proof. If $\{a_i\}_{i\in I}$ be a collection in Q_d then $d(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} d(a_i) = \bigvee_{i\in I} a_i$. Therefore $\bigvee_{i\in I} a_i \in Q_d$. Since $d(a * b) = a * d(b) = a * b \ \ \forall\ a, b \in Q_d$, Q_d is closed under the operation *. Also $d(0) = 0$ implies $0 \in Q_d$. Thus Q_d is a subquantale of Q. □

Example 6.3.3. *Let $Q = \{0, T\}$ and * is defined as follows*
$0 * T = T * 0 = 0 = 0 * 0$ *and* $T * T = T$.
$(Q, *)$ is a quantale. Let $d_1 : Q \to Q$ be a deduction defined as $d_1(0) = 0$ and $d_1(T) = 0$, then $Q_{d_1} = \{0\}$ is a subquantale of Q.

Example 6.3.4. *Let Q be the chain $0 \leqslant a \leqslant T$ and the binary operation $*$ defined as follows*
$0 * 0 = 0 * a = 0 * T = a * 0 = T * 0 = 0$
$a * a = a, a * T = T$
$T * a = T, T * T = T$
Consider the deduction $d : Q \to Q$ defined as $d(0) = 0, d(a) = T, d(T) = T$, then $Q_d = \{0, T\}$ is a subquantale of Q.

6.4. Properties of Deduction Maps

This section deals with some more properties of deduction maps. In particular, we deal with how the ideals and filters of a quantale are mapped by the deduction maps. We have a few definitions on deductions which are analogous to that of derivations on quantales.

Definition 6.4.1. Let d be a deduction on a quantale Q then

i) d is right sided if and only if $d(T * a) \leqslant d(a)$.

ii)d is left sided if and only if $d(a * T) \leqslant d(a)$.

iii)d is two sided if and only if d is both right and left sided.

iv)d is idempotent if and only if $d(a * a) = d(a)$.

v) d is commutative if and only if $d(a * b) = d(b * a)$.

Theorem 6.4.2. *Let d be a deduction on a quantale Q. Then the following two conditions are equivalent*

i) d is two sided

ii) $d(a * b) \leqslant d(a) \wedge d(b)\ \forall\ a, b \in Q$.

Proof. If d is two sided then $d(a * b) \leqslant d(a * T) \leqslant d(a)$ and $d(a * b) \leqslant d(T * b) \leqslant d(b)$. Therefore $d(a * b) \leqslant d(a) \wedge d(b)\ \ \forall a, b \in Q$.

If ii) holds then $d(a * T) \leqslant d(a) \wedge d(T) = d(a)$, hence d is left sided. Similarly $d(T * a) \leqslant d(T) \wedge d(a) = d(a)$, that is d is right sided. □

Theorem 6.4.3. *Let $d : Q \to Q$ be a deduction on the commutative quantale Q then if I is a $*$-ideal of Q, so is $d(I)$.*

Proof. Let I be a $*$-ideal of the quantale and $\{n_i\}_{i \in J} \in d(I)$, for some indexed set J. There exists $\{x_i\}_{i \in J} \in I$ such that $d(x_i) = n_i$, for each n_i.

Also, $\bigvee_{i\in J} n_i = \bigvee_{i\in J} d(x_i) = d(\bigvee_{i\in J} x_i)$. Since I is an ideal of Q we have $\bigvee_{i\in J} x_i \in I$. Thus, $\bigvee_{i\in J} n_i = d(\bigvee_{i\in J} x_i) \in d(I)$. Let $a \in Q$ and $m \in d(I)$ then we have that $a * m = a * d(x) = d(a * x)$ implies that $a * m \in d(I)$. Q being commutative implies that $m * a \in d(I)$. Thus $d(I)$ is a $*$-ideal of Q. □

Theorem 6.4.4. *Let $d : Q \to Q$ be a strict quantic dual on a quantale. If F is any filter on Q, so is $d^{-1}(F)$.*

Proof. Since $d(0) = 0$, $0 \notin d^{-1}(F)$. If $x \in d^{-1}(F)$ and $x \leqslant y$ for some $y \in Q$, then $d(x) \leqslant d(y)$ which implies $y \in d^{-1}(F)$. Also for any $r, s \in d^{-1}(F)$ we have $d(r), d(s) \in F$. Since F is a filter and d is strict quantic dual we have $d(r) * d(s) = d(r * s) \in F$. Therefore $r * s \in d^{-1}(F)$. Hence $d^{-1}(F)$ is a filter. □

6.5. The congruence θ_d

In this section we define an equivalence relation on Q with respect to the deduction d. Also, we examine the possible existence of a congruence relation on Q.

Definition 6.5.1. Let d be a deduction on Q then define $[a]^d = \{x \in Q : d(x*a) = 0\}$. Or , equivalently, $[a]^d = \{x \in Q : x * d(a) = 0\}$.

Lemma 6.5.2. *We have the following observations*

i) If $a \in K_d$ then $[a]^d = Q$

ii) If d is idempotent then $a \notin K_d$ implies $a \notin [a]^d$.

Proof. i) If $a \in K_d$, then $d(a) = 0$ and $x * d(a) = 0, \forall x \in Q$. Hence, $[a]^d = Q$.

ii)Suppose $a \in [a]^d$. Since d is idempotent we have $d(a) = d(a * a) = a * d(a) = 0$, which is a contradiction to $a \notin K_d$.

□

Lemma 6.5.3. $[a]^d$*is a left sided* $*$*-ideal of* Q*.*

Proof. Let $\{a_i\}_{i\in J} \subseteq [a]^d$, for some indexed set J. Then $d(a_i * a) = 0, \forall\ a_i$.
Also $d((\bigvee_{i\in I} a_i) * a) = d(\bigvee_{i\in I}(a_i * a)) = \bigvee_{i\in I} d(a_i * a) = 0$. Therefore $\bigvee_{i\in I} a_i \in [a]^d$.
For $x \in Q$ and $b \in [a]^d$, $d((x * b) * a) = x * d(b * a) = x * (b * d(a)) = 0$. Therefore $x * b \in [a]^d$.

□

Theorem 6.5.4. *If* Q *is commutative then* $[a]^d$ *is a* $*$*-ideal of* Q*.*

Proof. Follows from definition and the above lemma. □

Theorem 6.5.5. *Let* d *be a deduction on* Q*, then for any* $a, b \in Q$*, we have the following*
i) $a \leqslant b \Rightarrow [b]^d \subseteq [a]^d$
ii) $[a \vee b]^d = [a]^d \cap [b]^d$.

Proof. i) If $a \leqslant b$ and $y \in [b]^d$ then $y * d(b) = 0$. Also, $d(a) \leqslant d(b)$ implies $y * d(a) = 0$. Thus $[b]^d \subseteq [a]^d$.
ii) For $x \in [a \vee b]^d$, $0 = x * d(a \vee b) = (x * d(a)) \vee (x * d(b))$. That is, $x \in [a]^d$ and $x \in [b]^d$. Therefore $[a \vee b]^d \subseteq [a]^d \cap [b]^d$.
If $z \in [a]^d \cap [b]^d$ then $0 = z * (d(a) \vee d(b)) = z * d(a \vee b)$. Thus $z \in [a \vee b]^d$ and $[a \vee b]^d = [a]^d \cap [b]^d$. □

Definition 6.5.6. Let d be a deduction on Q. For $x, y \in Q$, define a relation θ_d on Q such that $(x, y) \in \theta_d$ if and only if $[x]^d = [y]^d$.

Theorem 6.5.7. *For any deduction* d, θ_d *is an equivalence relation on* Q*.*

Proof of the theorem is direct.

Definition 6.5.8. A binary relation θ defined on Q is a left congruence on Q if and only if

i) θ is an equivalence relation on Q

ii) If $(x_i, y_i) \in \theta$, for all $i \in I$ then $(\bigvee_{i\in I} x_i, \bigvee_{i\in I} y_i) \in \theta$, where I is some indexed set

iii) If $(a, b) \in \theta$ then $(c * a, c * b) \in \theta$.

Theorem 6.5.9. *For any deduction d on Q, θ_d is a left congruence.*

Proof. We have already shown in theorem 6.5.7 that θ_d is an equivalence relation on Q. If $(x_i, y_i) \in \theta_d$, then $[x_i]^d = [y_i]^d$. Theorem 6.5.5 shows that $[\bigvee_{i\in I} x_i]^d = \bigcap_{i\in I}[x_i]^d$. Hence $(\bigvee_{i\in I} x_i, \bigvee_{i\in I} y_i) \in \theta_d$. Let $(a, b) \in \theta_d$. By definition $x \in [c * a]^d$ if and only if $d(x * (c * a)) = 0$. Also $0 = d(x * (c * a)) = (x * c) * d(a)$ if and only if $x * c \in [a]^d$. Since $(a, b) \in \theta_d$, $x * c \in [b]^d$. And $(x * c) * d(b) = 0$ if and only if $d(x * (c * b)) = 0$. Thus $x \in [c * a]^d$ if and only if $x \in [c * b]^d$ showing that $[c * a]^d = [c * b]^d$. □

Theorem 6.5.10. *If d is a commutative deduction on Q then θ_d is a congruence on Q.*

Proof. Direct computations will give the result. □

Theorem 6.5.11. *If d is a commutative deduction on Q, then Q/θ_d defines a quotient quantale.*

Proof. θ_d is a congruence and hence the theorem follows. □

6.6. Graphs on Quantales with respect to the map Deduction

In this section we introduce the theory of graphs into the domain of quantales. We are familiar with the interplay of ring theory and graphs. Similarly, we investigate the possible development of graphs in the theory of quantales.

The Graph $G(Q,d)$

Definition 6.6.1. Let Q be a quantale. A graph $G = (V_G, E_G)$ where the vertex set $V_G = Q$ and the edge set $E_G = \{(a,b) : [a]^d = [b]^d, a \neq b\}$ is called the 'graph with respect to the deduction d' and is denoted as $G(Q,d)$.

Theorem 6.6.2. *$G(Q,d)$ is always a disconnected graph.*

Proof. Let us define a relation θ_d on Q such that $(a,b) \in \theta_d$ if and only if $[a]^d = [b]^d$. 6.5.7 shows that θ_d is an equivalence relation on Q. If $(a,b) \in E_G$ then $(a,b) \in \theta_d$ and thus belongs to the same equivalence class. Every equivalence class is either equal or disjoint. Hence, $G(Q,d)$ is always a disconnected graph. □

Let $\mathbb{E}(Q,\theta_d)$ denote the set of all equivalence classes with the respect to θ_d and $\mathbb{C}(G)$ denote the collection of components of $G(Q,d)$.

Theorem 6.6.3. *$\mathbb{E}(Q,\theta_d)$ and $\mathbb{C}(G)$ are equivalent sets.*

Proof. Let $X \in \mathbb{E}(Q,\theta_d)$. Since X is nonempty, there exists some $a \in X$. The vertex a is in one of the connected components of $G(Q,d)$, say $C(a)$. All other vertices in $C(a)$ will fall in the same equivalence class X. We define $f : \mathbb{E}(Q,\theta_d) \to \mathbb{C}(G)$ as $f(X) = C(a)$. Clearly, f is well defined. Let $X, Y \in \mathbb{E}(Q,\theta_d)$ such that $f(X) = f(Y)$. Let $a \in X$ and $b \in Y$, then $C(a) = f(X) = f(Y) = C(b)$. That is, a and b belong to the same connected component, and so $[a]^d = [b]^d$ implying that they lie in the same equivalence class. Hence $X = Y$. To prove f is onto, consider any connected component, say D. Let p be any vertex in D. Then $D = C(p)$. Let $Z \in \mathbb{E}(Q, \theta_d)$ be such that $p \in Z$. Hence $f(Z) = D$. Therefore $\mathbb{E}(Q, \theta_d)$ and $\mathbb{C}(G)$ are equivalent sets. □

Corollary 6.6.4. *For any quantale Q and a deduction d on Q,* $\omega(G(Q,d)) = |\mathbb{E}(Q, \theta_d)|$

Observation. *For the graph $G(Q,d)$, we observe the following properties:*

1. *Each equivalence class $X \in \mathbb{E}(Q, \theta_d)$ will become a connected component.*
2. *Each component is a complete subgraph of $G(Q,d)$.*
3. *The number of vertices in each component is equal to $|X|$.*

Proof. 1) and 3) follow easily from the previous theorem, so we prove only the second property. Every $X \in \mathbb{E}(Q, \theta_d)$ is mapped to a component $f(X)$ of $G(Q,d)$. Let $X = \{u_1, u_2, u_3, \cdots, u_k\}$, then $[u_i]^d = [u_j]^d, \forall i, j \in \{1, 2, \cdots, k\}$. Therefore $(u_i, u_j) \in E_G$, $\forall i, j \in \{1, 2, \ldots k\}$. Thus the component $f(X)$ is complete. □

6.7. The Graph Z_G

In this section, we introduce a different type of graph using the map deduction on a commutative quantale Q. We observe that the graph developed is always connected, in particular it can be a star graph.

Definition 6.7.1. An element a of a commutative quantale Q is said to be a zero divisor if there exist $b \neq 0$ such that $a * b = 0$.

Definition 6.7.2. If the deduction d is the identity map on a commutative quantale Q then $[a]^d = \{x \in Q | d(x * a) = 0\} = \{x \in Q | x * a = 0 = a * x\}$. We introduce two new terminologies the zero set of a and the closure set of a.
The zero set of a is defined and denoted as $[a]^\circ = \{x \in Q | x \neq 0, x * a = 0\}$.
The closure set of a is defined and denoted as $(\overline{[a]}) = \{a\} \cup [a]^\circ$.

Remark.

1. If $[a]^\circ \neq \emptyset$, then both the sets $[a]^\circ$ and $\overline{[a]}$ are lower sets of Q.

2. If $[a]^\circ \neq \emptyset$ then $[a]^\circ$ is a Q ideal.

Definition 6.7.3. Let Q be a commutative quantale. A graph $G = (V, E)$ where the vextex set $V = Q$ and the edge set $E = \{(a, b) : \overline{[a]} \cap \overline{[b]} \neq \emptyset, a \neq b\}$ is called the 'Zero-graph of Q' and is denoted as $\mathbf{Z}_{\boldsymbol{G}}$.

Lemma 6.7.4. *For a commutative quantale Q, $\mathbf{Z}_{\boldsymbol{G}}$ is always connected.*

Proof. Since $0 * a = 0, \forall a \in Q$, we have $[0]^\circ = Q$. Hence 0 has an edge with every other vertices of $\mathbf{Z}_{\boldsymbol{G}}$. □

Theorem 6.7.5. *For a commutative quantale Q, $\mathbf{Z}_{\boldsymbol{G}}$ is either a star graph or contains a cycle C_3.*

Proof. Since, 0 has an edge with every other vertices of $\mathbf{Z}_{\boldsymbol{G}}$, it will always contain a star graph with internal node at 0. If there is any other edge $(a, b) \in E$ in the graph then $0 - a - b - 0$ will form the triangle C_3 □

Corollary 6.7.6. *If Q has zero divisors then $\mathbf{Z}_{\boldsymbol{G}}$ is not bipartite.*

The following theorem is obtained from the above observations.

Theorem 6.7.7. *If Q does not have any nonzero divisors then $\mathbf{Z_G}$ will have the following properties:*

1. *$\mathbf{Z_G}$ is a star graph.*

2. *$\mathbf{Z_G}$ is a bipartite graph.*

3. *The chromatic number $\chi(\mathbf{Z_G})$=2.*

Theorem 6.7.8. *Let $y \in Q$ be such that $[y]^\circ \neq \varnothing$, then the subgraph induced by the set $\downarrow y \subseteq V$, is a clique of $\mathbf{Z_G}$.*

Proof. Let $x \in [y]^\circ$. For $a, b \in \ \downarrow y$, $a * x \leqslant y * x$, implies $a * x = 0$. Similarly $b * x \leqslant y * x$ implies $b * x = 0$. Hence $x \in [a]^\circ \cap [b]^\circ$ and $\overline{([a])} \cap \overline{([b])} \neq \varnothing$. Therefore any two vertices of $\downarrow y \subseteq V$ is always connected by an edge. Thus the subgraph induced by the set $\downarrow y \subseteq V$ is a clique of $\mathbf{Z_G}$. □

Theorem 6.7.9. *If $\mathbf{Z_G}$ is not a star graph then $\chi(\mathbf{Z_G}) \geqslant 3$.*

Proof. If $\mathbf{Z_G}$ is not a star graph then it will contain an odd cycle. Hence the result follows. □

These results motivated us to look into the possibility of a graph theoretic study on L-slices. The next chapter is a detailed study on such graphs.

6.8. The Q-slice $\mathfrak{D}(Q)$

The discussions in the previous sections clarify the inevitable link between locales and quantales. As L-slices are defined on locales, we may define generalised L-slices on quantales. We coin the term Q-slices and investigate the basic structural differences between L-slices and Q-slices. Also we give a specific example for the generalised L-slices through the collections of deductions on a quantale.

Definition 6.8.1. Let $(J, \leqslant, \vee)$ be a join semilattice with bottom element 0_J. Let$(Q, \preccurlyeq, \curlyvee, *)$ where $\preccurlyeq$ is the partial order on Q and $\curlyvee, *$ denote the join and associative binary operation respectively. Also, let 0_Q be the bottom element and top element 1_Q. By the "action of Q on J" we mean a function $\sigma_Q : Q \times J \to J$ such that the following conditions are satisfied.

1. $\sigma_Q (a, x_1 \vee x_2) = \sigma_Q (a, x_1) \vee \sigma_Q (a, x_2)$ for all $a \in Q$ and for all $x_1, x_2 \in J$ for all $a \in Q$. Also, if J is a sup lattice then $\sigma_Q (a, \vee_{i\in I} x_i) = V_{i\in I} \sigma_Q (a, x_i)$.

2. $\sigma_Q (a, 0_J) = 0_J$ for all $a \in Q$.

3. $\sigma_Q(a * b, x) = \sigma_Q(a, \sigma(b, x))$) for all $a, b \in Q, x \in J$. Also, if Q is commutative then $\sigma(a * b, x) = \sigma_Q(a, \sigma(b, x)) = \sigma_Q(b, \sigma(a, x))$) for all $a, b \in Q, x \in J$.

4. $\sigma_Q (1_Q, x) = x$ and $\sigma_Q (0_Q, x) = 0_J$ for all $x \in J$

4'. If Q is unital with unit e then $\sigma_Q(e, x) = x$ for all $x \in J$.

5. $\sigma_Q(a \curlyvee b, x) = \sigma_Q(a, x) \vee \sigma_Q(b, x)$ for $a, b \in Q, x \in J$.

If σ_Q is an action of the quantale Q on a join semilattice J, then we call (σ_Q, J) as Q-slice.

Example 6.8.2. *All locales are quantales with $* = \sqcap$. Thus all L-slices are Q-slices.*

Example 6.8.3. *The quantale itself can be treated as a Q-slice with $\sigma_Q = *$ that is, action can be considered as $*(q, x) = q * x$, for $q, x \in Q$. Thus $(*, Q)$ is a Q-slice.*

Definition 6.8.4. (σ_Q, J) and (ω_Q, K) be any two Q-slices then a map from $f : (\sigma_Q, J) \to (\omega_Q, K)$ is said to be Q-slice morphism if f preserves finite joins and $f(\sigma_Q(q,x)) = \omega_Q(q, f(x))$.

Theorem 6.8.5. *Every deduction on Q is a slice morphism on the Q-slice $(*, Q)$.*

Proof. A deduction map preserves finite joins. Also $d(*(q,x)) = d(q * x)$.

$$= q * d(x) = *(q, d(x))$$

Thus deductions are definitely Q-slice morphisms. □

Differences between L-slices and Q-slices

Let $(J, \leqslant)$ be a join semilatice. If (σ, J) is a L-slice and (σ_Q, J) be a Q-slice on J then we observe the following differences in (σ, J) and (σ_Q, J)

1. For the L-slice (σ, J), we know that $\sigma(a, \sigma(a,x)) = \sigma(a \sqcap a, x) = \sigma(a, x)$ for $a \in L$. But,for a Q- slice (σ_Q, J) , $\sigma_Q(q * q, x) \neq \sigma_Q(q, x)$, for $q \in Q$.

2. For the L-slice $(\sigma, J), \sigma(a,x) \leqslant x \;\forall\; x \in J$. But the same need not hold for $\sigma_Q(a,x)$.

3. The set F_x is a filter in the locale L for $x \in J$ but for the Q-slice the set $\{q \in Q : \sigma_Q(q,x) = x\}$ need not be a filter.

Let $\mathfrak{D}(Q)$ denote the collection of all deductions on $(Q, \leqslant, \curlyvee, *)$.

Lemma 6.8.6. $\mathfrak{D}(Q)$ *is a join semilattice*

Proof. Define $\lesssim$ on $\mathfrak{D}(Q)$ as $d_1 \lesssim d_2$ if and only if $d_1(x) \leqslant d_2(x)$ for all $x \in Q$.
Thus $\lesssim$ is a partial order on $\mathfrak{D}(Q)$.

Also, the join is defined as $d_1(x)\tilde{\sqcup}d_2(x) = d_1(x) \curlyvee d_2(x)$.We show that $d_1\tilde{\sqcup}d_2$ is a deduction on Q.

$$\begin{aligned} d_1\tilde{\sqcup}d_2(\curlyvee_{i\in I}x_i) &= d_1\left(\curlyvee_{i\in I}x_i\right) \curlyvee d_2\left(\curlyvee_{i\in I}x_i\right) \\ &= \curlyvee_{i\in I}d_1\left(x_i\right) \curlyvee \curlyvee_{i\in I}d_2\left(x_i\right) \\ &= \curlyvee_{i\in I}\left(d_1\left(x_i\right) \curlyvee d_2\left(x_i\right)\right) \\ &= \curlyvee_{i\in I}d_1\tilde{\sqcup}d_2\left(x_i\right). \end{aligned}$$

Therefore $d_1\tilde{\sqcup}d_2$ preserves arbitrary joins.

$$\begin{aligned} d_1\tilde{\sqcup}d_2(a * b) &= d_1(a * b) \curlyvee d_2(a * b) \\ &= (a * d_1(b)) \curlyvee (a * d_2(b)) \\ &= a * (d_1(b) \curlyvee d_2(b)) \\ &= a * (d_1\tilde{\sqcup}d_2(b)) \end{aligned}$$

Thus $d_1\tilde{\sqcup}d_2$ is a deduction on Q.

Define $\mathbf{O}_{\mathfrak{D}(Q)}(x) = 0_Q, \forall x \in Q$. Clearly, $\mathbf{O}_{\mathfrak{D}(Q)}$ is a deduction and is the bottom element of $(\mathfrak{D}(Q), \lesssim)$. Thus $(\mathfrak{D}(Q), \lesssim)$ is a join semilattice with bottom element $\mathbf{O}_{\mathfrak{D}(Q)}$. □

The join semilattice $(\mathfrak{D}(Q), \lesssim)$ allows the construction of a Q-slice . Let Q be a commutative quantale. Consider the Q- slice $(*, Q)$ and the collection of all deductions on the quantale , $\mathfrak{D}(Q)$. We will transform $\mathfrak{D}(Q)$ into a Q-slice.

To define an action we consider a map $\theta : Q \times \mathfrak{D}(Q) \to \mathfrak{D}(Q)$ which is defined as $\theta(q, d)(x) = d(*(q, x))$ for $x, q \in Q$ and $d \in \mathfrak{D}(Q)$.

Theorem 6.8.7. *The map $\theta : Q \times \mathfrak{D}(Q) \to \mathfrak{D}(Q)$ is an action of Q on $\mathfrak{D}(Q)$ and thus $(\theta, \mathfrak{D}(Q)$ is a Q-slice.*

Proof. First we show that $\theta(q, d)$ is a deduction on Q.

$$\begin{aligned}
\theta(q,d)\left(\curlyvee_{i\in I} x_i\right) &= d\left(*\left(q, \curlyvee_{i\in I} x_i\right)\right) \\
&= d\left(\curlyvee_{i\in I} *\left(q, x\right)\right) \\
&= \curlyvee_{i\in I} d\left(*\left(q, x_i\right)\right) \\
&= \curlyvee_{i\in I} \theta(q,d)\left(x_i\right) \\
\theta(q,d)(a*b) &= d(*(q, a*b)) \\
&= q * d(a*b) \\
&= q * a * d(b) (\because Q \text{ is commutative }) \\
&= a * q * d(b) \\
&= a * d(q*b) \\
&= a * d(*(q,b)) \\
&= a * \theta(q,d)(b)
\end{aligned}$$

Thus $\theta(q, d)$ is satisfies the conditions to be a deduction map and thus belongs to $\mathfrak{D}(Q)$.

Now we prove the conditions for $\mathfrak{D}(Q)$ to become a Q-slice.

1. $\theta\left(0_Q, d\right)(x) = d\left(0_Q * x\right) = d\left(0_Q\right) = 0_Q, \forall x \in Q$. Thus $\theta\left(0_Q, d\right) = \mathbf{0}_{\mathfrak{D}(Q)}$.

2. Let Q be unital with unit e. $\theta(e,d)(x) = d(e*x) = d(x), \forall x \in Q$

   Therefore $\theta(e, d) = d$.

3. $\theta\left(q, \mathbf{0}_{\mathfrak{D}(Q)}\right)(x) = \mathbf{0}_{\mathfrak{D}(Q)}(q*x) = 0_Q, \forall x \in Q$. Therefore $\theta\left(a, \mathbf{0}_{\mathfrak{D}(Q)}\right) = \mathbf{0}_{\mathfrak{D}(Q)}$.

4.

$$\begin{aligned}
\theta(q_1 \curlyvee q_2, d)(x) &= d(*(q_1 \curlyvee q_2, x)) \\
&= d((q_1 \curlyvee q_2) * x) \\
&= (q_1 \curlyvee q_2) * d(x) \\
&= (q_1 * d(x)) \curlyvee (q_2 * d(x)) \\
&= d(q_1 * x) \curlyvee d(q_2 * x) \\
&= \theta(q_1, d)(x) \curlyvee \theta(q_2, d)(x) \\
&= \theta(q_1, d) \,\tilde{\sqcup}\, \theta(q_2, d)(x) \\
\text{Thus } \ \theta(q_1 \curlyvee q_2, d) &= \theta(q_1, d) \,\tilde{\sqcup}\, \theta(q_2, d)
\end{aligned}$$

5.

$$\begin{aligned}
\theta(q, d_1 \tilde{\sqcup} d_2)(x) &= d_1 \tilde{\sqcup} d_2(q * x) \\
&= d_1(q * x) \curlyvee d_2(q * x) \\
&= q * d_1(x) \curlyvee q * d_2(x) \\
&= \theta(q_1, d)(x) \curlyvee \theta(q_2, d)(x) \\
&= \theta(q_1, d) \,\tilde{\sqcup}\, \theta(q_2, d)(x) \\
\text{Therefore } \ \theta(q, d_1 \tilde{\sqcup}\, d_2) &= \theta(q_1, d) \,\tilde{\sqcup} \theta(q_2, d)
\end{aligned}$$

6. $\theta(q_1 * q_2, d)(x) = d((q_1 * q_2) * x) = d((q_1 * q_2 * x))$

$$\begin{aligned}
\theta(q_1, \theta(q_2, d))(x) &= \theta(q_2, d)(q_1 * x) \\
&= d(q_2 * (q_1 * x)) \\
&= d(q_1 * q_2 * x) \ \ (\because \ \text{Q is commutative})
\end{aligned}$$

Similarly, $\theta(q_2, \theta(q_1, d))(x) = d(q_1 * q_2 * x)$

Thus $\theta(q_1 * q_2, d) = \theta(q_1, \theta(q_2, d)) = \theta(q_2, \theta(q_1, d))$.

Therefore θ is an action of Q on $\mathfrak{D}(Q)$. Hence $(\theta, \mathfrak{D}(Q))$ is a Q-slice.

$\square$

This last section reveals that the deductions which have been defined on quantales can be structurally viewed in a different way. We have shown that deductions are Q-slice morphisms on $(*, Q)$. In L-slices we have already proved that $\mathrm{Hom}(J, K)$ is a L-slice. The above theorem proves the possibility of the existence of a generalised L-slice on the collection of all deductions.

Here we have given only the basics of the generalised L-slices. The Q-slices which are the generalisation of L-slices can be further studied. Also, this chapter envisaged the development of the succeeding chapter.

Chapter 7

Graphs Associated with L-slices

This chapter deals with the graph theoretic approach to L-slices. The idea of relating graphs with algebraic structures was started by the work of Beck in [11]. The algebraic properties of L-slices prompted us to consider the possibility of various graphs that could be associated with it. The chapter introduces two different graphs on L-slices. The total graph $\Gamma((T(\sigma, J))$ is defined. We derive a characterisation for such graphs to be nonempty. The structural properties of $\Gamma((T(\sigma, J))$ is studied. The weak Zariski Topolgy on (σ, J) gives us the graph $G_T(\omega^*)$. The conditions under which the graph is nonempty is examined. Also some of the structural properties of $G_T(\omega^*)$ is obtained. Here we consider only finite L-slices and consequently the graphs under consideration woud be the finite ones.

7.1. Total Graph of L-Slice

Definition 7.1.1. Let (σ, J) be an L-slice and $L^* = L \backslash \{0_L\}$. We define the torsion elements of a L-slice $T(\sigma, J)$ as $T(\sigma, J) = \{x \in (\sigma, J) : \sigma(a, x) = 0_J \ for\ some\ a \in L^*\}$. It is evident that $T(\sigma, J)$ is always nonempty

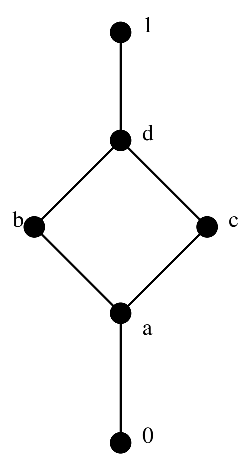
.

The structure of $T(\sigma, J)$.

Theorem 7.1.2. *$T(\sigma, J)$ is an ideal of L-slice (σ, J).*

Proof. If $x, y \in T(\sigma, J)$ then there exists $a, b \in L^*$ such that $\sigma(a, x) = 0_J$ and $\sigma(b, y) = 0_J$. Also, $\sigma(a \sqcap b, x \vee y) = \sigma(a, \sigma(b, x \vee y)) = \sigma(a, \sigma(b, x) \vee \sigma(b, y)) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, 0_J) = 0_J$. Therefore $x \vee y \in T(\sigma, J)$.
If $z \leqslant x$ then $\sigma(a, z) \leqslant \sigma(a, x)$ implies $\sigma(a, z) = 0_J$. Therefore $z \in T(\sigma, J)$. Consider $\sigma(b, x) \in (\sigma, J)$, then $\sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, 0_J) = 0_J$. Hence $T(\sigma, J)$ is an ideal of (σ, J). □

Examples 7.1.3. *i) Let L be a locale and let $J = \downarrow x$ for some $x \in L$. Then $(\sqcap, J)$ is a slice and $T(\sqcap, J) = \{y \in J : \sqcap(a, y) = 0_J\} = \{y \in J : a \sqcap y = 0_J\}$.*
ii) Consider the locale represented by the following Hasse diagram



Let $J = \downarrow b = \{0, a, b\{\}\}$ then $T(\sqcap, J) = \{0_J\}$.

Note that for any L-slice (σ, J) the annihilator $Ann(J) = \{x \in (\sigma, J) : \sigma(a, x) = 0_J \ \ \forall x \in (\sigma, J)\} \subseteq T(\sigma, J)$. We now define the total graph of the L-slice (σ, J).

Definition 7.1.4. The vertex set V_T of $\Gamma(T(\sigma, J))$ is the set of all elements of the L-slice and the edge set E_T of $\Gamma(T(\sigma, J)) = \{(x, y) : x \vee y \in T(\sigma, J)\}$.

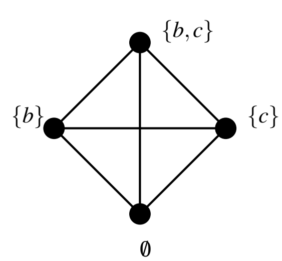
Theorem 7.1.5. *The total graph $\Gamma(T(\sigma, J))$ is complete if and only if $T(\sigma, J) = (\sigma, J)$.*

Proof. Suppose $\Gamma(T(\sigma, J))$ is complete then there exists an edge between every $x, y \in V_T$. That is, $x \vee y \in T(\sigma, J)$. In particular, every vertex is adjacent to 0_J. Hence $x \vee 0_J = x \in T(\sigma, J)\ \forall x \in (\sigma, J)$. Thus $T(\sigma, J) = (\sigma, J)$. Conversely suppose that $T(\sigma, J) = (\sigma, J)$. Since J is a join semilattice, for any two vertices $u, v \in V_T$ implies $u \vee v \in J = T(\sigma, J)$. Thus $\Gamma(T(\sigma, J))$ is complete. □

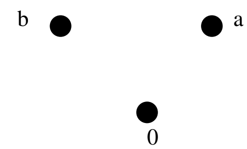
Corollary 7.1.6. *The above theorem is necessarily satisfied if $Ann(J) \neq \{0_J\}$.*

Proof. Suppose $a \in Ann(J)$. The definition of $Ann(J)$ shows that $\sigma(a, x) = 0_J, \quad \forall x \in (\sigma, J)$. Evidently, $T(\sigma, J) = (\sigma, J)$ and $\Gamma(T(\sigma, J))$ is complete. □

Examples 7.1.7. *i) Let $X = \{a, b, c\}$. Then $\mathfrak{P}(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{b, c\}$. $\downarrow A = \{C \in \mathfrak{P}(X) : C \subseteq A\}$ implies $\downarrow A = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. $\downarrow A$ is a join semilattice under the partial ordering $\subseteq$. Also, $(\mathfrak{P}(X), \subseteq)$ is a locale. Define the action $\sqcap$ on $\downarrow A$ as $\sqcap(B, A_1) = B \cap A_1$ where $A_1 \in \downarrow A$. The annihilator $Ann(\downarrow A) = \{\phi, \{a\}\}$ and $T(\sqcap, \downarrow A) = \{\phi, \{b\}, \{c\}, \{b, c\}\} = \downarrow A$. Therefore $\Gamma(T(\sqcap, \downarrow A))$ is complete. Also, $\Gamma(T(\sqcap, \downarrow A))$ is the complete graph K_4.*



ii) In Example 7.1.3 ii), we observed that $T(\sqcap, \downarrow b) = \{0_J\}$. Then the total graph $\Gamma(T(\sqcap, \downarrow b))$ is totally disconnected.



Now we can generalise the above as follows.

Proposition 7.1.8. *$\Gamma(T(\sigma, J))$ is totally disconnected if and only if $T(\sigma, J) = \{0_J\}$.*

Proof. Since $\Gamma(T(\sigma, J))$ is totally disconnected we have that 0_J is not connected with any other vertices. Hence $x \vee 0_J = x \notin T(\sigma, J)$ implying that $T(\sigma, J) = \{0_J\}$. Conversely, let $T(\sigma, J) = \{0_J\}$. Any two vertices x, y of $\Gamma(T(\sigma, J))$ is connected if and only if $x \vee y = 0_J$ and that is possible if and only if $x = 0_J, y = 0_J$.Hence $\Gamma(T(\sigma, J))$ is totally disconnected. □

We have already shown that $T(\sigma, J)$ is an ideal of (σ, J). Now we propose the next two theorems which is a consequence of the structure of $T(\sigma, J)$.

Theorem 7.1.9. *The subgraph induced by the set $T(\sigma, J)$ is always complete. In particular, if $| T(\sigma, J) | = n$ then the subgraph induced will be the complete graph K_n.*

Proof. Since $T(\sigma, J)$ is an ideal of (σ, J), if $x, y \in T(\sigma, J)$ then $x \vee y \in T(\sigma, J)$. Therefore the subgraph induced by $T(\sigma, J)$ is always complete. □

Corollary 7.1.10. *The clique number $\omega(\Gamma(T(\sigma, J)))$ is $| T(\sigma, J) |$.*

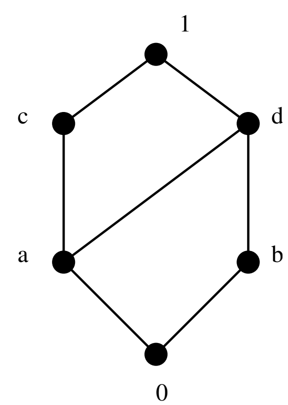
Corollary 7.1.11. *The subgraph induced by $(\sigma, J) \backslash T(\sigma, J)$ is totally disconnected and the independence number $\beta(\Gamma(T(\sigma, J))) = | (\sigma, J) \backslash T(\sigma, J) |$.*

Theorem 7.1.12. *If $T(\sigma, J)$ is a proper ideal of (σ, J) then $\Gamma(T(\sigma, J))$ is always disconnected.*

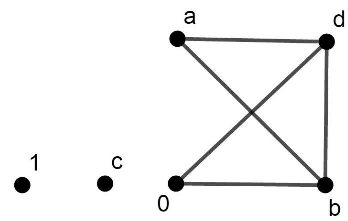
Proof. Let $x, y \in (\sigma, J)$ such that $x \in T(\sigma, J)$ and $y \in (\sigma, J)\backslash T(\sigma, J)$. The subgraph induced by $T(\sigma, J)$ and $(\sigma, J)\backslash T(\sigma, J)$ are disjoint. Suppose they are connected then $x \vee y \in T(\sigma, J)$. But $T(\sigma, J)$ is an ideal would imply that $y \in T(\sigma, J)$, which is a contradiction. Thus the subgraphs induced by $T(\sigma, J)$ and $(\sigma, J)\backslash T(\sigma, J)$ will always be disjoint. Thus $\Gamma(T(\sigma, J))$ is always disconnected. □

Let us consider some L-slices and examine the properties of total graph associated with them.

Example 7.1.13. *Let J be given by the Hasse diagram*

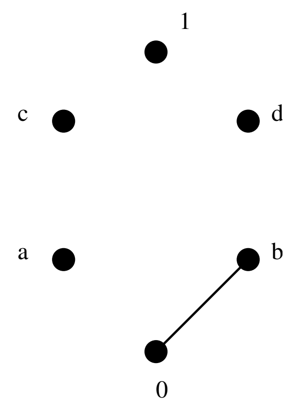


And let $L = \{0 \leqslant c \leqslant 1\}$. Define the action on J as $\sqcap(a, x) = a \sqcap x$ for $a \in L$ and $x \in J$. The ideal $T(\sigma, J) = \{0, b, d\}$ and it is a proper ideal of $(\sqcap, J)$. $\Gamma(T(\sqcap, J))$ is disconnected and the graph is



Thus the total graph of the L-slice is the union of two K_3 graphs and two K_1 graphs.

If we consider $L = \{0 \leqslant a \leqslant 1\}$ then $T(\sqcap, J) = \{0, b\}$. Then the total graph of L-slice is



The total graph of the L-slice is the union of one K_2 graph and four K_1 graphs.

If we let $L = \{0 \leqslant d \leqslant 1\}$ then $T(\sqcap, J) = \{0\}$ and $\Gamma(T(\sqcap, J))$ is disconnected.

Remark. A L-slice is said to be a σ-domain over L if there exists no torsion elements for the L-slice. In other words, there exists no $a \in L^*$ such that $\sigma(a, x) = 0_J$.

Examples 7.1.14. *i) If L is a chain then the L-slice $(\sqcap, L)$ is a σ-domain over L. Let $T^*(\sigma, J) = \{x \neq 0_J : \exists a \in L^*$ such that $\sigma(a, x) = 0_J\}$ then the corresponding total graph is denoted by $\Gamma(T^*(\sigma, J))$.*

In this case, if $T^*(\sigma, J)$ is nonempty then $T^*(\sigma, J)$ is a subslice. Also, if an L-slice is a σ-domain over L then the corresponding $\Gamma(T^*(\sigma, J))$ is an empty graph.

ii) If L is a chain then $\Gamma(T^*(\sqcap, L))$ is an empty graph.

Definition 7.1.15. A locale L is said to have zero divisors if for $a \in L^*$ there exists $b \in L^*$ such that $a \sqcap b = 0_L$.

Lemma 7.1.16. *Let a be a zero divisor of the locale L. If $x \in (\sigma, J)$ then $\sigma(a, x) \in T(\sigma, J)$.*

Proof. If a is a zero divisor of L then there exists $b \in L$ such that $a \sqcap b = 0_L$. Therefore $\sigma(b, \sigma(a, x)) = \sigma(a \sqcap b, x) = \sigma(0_L, x) = 0_J$ implies that $\sigma(a, x) \in T(\sigma, J)$. □

Lemma 7.1.17. *If the top element 1_L of the locale L is the join of n zero divisors of L then every element of the L-slice (σ, J) is the join of n torsion elements.*

Proof. Let $1_L = z_1 \sqcup z_2 \sqcup \ldots.. \sqcup z_n$, where each z_i is a zero divisor of L.

For any $x \in (\sigma, J)$, $\sigma(1_L, x) = \sigma(z_1 \sqcup z_2 \sqcup \ldots.. \sqcup z_n, x)$

$$= \sigma(z_1, x) \vee \sigma(z_2, x) \ldots\ldots..\sigma(z_n, x)$$

$$\Rightarrow x = \sigma(z_1, x) \vee \sigma(z_2, x) \ldots\ldots..\sigma(z_n, x)$$

The above lemma states that each $\sigma(z_i, x) \in T(\sigma, J)$. Hence the result. □

A characterisation of the total graph of an L-slice based on the zero divisors of the locale L.

Theorem 7.1.18. *If L has a finite basis of zero divisors then the total graph of the L-slice (σ, J) is complete.*

Proof. If $\{z_1, z_2, \ldots.z_n\}$ be the finite basis of zero divisors then from the above lemma $x = \sigma(z_1, x) \vee \sigma(z_2, x) \ldots\ldots..\sigma(z_n, x)$, where each $\sigma(z_i, x) \in T(\sigma, J)$. And the fact that $T(\sigma, J)$ is an ideal will give us the theorem. □

Proposition 7.1.19. *The chromatic number* $\chi\Gamma(T(\sigma, J))$ *of* $\Gamma(T(\sigma, J))$ *is such that either always* $\chi\Gamma(T(\sigma, J)) = 1$ *or* $\chi\Gamma(T(\sigma, J)) = n + 1$*, where* $n =| T(\sigma, J) |$*.*

Proof. If $T(\sigma, J) = \{0_J\}$ then graph $\Gamma(T(\sigma, J))$ is totally disconnected and $\chi\Gamma(T(\sigma, J))$ is one. We know that the subgraph induced by $T(\sigma, J)$ is the complete graph K_n. Theorem 7.1.12 shows that if $T(\sigma, J)$ is a proper ideal then $\Gamma(T(\sigma, J))$ is always disconnected. Thus if $T(\sigma, J) \neq \{0_J\}$ then $\chi\Gamma(T(\sigma, J)) = n + 1$. □

Remark. Theorem 7.1.8 ($t = \{0\}$) and Theorem 7.1.12 ($t \neq \{0\}$) shows that $\Gamma(T(\sigma, J))$ is never a critical graph.

Property 8. *The diameter of the graph* $diam(\Gamma(T(\sigma, J))) \in \{1, \infty\}$

Proof. The theorem 7.1.5 shows that $diam(\Gamma(T(\sigma, J))) = 1$. If $T(\sigma, J) = \{0_J\}$ then the graph is totally disconnected and $diam()\Gamma(T(\sigma, J))) = \infty$. □

Property 9. *The radius* $r(\Gamma(T(\sigma, J))) \in \{0, 1\}$*.*

Proof. The graph $\Gamma(T(\sigma, J))$ is either complete or disconnected. Hence the radius of the total graph of the L-slice will be either 0 or 1. □

Let us denote the subgraph induced by $T(\sigma, J)$ as $\Gamma_n(T(\sigma, J))$, where n denotes the cardinality of the set $T(\sigma, J)$.

Property 10. *If* $T(\sigma, J)$ *is proper ideal then the diameter and radius of* $\Gamma_n(T(\sigma, J))$ *will be the same and equal to 1.*

Proof. The property is obtained through the completeness of the subgraph. □

Property 11. *Whenever* $|T(\sigma, J)|= n \geqslant 3$*, then the girth of* $\Gamma_n(T(\sigma, J))$ *denoted as* $gr(\Gamma_n(T(\sigma, J))) = 3$ *and the circumference of* $\Gamma_n(T(\sigma, J))$*,* $c(\Gamma_n(T(\sigma, J))) = n$*.*

Proof. Follows from the completeness of $\Gamma_n(T(\sigma, J))$. □

The above property can also be restated as

Property 12. *If* $|T(\sigma, J)| = n \geqslant 3$ *if and only if* $gr(\Gamma_n(T(\sigma, J))) = 3$

In this section we have defined and studied various properties of the total graph. We have observed that $\Gamma(T(\sigma, J))$ is complete if $|Ann(J)| \geqslant 2$. If $|T(\sigma, J)| \geqslant 2$ then $\Gamma(T(\sigma, J))$ disconnected. Thus $\Gamma(T(\sigma, J))$ is either complete or disconnected.

7.2. Graphs associated with weak Zariski topology ω^* on $Spec(\sigma, C)$

We have shown in Chapter 5 that the sets $C(n) = \{p \in Spec(\sigma, C) : n \leqslant p\}$ forms basis for a topology on prime spectrum $Spec(\sigma, C)$. Also, if $C(n) \cup C(l) = C(n \wedge l)$ the collection $\nu = \{C(n) : n \in (\sigma, C)\}$ will then be the collection of closed sets on $Spec(\sigma, C)$ and the topology so formed may be called weak Zariski topology ω^* on $Spec(\sigma, C)$.

This section deals with graphs associated with this weak Zariski topology ω^*. For a subset T of $Spec(\sigma, C)$ we introduce a graph $G_T(\omega^*)$. We study some of its properties and show that it has a bipartite subgraph.

Definition 7.2.1. Let T be a nonempty subset of $Spec(\sigma, C)$. The graph $G_T(\omega^*)$ has as vertex set $V(G_T(\omega^*)) = \{n \in (\sigma, C) : \exists \;\; l \in (\sigma, C) \;\; such \;\; that \;\; C(n) \cup C(l) = T\}$. Also, two vertices n and k are adjacent if and only if $C(n) \cup C(k) = T$. In other words, the graph $G_T(\omega^*)$ has n as vertex if and only if there exists a $l \in (\sigma, C)$ such that $C(n \wedge l) = T$.

Remark. We study the properties of graphs associated with the weak Zariski topology. The definition itself gives us two conditions for such a graph to exist. We state them as our next two propositions.

Proposition 7.2.2. *$G_T(\omega^*) \neq \phi$ if and only if T is closed and is not an irreducible subset of $Spec(\sigma, C)$.*

Proof. Follows directly from the definition of $G_T(\omega^*)$. □

The above proposition can be rephrased as follows.

Proposition 7.2.3. *$G_T(\omega^*) \neq \phi$ if and only if $T = C(\bigwedge T)$ and T is not an irreducible subset of $Spec(\sigma, C)$.*

Proof. Suppose $G_T(\omega^*) \neq \phi$. The above proposition shows that T is closed.So it remains to show that $T = C(\bigwedge T)$. We know that $T \subseteq C(\bigwedge T)$. Let $C(n)$ be any closed subset of $Spec(\sigma, C)$ containing T. Then $n \leqslant p \;\; \forall p \in T$ implies that $n \leqslant \bigwedge_{p \in T} p = q$. Therefore for evey $l \in C(q)$ implies $l \in C(n)$. That is, $C(n) \supseteq C(q)$. Hence $C(q)$ is the smallest closed set containing T. Thus $T = C(q) = C(\bigwedge_{p \in T} p) = C(\bigwedge T)$. □

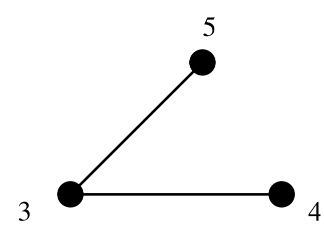
Theorem 7.2.4. *The weak Zariski topology graph $G_T(\omega^*)$ is connected and the diameter of the graph , $diam(G_T(\omega^*)) \leqslant 2$.*

Proof. If n and k are not adjacent then $C(n) \cup C(k) \neq T$. Now there exists vertices l and k such that $C(n) \cup C(l) = C(n \wedge l) = T$ and $C(m) \cup C(k) = C(m \wedge k) = T$. If $l = m$ then $n - l - k$ is a path of length two. If $l \neq m$ then $n - (l \wedge m) - k$ is a path of length two. Hence $G_T(\omega^*)$ is connected and $diam(G_T(\omega^*)) \leqslant 2$. □

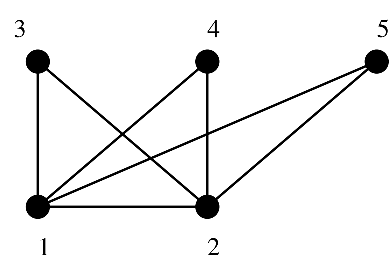
Corollary 7.2.5. *If $G_T(\omega^*)$ contains a cycle then the girth $g(G_T(\omega^*)) \leqslant 3$.*

Proof. Suppose $g(G_T(\omega^*)) = k > 3$. Let $n_1 - n_2 - n_3 - \cdots - n_{k-1} - n_k - n_1$ be a cycle with length k. Then clearly $n_1 - (n_2 \wedge n_{k-1}) - n_k - n_1$ is a cycle of length 3. Hence a contradiction. Therefore $g(G_T(\omega^*)) \leqslant 3$. □

Examples 7.2.6. *i) Let $C = \{1, 2, 3, 4, 5\}$ and $(C, \leqslant)$ be complete lattice with $\leqslant$ as the usual ordering 'less than or equal to'. Let the locale be $(L = \{1, 2, 5\}, \leqslant)$. The action σ defined as $\sigma(a, x) = a \sqcap x$ will make C an L-component $(\sqcap, C)$.*
In this case $Spec(\sigma, C) = \{2, 3, 4\}$ and $C(1) = Spec(\sigma, C)$, $C(2) = \{2, 3, 4\}$, $C(3) = \{3, 4\}$, $C(4) = \{4\}$, $C(5) = \phi$. Also, $C(n) \cup C(m) = C(n \wedge m)$ for every $n, m \in (\sqcap, C)$.
If $T = \{3, 4\}$ then $V(G_T(\omega^*)) = \{3, 4, 5\}$. The graph $G_T(\omega^*)$ is $K_{1,2}$.

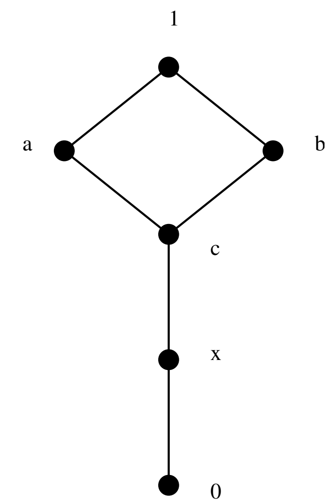


If $T = \{2, 3, 4\}$ then $V(G_T(\omega^*)) = \{1, 2, 3, 4, 5\}$ and the graph $G_T(\omega^*)$ is $K_{2,3}$.

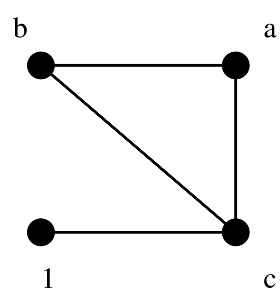


Also if $T = \{2, 4\}$ then $G_T(\omega^* = \phi)$

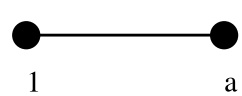
ii) Consider the complete lattice



Let $L = \{0, a, 1\}$. *The spectrum* $Spec(\sigma, C) = \{x, a, b\}$. $C(0) = Spec(\sigma, C)$, $C(1) = \phi$, $C(x) = \{x, a, b\}$, $C(c) = \{a, b\}$, $C(a) = \{a\}$, $C(b) = \{b\}$. *It can be easily verified that* $C(n) \cup C(m) = C(n \wedge m)$ *for every pair* $n, m \in (\sigma, C)$. *For* $T = \{x, a\}$ *then* $G_T(\omega^*) = \phi$. *If* $T = \{a, b\}$ *then* $V(G_T(\omega^*)) = \{1, c, a, b\}$ *and the graph is*



If $T = \{a\}$ *then* $V(G_T(\omega^*)) = \{1, a\}$ *and the graph is*



Remark. Since $C(1_C) = \varnothing$ the top element 1_C will always belong to the vertex set and $deg(1_C) \geqslant 1$. Also $deg(1_C)$ is the cardinality of the set $\{n \in (\sigma, C) : C(n) = T\}$

Proposition 7.2.7. *For any finite set T and $G_T(\omega^*) \neq \phi$ we have that $T \cap V(G_T(\omega^*) \neq \phi$.*

Proof. Let $p \in T$ then we have $C(p) \cup C(\bigwedge_{q \in T, q \neq p} q) = T$.Therefore, $p \in V(G_T(\omega^*))$. □

7.3. The subgraph $G_T^{'}(\omega^*)$

Definition 7.3.1. The subgraph $G_T^{'}(\omega^*)$ of $G_T(\omega^*)$ has vertex set $V(G_T^{'}(\omega^*))$ defined as $\{n \in (\sigma, C)$: there exist $l \in (\sigma, C)$ such that $C(n) \cup C(l) = T$, $C(n), C(l) \neq T, C(n) \cap C(l) = \phi\}$, where $(u, v) \in E(G_T^{'}(\omega^*))$ if and only if $C(u) \cup C(v) = T, C(u) \cap C(v) = \phi$.

Note that the degree of u is the number of vertices k with $C(v) = C(k)$.

Proposition 7.3.2. *$G_T^{'}(\omega^*) \neq \phi$ if and only if $T = C(\bigwedge_{q \in T} q)$ and is disconnected.*

Proof. We have already shown $G_T(\omega^*) \neq \phi$ then $T = C(\bigwedge_{q \in T} q)$. Let $n, l \in V(G_T^{'}(\omega^*))$, then $C(n) \cup C(l) = T, C(n) \cap C(l) = \phi$. Thus, T is disconnected. The converse follows easily from the definition. □

Theorem 7.3.3. *$G_T^{'}(\omega^*)$ is a bipartite graph.*

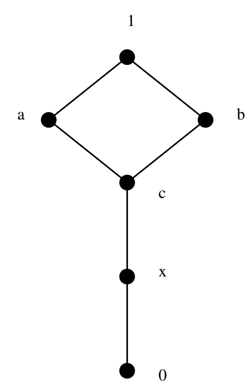
Proof. A graph is bipartite if and only if it does not contain an odd cycle [23]. We will show that $G_T^{'}(\omega^*)$ does not have an odd cycle. Suppose $g(G_T^{'}(\omega^*)) = k > 4$. Consider the cycle $n_1 - n_2 - n_3 - \cdots - n_{k-1} - n_k - n_1$ of length k. It is evident that $C(n_{k-1}) = C(n_1)$. The cycle $n_1 - n_2 - n_3 - \cdots - n_{k-2} - n_1$ is of length $k - 1$.

Thus $g(G'_T(\omega^*)) \leqslant 4$. We show that $g(G'_T(\omega^*)) \neq 3$. Suppose $n_1 - n_2 - n_3 - n_1$ is 3-cycle. Then $\phi = (C(n_1) \cap C(n_2)) \cup (C(n_3) \cap C(n_1)) = C(n_1) \cap (C(n_2) \cup C(n_3)) = C(n_1) \cap T = C(n_1)$.Thus we arrive at a contradiction. Hence the graph does not contain an odd cycle. □

Corollary 7.3.4. *If $G'_T(\omega^*)$ contains a cycle then $gr(G'_T(\omega^*)) = 4$*

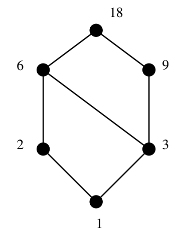
Remark. $G'_T(\omega^*)$ is a complete bipartite graph if and only if $C(n) = C(l)$ for every vertices l, n belonging to same vertex set.

Examples 7.3.5. *i) Consider the complete lattice C to be*

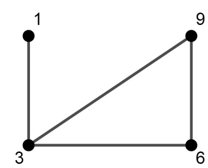


For $L = \{0, a, 1\}, Spec(\sigma, C) = \{x, a, b\}$.If $T = \{a, b\}$, then $V(G'_T(\omega^*)) = \{a, b\}$. Hence $G'_T(\omega^*)$ is $K_{1,1}$.Also if $T = \{a\}$, then $V(G'_T(\omega^*)) = \phi$.

ii) Consider the complete lattice C to be

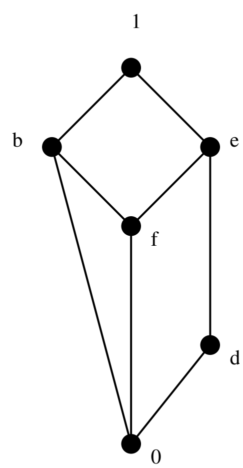


For $L = \{1, 2, 18\}, Spec(\sigma, C) = \{2, 6, 9\}$. *If* $T = \{6, 9\}$, *then* $V(G_T(\omega^*)) = \{3, 6, 9, 18\}$ *and the corresponding graph is*

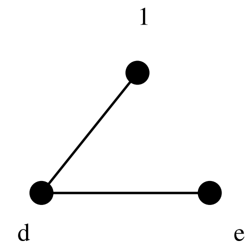


and $V(G_T^{'}(\omega^*)) = \{6, 9\}$ *and the graph is* $K_{1,1}$.

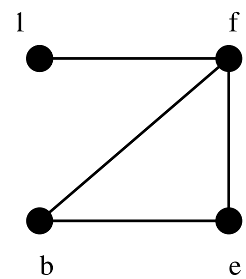
iii) Consider the complete lattice C to be



$L = \{0, d, e, 1\}, Spec(\sigma, C) = \{b, d, e\}$. *If* $T = \{d, e\}$, *then* $V(G_T(\omega^*)) = \{d, e, 1\}$ *and the corresponding graph is*



and $V(G'_T(\omega^*)) = \phi$. *If* $T = \{b, e\}$, *then* $V(G_T(\omega^*)) = \{f, b, e, 1\}$ *and the corresponding graph is*



$V(G'_T(\omega^*)) = \{b, e\}$ *and* $G'_T(\omega^*)$ *is* $K_{1,1}$.

This chapter introduces the concepts of total graphs and that of graphs associated with the weak Zariski topology. The introduction of concepts of algebraic graph theory into L-slices is initiated through this chapter.Different types of graphs can be studied in the background of L-slices.The topological properties of L-slices can be used to study the graphs associated with them.

Conclusion

The classical topology involves the concept of a point and its neighbourhood. From Stone's representation theorem there envisaged a journey from the realm of point-set topology to point free topology.Isbell [31]emphasises the importance of point-free topology. The benefit of thinking from a point free domain is that both lattice theoretic, in turn algebraic tools could be brought into play. Thus frame/locale became a breakthrough for topologists. The algebraic concept of 'group action' in the background of locales is studied by Sabna K.S. and Mangalambal N.R [58].The structure thus developed is called L-slices. The benefit of L-slices is that both algebraic and topological tools were put into action. Moreover [59] showcases many algebraic concepts like action,annihilator,isomorphism theorem and so on. Similarly, the topological properties like compactness were also discussed.Thus L-slices ensures the availability of both topological and algebraic tools.This frame work of L-slices is used in our thesis.

We have investigated L-slices from different perspectives. For each $x \in (\sigma, J)$, the filter F_x has assisted in the development of category **Batch**. We have defined associated filter , R-A slice on the basis of F_x. Also it enables us to define F-continuous slice morphisms parallel to the concept of sequential continuity in classical topology. Further, F_x prompted the development of the concept called Box $\mathfrak{S}$ and the stack of filters $\mathfrak{S}_x$ which consequently led to **Batch** .

The study on Hom slice $Hom(\sigma, J)$ led to the development of class of expansive and contractive operators. The class of contractive operators $(a : x)_{Hom}$ forms a basis for the topology on $Hom(\sigma, J)$. Also through the system of contractive operators J_a, evolved the quotient slice $(\gamma, J/\sim_a)$. The subslice $[a : f]_{(\sigma,J)}$ of (σ, J) permits the topological notion of continuity in the dominion of L-slices.

The ring $C(X)$ is the study of continuous real valued functions on the topological space X. In other words,$C(X)$ deals with all those continuous function with range set in R. Equivalently, in $Hom(L, J)$ we studied the properties of all L-slice morphisms with the domain fixed as the locale L. We defined the collection of zero sets and fixed ideals of L-slices and found that they have the structure of an L-slice.

The similarity between the structure of L-slices and modules prompted us to consider the possibility of Zariski topology on L-slices. There we introduced L-component thus succeeding in the definition of the Zariski topology Ω^*. Some properties of Ω^* were also investigated.

The possibility of generalising L-slices led to the development of Q-slices.Quantales are well known to be the generalisation of locales. We have studied a particular type of maps called deduction on quantales. The ideals constructed using deductions led to a quotienting of quantales. Also we defined Q-slices and showed that the collection of all deductions $\mathfrak{D}(Q)$ is a Q-slice.

Graph theoretic development of L-slices led us to the total graph $\Gamma(T(\sigma, J))$ and $G_T(\omega^*)$ on L-slices. We have shown that if $T(\sigma, J)$ is a proper ideal of (σ, J) then $\Gamma(T(\sigma, J))$ is disconnected. Also we showed that $\Gamma(T(\sigma, J))$ is complete if and only if the L-slice (σ, J) is not faithful. We were also able to prove that the weak Zariski topology graph $G_T(\omega^*)$ is connected and $diam(G_T(\omega^*)) \leqslant 2$.

The first, third and last chapters leave ample scope for further study. The first chapter introduced the category **Batch** and some of its basic properties. The **Batch** can be studied for its categorical properties like subobjects, monomorphism,epimorphism,limit and so on. Chapter 3 leaves open a far more wide area of studying the properties of fixed ideals.The study can be developed in the direction of relationship between fixed ideals and maximal ideals. Similarly, last chapter paves way for a graph theoretic approach to L-slices.More concepts of graph theory can be developed in the background of L-slices. A characterisation of L-slices and different types of graphs can be studied.

Research Papers

1. Mary Elizabeth Antony, Sabna K.S, Mangalambal N.R, *The Topology* Ω^* *on Spec*(σ, C) of L-component (σ, C), Malaya Journal of Matematik, Special Issue,No.1 (2019),University Press, 441-444

2. Mary Elizabeth Antony , Sabna K.S, Mangalambal N.R, *Regular Filter ,Associated Filter and Their Properties*, International Journal of Engineering and Advanced Technology, Vol 8, Number 4 (2019) , Blue Eyes Intelligence Engineering and Sciences Publication , 1645-1649.

3. Mary Elizabeth Antony , Sabna K.S, Mangalambal N.R, *Zariski Topology on L-slices*, International Journal of Innovative Technology and Exploring Engineering, Vol 8, Number 4 (2019) , Blue Eyes Intelligence Engineering and Sciences Publication , 274-276 .

4. Mary Elizabeth Antony, Mangalambal N.R, *Some notes on deductions on Quantales*, chapter 8 in the book *Topology and Fractals*, Cape Comorin Publisher,December 2018 82-91.

5. Mary Elizabeth Antony, Sabna K.S, Mangalambal N.R, *Some notes on Second countability in Frames*, IOSR Journal of Mathematics, Vol 9, Issue 2, Nov-Dec.2013, 29-32.

6. Mary Elizabeth Antony, Sabna K.S, Mangalambal N.R, *The continuity of L-slice homomorphism* σ_x, Communicated.

7. Mary Elizabeth Antony, Sabna K.S, Mangalambal N.R, *The fixed ideals of* $Hom(L, J)$, Communicated.

8. Mary Elizabeth Antony, Mangalambal N.R, *Graphs associated with quantales* , Communicated.

9. Mary Elizabeth Antony, Sabna K.S, Mangalambal N.R, *The Box* $\mathfrak{S}$ *and the category* **Batch**, Communicated.

10. Mary Elizabeth Antony, Sabna K.S, Mangalambal N.R, *The total graph of L-slices*, Communicated.

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