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CERTIFICATE

I hereby certify that the thesis entitled "TOTAL DOMINATION POLYNOMIALS: A NEW APPROACH" is a bonafide work carried out by **Sri. Latheeshkumar A. R.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Anil Kumar. V.

(Research Supervisor)

DECLARATION

I hereby declare that the thesis, entitled "**TOTAL DOMINATION POLYNOMIALS: A NEW APPROACH**" is based on the original work done by me under the supervision of **Dr. Anil Kumar V.**, Associate Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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University of Calicut,

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Contents

List of Symbols

$\cong$	is isomorphic to
$\deg u$	the degree of the vertex u
$\Delta(G)$	the maximum degree of G
$\delta(G)$	the minimum degree of G
$\overline{G}$	the complement of a simple graph G
$\gamma(G)$	the domination number of G
$\gamma_{gb}(G)$	the global bipartite domination number of G
$\gamma_t(G)$	the total domination number of G
$\gamma_{gbt}(G)$	the global bipartite total domination number of G
$\tau(G)$	the vertex cover number of G
C_n	the cycle of length n
$\mathcal{C}(G,i)$	the family of vertex covering sets of G with cardinality i
$c(G,i)$	the cardinality of $\mathcal{C}(G,i)$
$\mathcal{C}(G,x)$	the vertex cover polynomial of G.
$D(G,i)$	the family of dominating sets of G with cardinality i
$d(G,i)$	the cardinality of $D(G,i)$

$D_t(G,i)$	the family of total dominating sets of G with cardinality i
$d_t(G,i)$	the cardinality of $D_t(G,i)$
$D_{gb}(G,i)$	the family of global bipartite dominating sets of G with cardinality i
$d_{gb}(G,i)$	the cardinality of $D_{gb}(G,i)$
$D(G,x)$	the domination polynomial of G
$D_t(G,x)$	the total domination polynomial of G
$D_{gb}(G,x)$	the global bipartite domination polynomial of G
$E(G)$	the edge set of the graph G
$G-e$	the subgraph of G obtained by deleting the edge e
$G-v$	the subgraph of G obtained by deleting the vertex v
$G=(V,E)$	a graph G
$G^{(k)}$	the one point union of k copies of G
$G_1\Box G_2$	the Cartesian product of two graphs G_1 and G_2
$G_1\circ G_2$	the corona of two graphs G_1 and G_2
$G_1\cup G_2$	the union of two graphs G_1 and G_2
$G_1\oplus G_2$	the ring sum of two graphs G_1 and G_2
$G_1\vee G_2$	the join of two graphs G_1 and G_2
$H=(V,\mathcal{E})$	a hypergraph H
H_G	the open neighborhood hypergraph of G
K_n	the complete graph on n vertices
$K_{m,n}$	the complete bipartite graph with part size m and n
$K_{n_1,n_2,\ldots,n_m}$	the complete m-partite graph
$M(G)$	the middle graph of the graph G
$\mathscr{P}(X)$	the power set of X

P_n	the path on n vertices
$spl(G)$	the splitting graph of the graph G
$spl^k(G)$	the splitting graph of order k of the graph G
$S^k(G)$	the iterated splitting graph of order k of the graph G
$T(G)$	the total graph of the graph G
$V(G)$	the vertex set of the graph G

Introduction

Graph theory is one of the most relevant and fastest growing branches of mathematics. Owing to its exponential growth in the past few decades and its extensive applications in diverse fields, Graph theory has become the foundation stone of applied mathematics. This all alluring field of study has already established intricate connection with other branches of mathematics including Number theory, Algebra, Linear Algebra, Topology and Geometry. Moreover, cutting across disciplines and crossing the border of academia, Graph theory has granted a plethora of indispensable tools in the design and analysis of communication networks, mobile computing and social networks to mention a few. In fact, the varied applications of Graph theory in Engineering, Social science, Biological science etc. have immensely contributed to the progress and popularity of mathematics in general and Graph theory in particular.

One of the prime concerns of Graph theory today is the study of graph polynomials. Graph polynomial is defined as : "Let $\mathscr{G}$ be the class of graphs and let R be a ring and X be a (not necessarily finite) set of indeterminates. A graph polynomial is a function $\mathsf{P}\colon \mathscr{G} \to R[X]$ such that for isomorphic graphs H and

K, we have $\mathsf{P}(H) = \mathsf{P}(K)$" [39]. To put in simple terms, graph polynomials are polynomials assigned to graphs. The past few decades can be marked "seminal" while tracing the evolution of graph polynomials as an important field of research in Graph theory because many graph polynomials were studied and plenty of theoretical and practical approaches were developed during the period. The *edge difference polynomial,* a multi variate polynomial, introduced by J. J. Sylvester in 1878 is the first polynomial in Graph theory. Since then, several graph polynomials such as chromatic polynomial, Tutte polynomial, characteristic polynomial, matching polynomial, independence polynomial, interlace polynomial etc. have been introduced and studied extensively.

Studies pertaining to domination and related concepts constitutes another fascinating and productive area of research in Graph theory. For a graph G, dominating set of a given cardinality may not be unique. S. Alikhani's research in this field explored the concept of domination polynomial in graphs. Subsequently, S. Sanalkumar and A. Vijayan introduced the concept of total domination polynomial in graphs.

An overview of the thesis

The thesis successfully employs vertex cover polynomials to determine the total domination polynomials of some graphs. It is known that the problem of finding total dominating sets in simple graphs can be translated to the concept of vertex covers in hypergraphs. This prompted us to study total domination polynomials using vertex cover polynomials in hypergraphs. We establish that the interplay between total domination polynomial of a graph and vertex cover polynomial of its open neighborhood hypergraph produces several results on total domination polynomials.

The thesis comprises an introductory chapter and eight other chapters. The first section of every chapter is a brief introduction to the chapter. Along with providing a blue print of the thesis, the introductory chapter mentions what prompted the study and states the main results.

Chapter one contains the basic definitions and theorems that come in handy in the subsequent chapters.

Let $G = (V(G), E(G))$ be a graph. The *open neighborhood* of a vertex u of G is defined as $N_G(u) = \{v \in V(G) \colon uv \in E(G)\}$. A subset S of $V(G)$ is called a *total dominating set* of G if $N_G(S) = \bigcup_{u \in S} N(u) = V(G)$ and the *total domination number* [27] of G, denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. The *open neighborhood hypergraph* [27] of G, denoted by $ONH(G)$ or H_G is the hypergraph with vertex set $V(G)$ and edge set $\{N_G(u) \colon u \in V(G)\}$, consisting of the open neighborhoods of vertices in G. The *total domination polynomial* [31] or *TD-Polynomial* of G is defined as $D_t(G, x) = \sum_{i=\gamma_t(G)}^{|V(G)|} d_t(G, i)$, where $d_t(G, i)$ is the number of total dominating sets of G of cardinality i.

A subset S of $V(G)$ is called a *vertex cover* [18] of G if and only if S contains at least one end vertex of each edge in G and the *vertex cover number* of G, denoted by $\tau(G)$is the minimum cardinality of a vertex cover of G. The *vertex cover polynomial* [18] of G is defined as $\mathcal{C}(G, x) = \sum_{i=\tau(G)}^{|V(G)|} c(G, i)x^i$, where $c(G, i)$

is the number of vertex covers of cardinality i of G.

In **Chapter two**, we prove that the total domination polynomial of a graph G is the vertex cover polynomial of its open neighborhood hypergraph. In the second section of the chapter the TD-Polynomial of paths and cycles are expressed in terms of vertex cover polynomials of some graphs. In section 3, we classify the vertex cover polynomials of a graph G in terms of a given subset of the vertex set of G. Making use of these vertex cover polynomials, we obtain the total domination polynomials of certain classes of graphs in terms of the vertex cover polynomials of paths and cycles. The TD-Polynomial of the tree T_{n_1,n_2,n_3}, one point union of cycles $C_n^{(k)}$, the n-gon book graph $C_n^{2(k)}$, the theta graph $\theta_{(n)^k}$ are determined. Finally TD-Polynomial of $K_{n,n+1}^{(k)}$, the one point union of k copies of the complete bipartite graph $K_{n,n+1}$ is derived.

The inclusion of a particular vertex in every total dominating set of a graph is important in the study of total domination polynomials. Let v be a vertex of G. The polynomial $D_t^v(G,x)$ is defined as $D_t^v(G,x) = \sum_{i=1}^{|V(G)|} d_t^v(G,i)x^i$, where $d_t^v(G,i)$ is the number of total dominating sets of G containing the vertex v of cardinality i [4]. As in the previous chapter, **Chapter three** adopts the terminology of open neighborhood hypergraph to determine the total domination polynomial. In the second section, the polynomial $D_t^v(G,x)$ of paths, cycles, complete graphs and complete bipartite graphs are determined. Moreover, the TD-Polynomial of one point union of complete graphs is determined.

Section 3 deals with the total domination polynomials of ring sum of some graphs with the star graph $K_{1,m}$. By suitably selecting the common vertices, the ring sum of of the path P_n and and the star graph $K_{1,m}$, produces a number of different graphs. The TD-Polynomials of each of these graphs are obtained.

In **Chapter four**, we study the TD-Polynomials of total and middle graph of a graph. This chapter comprises three sections. In the second section, we determine the total domination polynomials of the complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$, the corona $G \circ K_1$ and $G \circ \overline{K}_m$, $G(m_1, m_2, \ldots, m_n)$, the caterpillar graph $T(m_1, m_2, \ldots, m_n)$, centipede, the bi-star graph etc. It is proved that TD-Polynomial of a connected $(n-1)$-regular bipartite graph is square of the total domination polynomial of the complete graph K_n. The TD-Polynomial of some Cayley graphs are also determined. Finally, the total domination polynomial of join of two graphs is determined.

In section 3, we discuss the TD-Polynomials of total graph and middle graph of graphs like star graph $K_{1,n}$, the caterpillar graph, $G(m_1, m_2, \ldots, m_n)$, the corona $G \circ \overline{K}_m$ etc.

Chapter five is an attempt to analyze the total domination polynomials of Cartesian products of some graphs. The chapter succeeds in establishing an interesting relation between domination polynomials and total domination polynomials. There are six sections in the chapter. In the second section, we prove that the TD-Polynomial of Cartesian product of a bipartite graph G with K_2 is square of domination polynomial of G. It is proved that for any non bipartite graph G with n vertices, there exists a bipartite graph H with $2n$ vertices such that $D_t(K_2 \Box G) = D(H, x)$.

In the third section of the chapter, TD-Polynomials of some cubic Cayley graphs are determined. We express total domination polynomials of cubic Cayley graphs in terms of domination polynomials of cycles.

The next section investigates the relation between domination polynomials of some regular graphs and the total domination polynomials of their Cartesian product with K_2. We have shown that TD-Polynomial of $K_2 \Box K_n$ is the domination polynomial of an $(n-1)$-regular bipartite graph with $2n$ vertices.

In the fifth section, we determine the TD-Polynomial of Cartesian product of friendship graph with K_2. It is proved that TD-Polynomial $K_2 \Box F_n$ is the domination polynomial of the theta graph $\theta_{\underbrace{3,3,\ldots,3}_{(2n\ times)}}$.

The sixth section of chapter five is on the total domination polynomials of Cartesian product of certain classes of graphs with the cycle C_4. It is proved that for any bipartite graph G, the TD-Polynomial of $C_4 \Box G$ is square of domination polynomial of $K_2 \Box G$. In addition to this, the section provides the TD-Polynomials of Cartesian products of some more classes of graphs.

In **Chapter six**, we discuss total domination polynomials of splitting graph of order k of a graph G. In the second section, it is proved that TD-Polynomial of splitting graph of order k of a graph G can be expressed in terms of the TD-Polynomial of G. Moreover, we have obtained TD-Polynomial of splitting graph of order k of paths, cycles, complete graphs, complete bipartite graphs etc.

In the third section, we introduce a terminology of iterated splitting graph of order k of a graph G. The iterated splitting graph $S^i(G)$ of a graph G is defined as $S^i(G) = S^1(S^{i-1}(G))$, where $i = 2, 3, \ldots, k$ and $S^1(G)$ denotes the splitting graph $spl(G)$ of G. In the section, the TD-Polynomial of $S^k(G)$ is obtained in terms of TD-Polynomial of G. Further, TD-Polynomial of iterated splitting graphs of order k of paths, cycles, complete graphs, complete bipartite graphs etc. are determined.

In **Chapter seven**, we introduce the concepts of *global bipartite domination number* $\gamma_{gb}(G)$ and *global bipartite total domination number* $\gamma_{gbt}(G)$ of a connected bipartite graph.

Let G be a connected spanning sub graph of $K_{m,n}$. The bipartite complement or relative complement of G in $K_{m,n}$, denoted by $\widehat{G}$ is the graph $K_{m,n} - E(G)$. That is, $\widehat{G}$ can be obtained from the complete bipartite graph $K_{m,n}$ by "rubbing out" all the edges of G. A *global bipartite dominating set* (GBDS) of G is a set S of vertices of G such that S dominates G and $\widehat{G}$. The global bipartite domination number $\gamma_{gb}(G)$ of G is the minimum cardinality of a global bipartite dominating set of G. Some general properties satisfied by this concept are studied. We also determine the global bipartite domination number of certain classes of graphs. Connected graphs with global bipartite domination number $m + n$ or $m + n - 1$ are characterized. We prove that for any two positive integers a and b with $a < b$, there exists a graph G with $\gamma(G) = a$ and $\gamma_{gb}(G) = b$.

In the third section, the concept of global bipartite total domination is studied. For a connected spanning subgraph G of $K_{m,n}$, a total dominating set S of G is called a *global bipartite total dominating set* (GBDTS) of G if S dominates $\widehat{G}$. The global bipartite total domination number $\gamma_{gbt}(G)$ of G is the minimum cardinality of a global bipartite total dominating set of G. We characterize global bipartite total dominating sets among total dominating sets of a graph G. It is proved that for any two positive integers a and b with $a < b$, there exists a graph G with $\gamma_t(G) = a$ and $\gamma_{gbt}(G) = b$. Moreover, the graphs having global bipartite total domination number $m + n$ or $m + n - 1$ are characterized.

In **Chapter eight**, we introduce the concept of global bipartite domination polynomials of graphs. Let $\mathcal{D}_{gb}(G,i)$ be the family of global bipartite dominating sets of G with cardinality i and let $d_{gb}(G,i) = |\mathcal{D}_{gb}(G,i)|$. Then the *global bipartite domination polynomial* $D_{gb}(G,x)$ is defined as $D_{gb}(G,x) = \sum_{i=\gamma_{gb}(G)}^{|V(G)|} d_{gb}(G,i)x^i$.

In the second section, some properties of $D_{gb}(G,x)$ are discussed. Further, we obtain global bipartite domination polynomial of $K_{m,n}, K_{m,n} - e, B_{m,n}$ etc. The third section deals with global bipartite domination polynomial of paths. A characterization of GBD set of P_n is also provided. The prime focus of the section is on the relation between domination polynomial and global bipartite domination polynomial of paths. Section four deals with the properties of global bipartite domination polynomial of cycles. In the section, we obtain global bipartite domination polynomial of even cycles.

The concluding part of the thesis enlists some suggestions for further study and incorporates the list of publications and bibliography.

Chapter 1

Preliminaries

The chapter aims at listing the terminology and notation that we use in the thesis. Most of the terms used in this study belong to the standard graph theoretic terminology. Some of the terms will be introduced later. The prime source of the definition, terminology and notation is [11] Bondy, J. A. and Murty, U.S.R, Graph theory with applications, (1976).

This chapter comprises six sections. The first section focuses on the definitions and terminologies of Graph theory, which are integral in the discussion of the topics in the forthcoming chapters. The second section deals with various graph operations that come in handy in the subsequent chapters. Basic properties of hypergraphs are discussed in the third section. Some of the fundamental ideas and basic results in domination and total domination are incorporated in section five. In section six, basic definitions and properties of domination and total domination polynomials are mentioned. Moreover, some properties of vertex covering set are discussed. The chapter is summed up by the definition of vertex cover polynomial.

1.1 Basic definitions and terminologies

A (undirected) *graph* [40] $G = (V(G), E(G))$ consists of a nonempty set $V(G)$ and $E(G)$, a binary symmetric relation on $V(G)$. The sets $V(G)$ and $E(G)$ are called *vertex set* and *edge set* [11] of G respectively. If the graph G is clear from the context, we simply write $G = (V, E)$ instead of $G = (V(G), E(G))$ and if $e = \{u, v\}$, where $e \in E$ and $u, v \in V$, we simply write $e = uv$. An element of V is called a *vertex,* and an element of E is called an *edge* [40]. We draw graphs in such a way that each vertex is indicated by a point, and each edge by a line joining the points representing ends of the edge [11]. For the graph $G = (V, E)$, the number of vertices of G is called the *order* [11] of G, denoted by n and the number of edges is called the *size* [11] of G, denoted by m.

Let $e = uv$ be an edge of G. Then, we say that the two vertices u and v are *adjacent* [40] to each other and the edge e is *incident* with (incident to or incident at) u and v. The vertices u and v are said to be the end vertices of the edge e. The vertex v is called a *neighbor* of u. We write $u \sim v$ for 'u adjacent to v'. Two edges are said to be adjacent if they have a common vertex [11]. An edge with identical ends is called a *loop* [11]. Two or more edges with the same pair of ends are said to be *parallel* edges or multiple edges and graph having multiple edges is a *multigraph* [11].

A finite graph [11] is one in which both vertex set and edge set are finite. A graph having exactly one vertex and no edges is called a trivial graph [11] and all other graphs are called nontrivial graphs. A graph having no loops or multiple edges is called a *simple graph* [11]. Unless otherwise stated the graphs considered in this thesis are simple.

For a vertex v in a graph G, the *degree* [22] of v, denoted by deg v, is the number of edges incident with v. A vertex of degree one is called an *end vertex* [40] or a *pendant vertex* and a vertex adjacent to a pendant vertex is called a *support vertex* [40]. A *pendant edge* [22] is the edge incident with a pendant vertex. A vertex of degree zero is called an *isolated vertex* [22]. In a graph G, $\delta(G)$ and $\Delta(G)$ denotes the minimum and maximum degrees of vertices in G. A k-regular graph [11] is one with $\delta(G) = \Delta(G) = k$. A 3-regular graph is also known as *cubic graph* [11].

Let $H = (V(H), E(H)$ and $K = (V(K), E(K)$ be two graphs. We say that H and K are *isomorphic* [8], denoted by $H \cong K$, if there exist bijections $f\colon V(H) \to V(K)$ and $g\colon E(H) \to E(K)$ such that $g(uv) = f(u)f(v)$ for all u, v in $V(H)$. In other words, $e = uv$ is an edge of H if and only if $g(e) = f(u)f(v)$ is an edge of K.

A *complete graph* [8] is a simple graph in which every pair of distinct vertices are adjacent. A complete graph [11] having n vertices is denoted by K_n. A graph is *bipartite* [11] if its vertex set can be partitioned into two subsets, X and Y so that every edge has one end in X and other end in Y; such a partition (X, Y) is called a *bipartition* of the bipartite graph. A *complete bipartite graph* [11] is a bipartite graph such that each vertex of X is adjacent to all vertices of Y and vice versa. $K_{m,n}$ denotes a complete bipartite graph with $\mid X \mid= m$ and $\mid Y \mid= n$. The *complement* [8] of a simple graph G, denoted by $\overline{G}$, is graph with $V(\overline{G}) = V(G)$ and two vertices u and v are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A *walk* [11] in a graph G is an alternating sequence $W\colon v_0e_1v_1e_2v_2 \dots e_nv_n$ of vertices and edges beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i; v_0 is the origin and v_n is the terminus of W. We call W a v_0 - v_n walk. The *length* [11] of a walk is the number of edges in it. If all the edges in a walk are distinct, it is called a *trail* [11]. A *path* [11] is a walk in which all vertices are distinct. Usually, we leave out the edges while writing a path. A *cycle* [11] is a closed trail in which all the vertices are distinct. A cycle of length n is denoted by C_n and a path with n vertices is denoted by P_n. Note that P_n has length $(n-1)$ [11].

Let u and v are two vertices in a graph G. If there is a u - v path in G, we say that u and v are *connected.* A graph G is said to be connected [22] if every pair of verices of G are connected. A disconnected graph [22] is one which is not connected. The *distance* [22] between u and v, denoted by $d(u, v)$, is the length of the shortest u-v path in G.

A graph H is called a *subgraph* [11] of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, then G is said to be a *supergraph* [11] of H. An *induced subgraph* [13] H of a graph G is a subgraph of G such that two vertices of H are adjacent if and only if they are adjacent in G. In this case, if $V(H) = S$, we write $H = G[S]$ or $H = \langle S \rangle$. A subgraph H of G is a *spanning subgraph* [19] of G, if $V(H) = V(G)$.

A graph is *acyclic* [22] if it has no cycles. A *tree* [22] is a connected acyclic graph. A *spanning tree* [11] of G is a spanning subgraph of G that is a tree.

If F is any set of edges in G, then $G-F$ is the graph $(V(G), E(G)-F)$ [51]. If $F = \{e\}$, then $G-F$ is written as $G-e$ [51]. For any subset S of $V(G)$, the graph $G-S$ is obtained from G by deleting all the vertices in S [51]. If

$S = \{v\}$, $G - S$ is written as $G - v$ [51]. A maximal connected subgraph of a graph G is called a *component* of G [11].

1.2 Operations on graphs

The *union* [50] of two graphs G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* [21] of two graphs G_1 and G_2 denoted by $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \colon u \in V(G_1) \text{ and } v \in V(G_2)\}$. The *corona* [21] of two graphs G_1 and G_2 denoted by $G_1 \circ G_2$ is the graph formed from one copy of G_1 and $\mid V(G_1) \mid$ copies of G_2, such that the i^{th} vertex of the copy of G_1 is adjacent to every vertex in the i^{th} copy of G_2. The *Cartesian product* [29] $G_1 \Box G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1v_2 \in E(G_2)$ and $u_1 = u_2$.

1.3 Hypergraphs

A *hypergraph* [9] H is a pair $(V, \mathcal{E})$, where V is finite non-empty set called the set of vertices and $\mathcal{E}$ is a collection of nonempty subsets of V called *hyperedges* or *edges*. That is, $\mathcal{E}$ is a subset of $\mathscr{P}(V) \setminus \{\phi\}$, where $\mathscr{P}(V)$ is the power set of V. In drawing hypergraphs, each vertex is drawn as a point in the plane. An edge E_1 with $\mid E_1 \mid > 2$, is drawn as a curve encircling all the vertices of E_1 [9]. An edge E_1 with $\mid E_1 \mid = 2$, is drawn as a curve connecting its two vertices and an edge E_1 with $\mid E_1 \mid= 1$, is drawn as a loop as in a graph [9].

Two vertices in a hypergraph are adjacent [9] if there is a hyperedge which contains both vertices. Two vertices x and y of a hypergraph are connected [27] if there is a sequence of vertices $x = v_0, v_1, \ldots, v_k = y$ such that v_{i-1} is adjacent to v_i for $i = 1, 2, \ldots, k$. A connected hypergraph [27] is a hypergraph in which every pair of vertices are connected. A maximal connected sub hypergraph of a hypergraph is called a component. Two hyperedges in a hypergraph are incident [9] if their intersection is nonempty. If $\mid E_i \mid= 2$ for all i, and if the hypergraph H is simple, then H is a simple graph without isolated vertices. A k-uniform hypergraph [9] or a k-hypergraph is a hypergraph in which every edge consists of k vertices. Since every simple graph is a 2-uniform hypergraph, graphs are special hypergraphs [49].

1.4 Open neighbourhood hypergraph

For a graph G be with vertex set V and edge set E, the *open neighborhood* [27] of a vertex u of G is $N_G(u) = \{v \in V : uv \in E\}$ and its *closed neighborhood* is the set $N_G[u] = N_G(u) \cup \{u\}$. The open neighborhood of a set [27] $S \subseteq V$ is the set $N_G(S) = \cup_{u \in S} N_G(u)$ and its closed neighborhood is the set $N_G[S] = N_G(S) \cup S$. If there is no ambiguity, we simply write $N(u), N[u], N(S)$ and $N[S]$ instead of $N_G(u), N_G[u], N_G(S)$ and $N_G[S]$, respectively.

The *open neighborhood hypergraph* [27] of a graph G, denoted by $ONH(G)$ or H_G, is the hypergraph with vertex set $V(G)$ and edge set $\{N_G(u) \colon u \in V(G)\}$ consisting of the open neighborhoods of vertices of V in G.

1.5 Domination in graphs

This section discusses some of the fundamental ideas and basic results in domination in graphs.

One of the fastest growing areas in Graph theory is the study of dominating sets and related properties. The study of domination in graphs began in 1960, when the problem of queen domination in an $n \times n$ chess board was studied [23].

For a graph G, a set $S \subseteq V(G)$ is called a *dominating set* [23] of G if every vertex $u \in V(G)$ is either an element of S or is adjacent to an element of S. Alternatively, we say that $S \subseteq V(G)$ is a dominating set of G if every element in $V \setminus S$ is adjacent to some element in S. Equivalently, $N[S] = V$. If S is a dominating set of a graph, then every superset of S is also a dominating set. On the other hand, not every subset of S is necessarily a dominating set. A dominating set S of G is a *minimal dominating set* [23] if no proper subset of S is a dominating set. The *domination number* [23] of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G and a γ-set is a dominating set with cardinality $\gamma(G)$.

After the introduction of the concept of domination, several types of dominating sets were introduced and studied in detail. An excellent treatment of this topic is available in [23] and [24].

The concept of *total domination* [16] was introduced by Cockayne, Dawes, and Hedetniemi . For a graph $G = (V, E)$, with no isolated vertices, a set $S \subseteq V$ is called a *total dominating set* [27] or *TD-set* if every vertex of G is adjacent to a vertex in S. A total dominating set S is said to be *minimal* [27], if no proper subset of S is a total dominating set. The *total domination number* [27] of a graph G, denoted by $\gamma_t(G)$, is defined as the minimum cardinality of a total dominating set of G. If S is a total dominating set of G with $|S| = \gamma_t(G)$, then S is called a γ_t-set of G [27]. Obviously, a TD-set is a dominating set and a dominating set S is a TD-set if the induced subgraph $\langle S \rangle$, has no isolated vertices.

In the next section, we discuss some graph polynomials which are essential for the study.

1.6 Graph polynomials

The concept of *domination polynomial* of a graph was introduced by S. Alikhani in 2009. For a graph G, let $\mathcal{D}(G, i)$ be the family of dominating sets with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$ [5]. If $\gamma(G)$ is the domination number of G, then the *domination polynomial* [5] $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$.

Theorem 1.6.1. *(see [1]) For every natural number n,*

$$(i)\ D(K_n, x) = (1 + x)^n - 1.$$

$$(ii)\ D(K_{1,n}, x) = x^n + x(1 + x)^n.$$

Theorem 1.6.2. *(see [1]) If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then $D(G, x) = D(G_1, x)D(G_2, x) \ldots D(G_m, x)$.*

In analogue to the domination polynomial, S. Sanalkumar introduced the concept of *total domination polynomial* of a graph. For a graph G, let $\mathcal{D}_t(G, i)$ be the family of total dominating sets with cardinality i and let $d_t(G,i) = |\mathcal{D}_t(G,i)|$ [3]. If $\gamma_t(G)$ is the total domination number of G, then the *total domination polynomial* [31] or *TD-Polynomial* of G, denoted by $D_t(G,x)$ is defined as $D_t(G,x) = \sum_{i=\gamma_t(G)}^{|V(G)|} d_t(G,i)x^i$.

A *vertex cover* [46] or transversal of a graph G is a set S of vertices of G such that each edge in G has at least one end in S. A vertex covering set with k vertices is called a k-vertex cover. A *minimum vertex cover* is a vertex cover having the smallest possible number of vertices for a given graph [41]. The number of vertices of a minimum vertex cover of a graph G is known as the *vertex cover number* and is denoted as $\tau(G)$ [50].

Let $\mathcal{C}(G,i)$ be the family of vertex covering sets of a graph G with cardinality i and let $c(G,i) = |\mathcal{C}(G,i)|$ [18]. The polynomial $\mathcal{C}(G,x) = \sum_{i=\tau(G)}^{|V(G)|} c(G,i)x^i$ is defined as *vertex cover polynomial* of G [18].

Chapter 2

TD-Polynomials- A New Approach

2.1 Introduction

This chapter deals with the relation between total domination polynomials and vertex cover polynomials. For a graph $G = (V, E)$, the *open neighborhood hypergraph* of G, denoted by $ONH(G)$, is the hypergraph with vertex set V and edge set $\{N_G(x)|x \in V\}$. A *vertex cover* in $ONH(G)$ is a set of vertices intersecting every edge of $ONH(G)$, which is equivalent to a *total dominating set* in G. Using the interplay between total dominating sets and vertex cover in hypergraphs, we determine the total domination polynomial of some classes of graphs. Here we need the following.

Definition 2.1.1. *(see [44]) A graph G in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called the root of G. Let G be a rooted graph. The graph $G^{(n)}$ obtained by identifying the roots of n copies*

[1] A part of this chapter has been published in *Global Journal of Pure and Applied Mathematics*. Volume 13, Number 10, 2017, Pages 7315-7319.

of G is called a one point union of the n copies of G.

Definition 2.1.2. *A one point union $C_n^{(k)}$ of k copies of C_n is the graph obtained by taking v as a common vertex such that any two cycles C_n^i and C_n^j $(i \neq j)$ are edge disjoint and do not have any vertex in common except v.*

Definition 2.1.3. *(see [38]) An n-gon book of k pages denoted by $C_n^{2(k)}$ is the graph obtained when k copies of the cycle C_n share a common edge.*

Definition 2.1.4. *(see [38]) Given k natural numbers $n_1, n_2, \ldots, n_k$, the generalized theta graph $\theta(n_1, n_2, \ldots, n_k)$ is obtained by connecting two vertices u and v by k parallel paths of length $n_1 - 1, n_2 - 1, \ldots, n_k - 1$.*

Definition 2.1.5. *Let $P_{n_1+1}, P_{n_2+1}, \ldots, P_{n_k+1}$ be k paths. For $i = 1, 2, \ldots, k$, let a_i be a pendant vertex of the path P_{n_i}. Then the tree $T_{n_1,n_2,\ldots,n_k}$ is obtained by identifying the vertices a_i for every i.*

Theorem 2.1.6. *(see [26]) The ONH of a connected bipartite graph consists of two components, while the ONH of a connected graph that is not bipartite is connected.*

Theorem 2.1.7. *(see [27]) If G is a graph with no isolated vertices and H_G is the ONH of G, then $\gamma_t(G) = \tau(H_G)$.*

Theorem 2.1.8. *(see [18]) Let G be a graph and $L = \{u \in V(G) | \ uu \in E(G)\}$. Then $\mathcal{C}(G, x) = x^{|L|}\mathcal{C}(G - L, x)$.*

Theorem 2.1.9. *(see [18]) Let G be a graph with $| \ V(G) \ | \geq 2$. Let $u \in V(G)$ and $d = |N_G(u)|$. If G has no loops at $N_G[u]$, then $\mathcal{C}(G, x) = x\mathcal{C}(G - u, x) + x^d\mathcal{C}(G - u - N_G(u), x)$.*

Lemma 2.1.10. *(see [18]) For $n \geq 5$, we have*

$$\mathcal{C}(C_n, x) = x\mathcal{C}(P_{n-1}, x) + x^2\mathcal{C}(P_{n-3}, x).$$

Theorem 2.1.11. *(see [18]) Let $G = G_1 \cup G_2 \ldots \cup G_n$ be the union of n graphs $G_1, G_2, \ldots, G_n$. Then $\mathcal{C}(G, x) = \mathcal{C}(G_1, x)\mathcal{C}(G_2, x)\ldots\mathcal{C}(G_n, x)$.*

Theorem 2.1.12. *(see [18]) For the path graph P_n, where $n \geq 2$, we have*

$$\mathcal{C}(P_n, x) = \sum_{i=0}^{n} \binom{i+1}{n-i} x^i.$$

Theorem 2.1.13. *(see [18]) For the cycle graph C_n, where $n \geq 3$, we have*

$$\mathcal{C}(C_n, x) = \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i.$$

Theorem 2.1.14. *The total domination polynomial of a connected bipartite graph G is the product of the vertex cover polynomials of the two components of its open neighborhood hypergraph, H_G, while the total domination polynomial of a connected graph that is not bipartite is the vertex cover polynomial of H_G.*

Proof. A set S of vertices of G is a total dominating set if and only if it is a vertex covering set of the open neighborhood hypergraph H_G of G. Therefore, if G is not bipartite, the result follows. If G is bipartite, its open neighborhood hypergraph, H_G has two components and its vertex cover polynomial is the product of the vertex cover polynomials of its components. Thus the proof follows from the definitions of total domination set of G and vertex cover polynomial of H_G. □

2.2 TD-Polynomials of paths and cycles

We observe that using the interplay between total dominating sets in graphs and transversals in hypergraphs, several results on total domination polynomials in graphs can be obtained that appear very difficult to obtain using purely graph theoretic techniques. In this section, we study the relation between total domination polynomials and vertex cover polynomials of paths and cycles.
We need the following to prove the main results of this section.

Lemma 2.2.1. *Let $P_n^{'}$ be the graph shown in figure 2.1. Then,*

$$\mathcal{C}(P_n^{'}, x) = x\mathcal{C}(P_{n-1}, x) = x\sum_{i=0}^{n-1}\binom{i+1}{n-(i+1)}x^i.$$

Proof. The proof follows immediately from Theorem 2.1.8 and Theorem 2.1.12. □

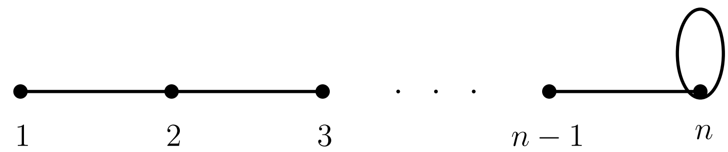

Figure 2.1: The Graph $P_n^{'}$

Lemma 2.2.2. *Let $P_n^{''}$ be the graph shown in Figure 2.2. Then,*

$$\mathcal{C}(P_n^{''}, x) = x^2\mathcal{C}(P_{n-2}, x) = x^2\sum_{i=0}^{n-2}\binom{i+1}{n-(i+2)}x^i.$$

Proof. The proof follows from Theorem 2.1.8 and Theorem 2.1.12. □

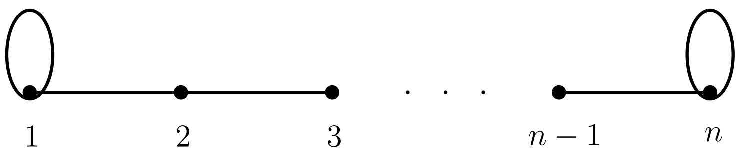

Figure 2.2: The Graph $P_n^{''}$

Theorem 2.2.3. *For the path graph P_n, where $n > 2$, we have*

$$\mathcal{C}(P_n, x) = x\left[\mathcal{C}(P_{n-1}, x) + \mathcal{C}(P_{n-2}, x)\right]$$

Proof. Let $(1, 2, \ldots, n)$ be the path P_n and S be a vertex covering set of P_n. If $n \in S$, then S is a vertex covering set of the graph $P_n^{'}$ shown in figure 2.1. If $n \notin S$, then the vertex $n-1 \in S$. Therefore, S is a vertex covering set of $P_{n-1}^{'}$. Conversely, any vertex covering set of $P_n^{'}$ or $P_{n-1}^{'}$ is a vertex covering set of P_n. This completes the proof. □

Theorem 2.2.4. *For $n \geq 1$, $D_t(P_{2n}, x) = \left[\mathcal{C}(P_n^{'}, x)\right]^2$.*

Proof. Let $(1, 2, \ldots, 2n)$ be the path P_{2n}. Since P_{2n} is bipartite, the open neighborhood hypergraph of $P_{2n}, ONH(P_{2n})$ has two components say G_1 and G_2.

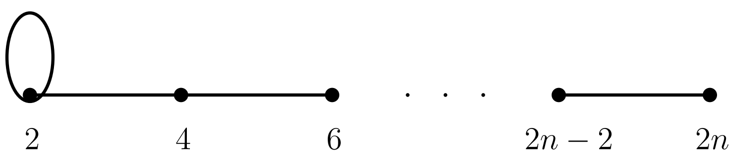

Figure 2.3: The Graph G_2

The edge sets of G_1 and G_2 are $E(G_1) = \{xy\colon x = 2i-1 \text{ and } y = 2i+1 \text{ where } 1 \leq i \leq n-1\} \cup \{\{2n-1\}\}$ and $E(G_2) = \{xy\colon x = 2i \text{ and } y =$

$2i+2$ where $1 \leq i \leq n\} \cup \{\{22\}\}$. Clearly G_1 is isomorphic to G_2. Using the terminology in [9], the graph G_2 can be drawn as shown in figure 2.3. Since G_2 is isomorphic to $P_n^{'}$ and G_1 is isomorphic to G_2, the proof follows from Theorem 2.1.8 and 2.1.11. □

Theorem 2.2.5. *For $n \geq 1$, the total domination polynomial of path P_{2n} is,*

$$D_t(P_{2n}, x) = x^2 \left[\sum_{i=0}^{n-1} \binom{i+1}{n-(i+1)} x^i \right]^2.$$

Proof. The proof follows from Lemma 2.2.1 and Theorem 2.2.4. □

Theorem 2.2.6. *For $n \geq 1$, $D_t(P_{2n+1}, x) = \mathcal{C}(P_{n+1}, x)\mathcal{C}(P_n^{''}, x)$.*

Proof. Let $(1, 2, 3, \ldots, 2n\text{-}1, 2n, 2n\text{+}1)$ be the path P_{2n+1}. Then the open neighborhood hypergraph of P_{2n+1} has two components G_1 and G_2 with edge sets $E(G_1) = \{xy \colon x = 2i-1 \text{ and } y = 2i+1, \text{ where } 1 \leq i \leq n\}$ and $E(G_2) = \{xy \colon x = 2i \text{ and } y = 2i+2, \text{ where } 1 \leq i \leq n-1\} \cup \{22, 2n2n\}$. The graph G_2 can be drawn as shown in figure 2.4.

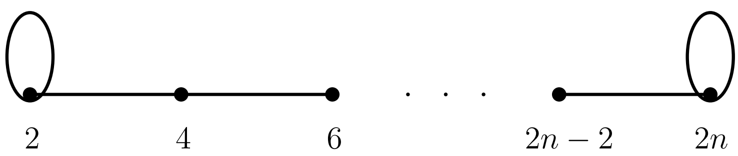

Figure 2.4: The Graph G_2

Let P_{n+1} be the path $(1, 3, 5, \ldots, 2n\text{-}1, 2n\text{+}1)$ Since $E(G_1) = E(P_{n+1})$, a set S is a vertex cover of G_1 if and only if S is a vertex cover of P_{n+1}. Since G_2 is isomorphic to $P_n^{''}$, by Theorem 2.1.14 and Lemma 2.2.2, we have $D_t(P_{2n+1}, x) = \mathcal{C}(G_1, x)\mathcal{C}(G_2, x) = \mathcal{C}(P_{n+1}, x)\mathcal{C}(P_n^{''}, x)$. This completes the proof. □

Theorem 2.2.7. *The total domination polynomial of the path P_{2n+1} is,*

$$D_t(P_{2n+1}, x) = x^2 \left[\sum_{i=0}^{n+1} \binom{i+1}{n+1-i} x^i\right] \left[\sum_{i=0}^{n-2} \binom{i+1}{n-(2+i)} x^i\right]$$

Proof. The proof follows immediately from 2.1.12, 2.1.14, 2.2.2 and 2.2.6. □

Theorem 2.2.8. *For $n \geq 3$, we have, $D_t(C_{2n}, x) = [\mathcal{C}(C_n, x)]^2$.*

Proof. Let $(1, 2, \ldots, 2n, 1)$ be the cycle C_{2n}. Since C_{2n} is bipartite, its open neighborhood hypergraph has two components. It can be observed that the components are cycles, say C' and C'', where $C' = (2, 4, 6, \ldots, 2n, 2)$ and $C'' = (1, 3, 5, \ldots, 2n\text{-}1, 1)$. Since the cycles C' and C'' are isomorphic to the cycle C_n, the proof follows from Theorem 2.1.11. □

Theorem 2.2.9. $D_t(C_{2n}, x) = \left[\sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i\right]^2.$

Proof. The proof follows from Theorems 2.1.13 and 2.2.8. □

Theorem 2.2.10. *If n is an odd positive integer, then*

$$D_t(C_n, x) = \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i.$$

Proof. Since the open neighborhood hypergraph of a cycle of odd length is isomorphic to itself, the proof follows from Theorem 2.1.13. □

Theorem 2.2.11. $D_t(C_{2n}, x) = [x\mathcal{C}(P_{n-1}, x) + x^2\mathcal{C}(P_{n-3}, x)]^2.$

Proof. From Theorem 2.2.8 we have, $D_t(C_{2n}, x) = [\mathcal{C}(C_n, x)]^2$. Then the proof follows from Lemma 2.1.10. □

Theorem 2.2.12. *If n is an odd positive integer, then*

$$D_t(C_n, x) = \left[x\mathcal{C}(P_{n-1}, x) + x^2\mathcal{C}(P_{n-3}, x)\right].$$

Proof. Here, $D_t(C_n, x) = \mathcal{C}(C_n, x)$. Then by Lemma 2.1.10, the proof follows. □

2.3 TD-Polynomials of some graph classes

In this section, we find the total domination polynomial of some classes of graphs using vertex cover polynomials of paths and cycles. Let G be a graph and A be a subset of the set of vertices of G. We define the following classes of vertex cover polynomials to obtain the main results of this chapter.

Definition 2.3.1. *Let $C^A(G, i) = \{S \subseteq V(G) \colon S \in C(G, i) \text{ and } S \cap A \neq \phi\}$. Then, the polynomial $\mathcal{C}^A(G, x)$ is defined as $\mathcal{C}^A(G, x) = \sum_{i=1}^{|V(G)|} c^A(G, i)x^i$, where $c^A(G, i) = |C^A(G, i)|$.*

Definition 2.3.2. *Let $C^{A^*}(G, i) = \{S \subseteq V(G) \colon S \in C(G, i) \text{ and } S \cap A = A\}$. Then, the polynomial $\mathcal{C}^{A^*}(G, x)$ is defined as $\mathcal{C}^{A^*}(G, x) = \sum_{i=1}^{|V(G)|} c^{A^*}(G, i)x^i$, where $c^{A^*}(G, i) = |C^{A^*}(G, i)|$.*

Definition 2.3.3. *Let $C_A(G, i) = \{S \subseteq V(G) \colon S \in C(G, i) \text{ and } S \cap A = \phi\}$. Then, the polynomial $\mathcal{C}_A(G, x)$ is defined as $\mathcal{C}_A(G, x) = \sum_{i=1}^{|V(G)|} c_A(G, i)x^i$, where $c_A(G, i) = |C_A(G, i)|$.*

Note 2.3.4. *If $A = \{a\}$, then we write $\mathcal{C}^a(G, x)$ and $\mathcal{C}_a(G, x)$ instead of $\mathcal{C}^A(G, x)$ and $\mathcal{C}_A(G, x)$ respectively.*

Lemma 2.3.5. *Let u be a vertex of degree d in G. Let G has no loops at $A \cup N_G[u]$ and $A \cap N_G[u] = \phi$. Then $\mathcal{C}_A(G,x) = x\mathcal{C}_A(G-u,x) + x^d\mathcal{C}_A(G-N_G[u],x)$.*

Proof. Let $S \in C_A(G,i)$. Then either $u \in S$ or $u \notin S$. Now, $u \in S$ if and only if $S \setminus \{u\} \in C_A(G-u,i-1)$. If $u \notin S$, then by definition, $N_G(u) \subseteq S$ and $S \setminus N_G(u)$ is an $(i-d)$-vertex cover of $G - N_G[u]$. Conversely, if $S \in C_A(G - N_G[u], i-d)$, then $u \notin S$ and $S \cup N_G[u] \in C_A(G,i)$. Therefore, $c_A(G,i) = c_A(G-u,i-1) + c_A(G - N_G[x], i-d)$. So,

$$
\begin{aligned}
\mathcal{C}_A(G,x) &= \sum_{i=1}^{|V(G)|} c_A(G,i)x^i \\
&= \sum_{i=1}^{|V(G)|} \left[c_A(G-u,i-1) + c_A(G-N_G[x],i-d)\right] x^i \\
&= \sum_{i=1}^{|V(G)|} c_A(G-u,i-1)x^i + \sum_{i=1}^{|V(G)|} c_A(G-N_G[x],i-d)x^i \\
&= x\mathcal{C}_A(G-u,x) + x^d\mathcal{C}_A(G-N_G[u],x).
\end{aligned}
$$

Thus the proof follows. □

Lemma 2.3.6. *If the path $P_n = (1,2,\ldots,n)$, then*

(*i*) $\mathcal{C}^{\{1\}}(P_n,x) = x\ \mathcal{C}(P_{n-1},x)$,

(*ii*) $\mathcal{C}_{\{1\}}(P_n,x) = x\ \mathcal{C}(P_{n-2},x)$,

(*iii*) $\mathcal{C}^{\{1,n\}^*}(P_n,x) = x^2\ \mathcal{C}(P_{n-2},x)$,

(*iv*) $\mathcal{C}_{\{1,n\}}(P_n,x) = x^2\ \mathcal{C}(P_{n-4},x)$.

Proof. **(i)** Let S be a subset of vertices of P_n. It is observed that S is a vertex covering set of P_n containing the vertex 1 if and only if S is a vertex covering set of the graph H shown in figure 2.5. Therefore, the proof follows from Theorem 2.1.8.

Figure 2.5: The graph H.

(ii) If S is a vertex covering set of P_n and $S \cap \{1\} = \phi$, then $2 \in S$. So, S is a vertex covering set of the graph K shown in figure 2.6. Therefore, from Theorem 2.1.8 the result follows.

Figure 2.6: The graph K.

(iii) If S is a vertex covering set of P_n containing the vertices 1 and n, then S is a vertex covering set of the graph P_n'' shown in figure 2.2. Therefore, the proof follows from theorem 2.1.8.

(iv) Let S be a vertex covering set of P_n such that $S \cap \{1, n\} = \phi$, then S is a vertex covering set of the graph K_1 shown in figure 2.7.

Therefore, from Theorem 2.1.8, the proof follows. □

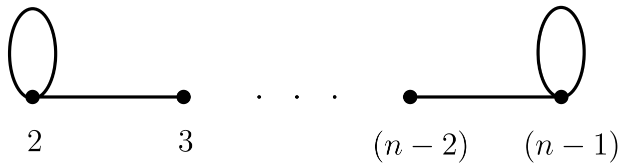

Figure 2.7: The graph K_1

Next, we find the total domination polynomial of the tree T_{n_1,n_2,n_3}.

Theorem 2.3.7. *If n_1, n_2, n_3 are even and T_1, T_2 are the components of the open neighborhood hypergraph of the tree T_{n_1,n_2,n_3}, then*

$$\begin{aligned} D_t(T_{n_1,n_2,n_3}, x) &= \mathcal{C}(T_1, x)\mathcal{C}(T_2, x), \ \textit{where} \\ \mathcal{C}(T_1, x) &= x\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}}, x) + x^3\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}-1}, x) \ \textit{and} \\ \mathcal{C}(T_2, x) &= x^3\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}-1}, x) - x^6\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}-3}, x). \end{aligned}$$

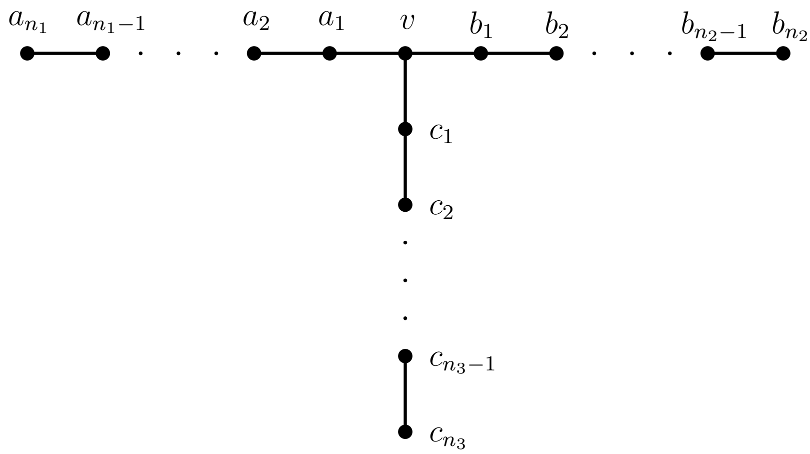

Figure 2.8: T_{n_1,n_2,n_3}.

Proof. Let $X = \{x_i : i \text{ is odd}\}$ and $Y = \{y_j : j \text{ is even}\} \cup \{v\}$ be the partite sets of

T_{n_1,n_2,n_3}. Let T_1 and T_2 be the components of the open neighborhood hypergraph of T_{n_1,n_2,n_3}, such that $E(T_1) = \{N(x)\colon x \in X\}$ and $E(T_2) = \{N(y)\colon y \in Y\}$. Then T_1 can be represented as shown in figure 2.9.

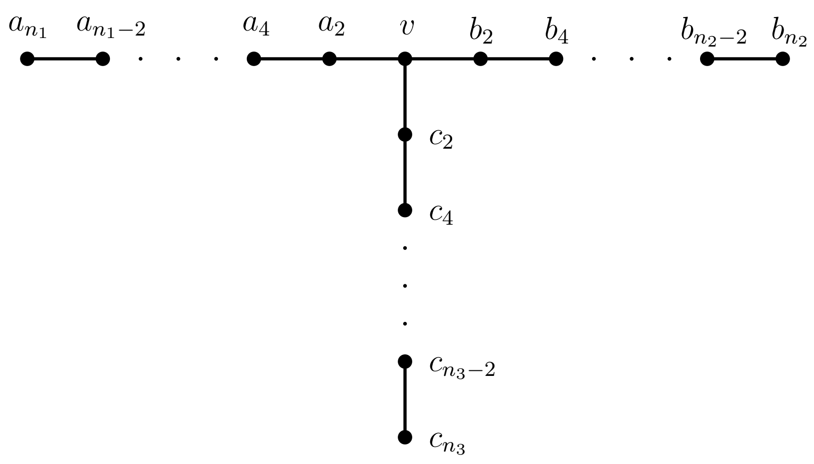

Figure 2.9: The graph T_1.

From Theorem 2.1.9, we have,

$$\begin{aligned}
\mathcal{C}(T_1, x) &= x\mathcal{C}(T_1 - v, x) + x^3\mathcal{C}(T_1 - v - \{a_2, b_2, c_2\}, x) \\
&= x\mathcal{C}(P_{\frac{n_1}{2}}, x)\mathcal{C}(P_{\frac{n_2}{2}}, x)\mathcal{C}(P_{\frac{n_3}{2}}, x) \\
&+ x^3\mathcal{C}(P_{\frac{n_1}{2}-1}, x)\mathcal{C}(P_{\frac{n_2}{2}-1}, x)\mathcal{C}(P_{\frac{n_3}{2}-1}, x) \\
&= x\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}}, x) + x^3\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}-1}, x).
\end{aligned}$$

Next, we find the vertex cover polynomial of T_2. It can be observed that $E(T_2) = \{a_1, b_1, c_1\} \cup E(T_a) \cup E(T_b) \cup E(T_c)$, where the graphs T_a, T_b and T_c are shown in figure 2.10. Let $A = \{a_1, b_1, c_1\}$. Then a set S is vertex covering set of T_2 if and only if $S \cap A \neq \phi$ and S is a vertex covering set of $T_a \cup T_b \cup T_c$. In other words

$\mathcal{C}(T_2, x) = \mathcal{C}^A(T_a \cup T_b \cup T_c, x)$. Therefore, from Theorem 2.1.11 and Lemma 2.3.6 we have,

$$\begin{aligned}
\mathcal{C}(T_2, x) &= \mathcal{C}^A(T_a \cup T_b \cup T_c, x) \\
&= \mathcal{C}(T_a \cup T_b \cup T_c, x) - \mathcal{C}_A(T_a \cup T_b \cup T_c, x) \\
&= \mathcal{C}(T_a, x)\mathcal{C}(T_b, x)\mathcal{C}(T_c, x) - \mathcal{C}_{a_1}(T_a, x)\mathcal{C}_{b_1}(T_b, x)\mathcal{C}_{c_1}(T_c, x) \\
&= x^3\mathcal{C}(P_{\frac{n_1}{2}-1}, x)\mathcal{C}(P_{\frac{n_2}{2}-1}, x)\mathcal{C}(P_{\frac{n_3}{2}-1}, x) \\
&- x^6\mathcal{C}(P_{\frac{n_1}{2}-3}, x)\mathcal{C}(P_{\frac{n_2}{2}-3}, x)\mathcal{C}(P_{\frac{n_3}{2}-3}, x) \\
&= x^3 \prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i}{2}-1}, x) - x^6 \prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i}{2}-3}, x).
\end{aligned}$$

This completes the proof. □

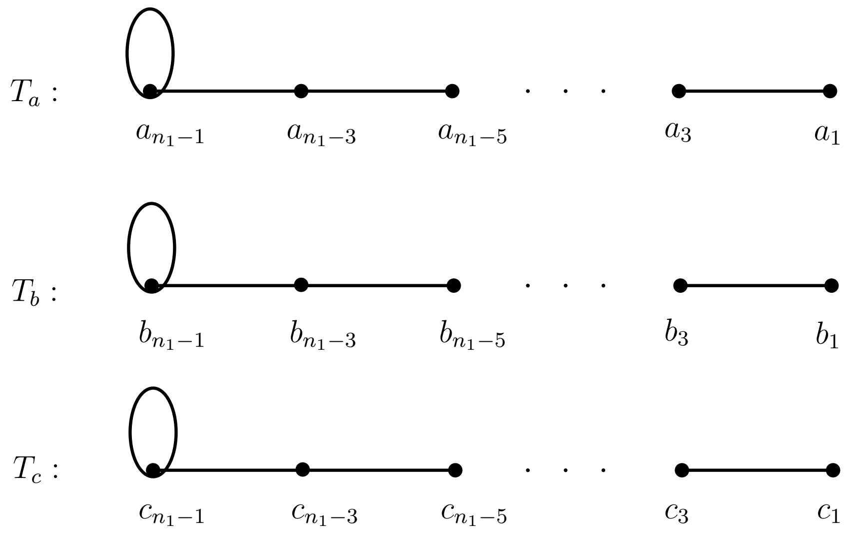

Figure 2.10: The Graphs T_a, T_b and T_c.

Corollary 2.3.8. *If $n_1 = n_2 = n_3 = 2n$, then*

$D_t(T_{n_1,n_2,n_3}, x) = x^4 \left[\mathcal{C}(P_n, x)\mathcal{C}(P_{n-1}, x)\right]^3 + x^7 \left[\mathcal{C}(P_n, x)\mathcal{C}(P_{n-3}, x)\right]^3 + x^6 \left[\mathcal{C}(P_n, x)\right]^6$

$- x^9 \left[\mathcal{C}(P_{n-1},x)\mathcal{C}(P_{n-3},x)\right]^3.$

Proof. The proof follows from Theorem 2.3.7. □

Theorem 2.3.9. *If n_1, n_2, n_3 are odd and T_1, T_2 are the components of the open neighborhood hypergraph of the tree T_{n_1,n_2,n_3}, then*

$$\begin{aligned}
D_t(T_{n_1,n_2,n_3},x) &= \mathcal{C}(T_1,x)\mathcal{C}(T_2,x), \text{ where} \\
\mathcal{C}(T_1,x) &= x^4\left[\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-3}{2}},x) + x^2\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-5}{2}},x)\right] \text{ and} \\
\mathcal{C}(T_2,x) &= \prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i+1}{2}},x) - x^3\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-3}{2}},x).
\end{aligned}$$

Proof. Let $X = \{x_i : i \text{ is odd}\}$ and $Y = \{y_j : j \text{ is even}\} \cup \{v\}$ be the bipartition of T_{n_1,n_2,n_3}. Let T_1 and T_2 be the components of the open neighborhood hypergraph of T_{n_1,n_2,n_3}, such that $E(T_1) = \{N(x) : x \in X\}$ and $E(T_2) = \{N(y) : y \in Y\}$. Then proceeding as in Theorem 2.3.7, we can represent T_1 as shown in figure 2.11. Let $T_1^{*} = T_1 - \{a_{n_1-1}, b_{n_2-1}, c_{n_3-1}\}$, $T_1^{**} = T_1 - \{v, a_{n_1-1}, b_{n_2-1}, c_{n_3-1}\}$, and $T_1^{***} = T_1 - \{v, a_2, b_2, c_2, a_{n_1-1}, b_{n_2-1}, c_{n_3-1}\}$. Then, from Theorem 2.1.8 and 2.1.9, we get,

$$\begin{aligned}
\mathcal{C}(T_1,x) &= x^3\mathcal{C}\left(T_1^{*},x\right) \\
&= x^3\left[x\mathcal{C}\left(T_1^{**},x\right) + x^3\mathcal{C}\left(T_1^{***},x\right)\right] \\
&= x^4\left[\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-1}{2}-1},x) + x^2\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-1}{2}-2},x)\right] \\
&= x^4\left[\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-3}{2}},x) + x^2\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-5}{2}},x)\right].
\end{aligned}$$

Let $P_{\frac{n_1+1}{2}} = (a_1, a_3, a_5, \ldots, a_{n_1-2}, a_{n_1})$, $P_{\frac{n_2+1}{2}} = (b_1, b_3, b_5, \ldots, b_{n_2-2}, b_{n_2})$ and

$P_{\frac{n_3+1}{2}} = (c_1, c_3, c_5, \ldots, c_{n_3-2}, c_{n_3})$ be three paths. Then the edge set of the graph T_2 is $E(T_2) = \{a_1, b_1, c_1\} \cup E(P_{\frac{n_1+1}{2}}) \cup E(P_{\frac{n_2+1}{2}}) \cup E(P_{\frac{n_3+1}{2}})$.

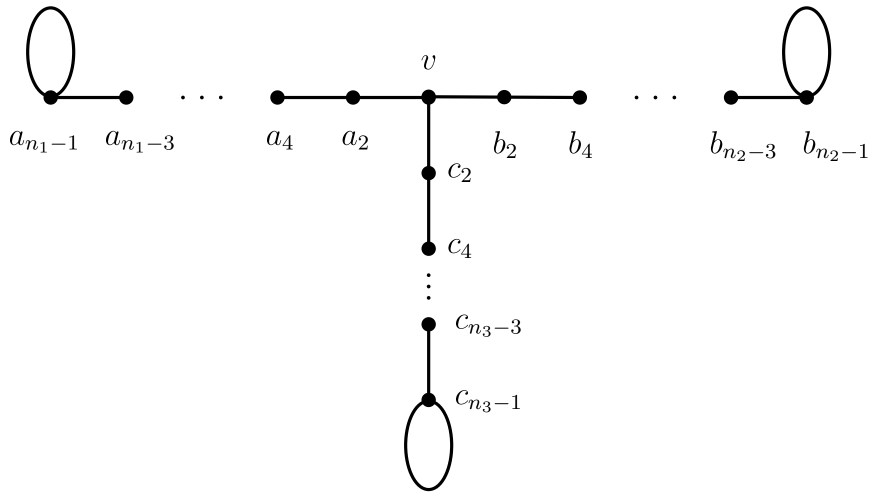

Figure 2.11: The Graph T_1.

Let $A = \{a_1, b_1, c_1\}$. Then a set S is vertex covering set of T_2 if and only if $S \cap A \neq \phi$ and S is a vertex covering set of $P_{\frac{n_1+1}{2}} \cup P_{\frac{n_2+1}{2}} \cup P_{\frac{n_3+1}{2}}$. Therefore, we need to calculate the polynomial $\mathcal{C}^A(P_{\frac{n_1+1}{2}} \cup P_{\frac{n_2+1}{2}} \cup P_{\frac{n_3+1}{2}}, x)$. That is $\mathcal{C}(T_2, x) = \mathcal{C}^A(P_{\frac{n_1+1}{2}} \cup P_{\frac{n_2+1}{2}} \cup P_{\frac{n_3+1}{2}}, x)$. Therefore, from Theorem 2.1.11 and Lemma 2.3.6 we have,

$$\begin{aligned}
\mathcal{C}(T_2, x) &= \mathcal{C}^A(P_{\frac{n_1+1}{2}} \cup P_{\frac{n_2+1}{2}} \cup P_{\frac{n_3+1}{2}}, x) \\
&= \mathcal{C}(P_{\frac{n_1+1}{2}} \cup P_{\frac{n_2+1}{2}} \cup P_{\frac{n_3+1}{2}}, x) - \mathcal{C}_A(P_{\frac{n_1+1}{2}} \cup P_{\frac{n_2+1}{2}} \cup P_{\frac{n_3+1}{2}}, x) \\
&= \mathcal{C}(P_{\frac{n_1+1}{2}}, x)\mathcal{C}(P_{\frac{n_2+1}{2}}, x)\mathcal{C}(P_{\frac{n_3+1}{2}}, x) \\
&- \mathcal{C}_{a_1}(P_{\frac{n_1+1}{2}}, x)\mathcal{C}_{b_1}(P_{\frac{n_2+1}{2}}, x)\mathcal{C}_{c_1}(P_{\frac{n_3+1}{2}}, x) \\
&= \prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i+1}{2}}, x) - x^3 \prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i-3}{2}}, x).
\end{aligned}$$

Thus the result follows. □

Lemma 2.3.10. *If $d \geq 3$ is a positive integer, the vertex cover polynomial of the tree* $T\underbrace{n, n, \ldots, n}_{d\ times}$, *denoted by* $T_{[n]^d}$ *is* $\mathcal{C}(T_{[n]^d}, x) = x\left[\mathcal{C}(P_n, x)\right]^d + x^d\left[\mathcal{C}(P_{n-1}, x)\right]^d$.

Proof. Let v be the vertex of degree d in $T_{[n]^d}$. Using Theorem 2.1.9 and 2.1.11 we have,

$$\begin{aligned}\mathcal{C}(T_{[n]^d}, x) &= x\left[\mathcal{C}\left(T_{[n]^d} - v, x\right)\right] + x^d\left[\mathcal{C}\left(T_{[n]^d} - N_{T_{[n]^d}}[v], x\right)\right]\\ &= x\left[\mathcal{C}(P_n, x)\right]^d + x^d\left[\mathcal{C}(P_{n-1}, x)\right]^d.\end{aligned}$$

Thus the proof is complete. □

Lemma 2.3.11. *If A is the set of all pendant vertices of the tree $T_{[n]^d}$, then*

$$\begin{aligned}\mathcal{C}_A\left(T_{[n]^d}, x\right) &= x\left[x\mathcal{C}(P_{n-2}, x)\right]^d + x^d\left[x\mathcal{C}(P_{n-3}, x)\right]^d.\\ \mathcal{C}^A\left(T_{[n]^d}, x\right) &= x\left[\mathcal{C}(P_n, x)\right]^d + x^d\left[\mathcal{C}(P_{n-1}, x)\right]^d\\ &- x\left[x\mathcal{C}(P_{n-2}, x)\right]^d - x^d\left[x\mathcal{C}(P_{n-3}, x)\right]^d.\end{aligned}$$

Proof. From Lemma 2.3.5, 2.3.6 and 2.3.10 we have,

$$\begin{aligned}\mathcal{C}_A(T_{[n]^d}, x) &= x\left[\mathcal{C}_A(P_n, x)\right]^d + x^d\left[\mathcal{C}_A(P_{n-1}, x)\right]^d\\ &= x\left[x\mathcal{C}(P_{n-2}, x)\right]^d + x^d\left[x\mathcal{C}(P_{n-3}, x)\right]^d.\\ \text{Since } \mathcal{C}^A(T_{[n]^d}, x) &= \mathcal{C}(T_{[n]^d}, x) - \mathcal{C}_A(T_{[n]^d}, x), \text{ from Lemma 2.3.10 we have,}\\ \mathcal{C}^A\left(T_{[n]^d}, x\right) &= x\left[\mathcal{C}(P_n, x)\right]^d + x^d\left[\mathcal{C}(P_{n-1}, x)\right]^d\\ &- x\left[x\mathcal{C}(P_{n-2}, x)\right]^d - x^d\left[x\mathcal{C}(P_{n-3}, x)\right]^d.\end{aligned}$$

This completes the proof. □

Next, we find the TD-Polynomial of one point union of cycles $C_n^{(k)}$.

Theorem 2.3.12. *If $n = 2m+1$ for some positive integer m, then the TD-Polynomial of $C_n^{(k)}$ is*

$$\begin{aligned} D_t(C_n^{(k)}, x) &= x\left[\mathcal{C}(P_m, x)\right]^{2k} + x^{2k}\left[\mathcal{C}(P_{m-1}, x)\right]^{2k} \\ &- x\left[x\mathcal{C}(P_{m-2}, x)\right]^{2k} - x^{2k}\left[x\mathcal{C}(P_{m-3}, x)\right]^{2k}. \end{aligned}$$

Proof. For $j = 1, 2, \ldots, k$ let v be the vertex common to the cycles C_n^j. Let $\left(v, a_1^i, a_2^i, a_3^i, \ldots, a_{n-1}^i, v\right)$ be the cycle C_n^i. If G represents the open neighborhood hypergraph of $C_n^{(k)}$, then $E(G) = N_{C_n^{(k)}}(v) \cup E\left(T_{[m]^{2k}}\right)$, where the tree $T_{[m]^{2k}}$ is shown in figure 2.12.

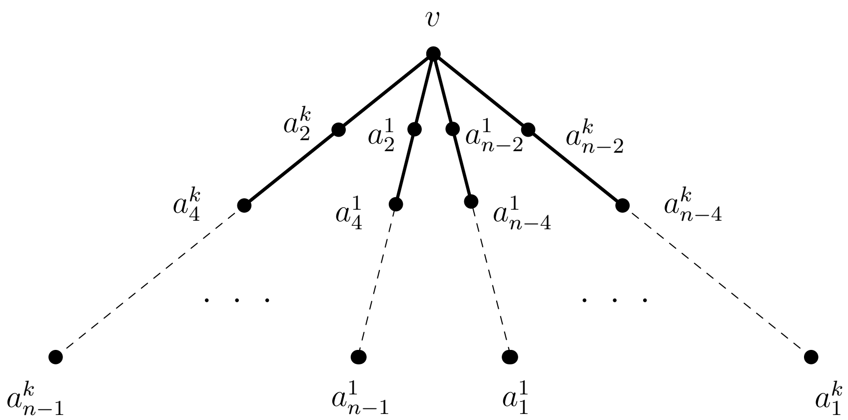

Figure 2.12: The Tree $T_{[m]^{2k}}$

If A is the set of all pendent vertices of $T_{[m]^{2k}}$, then a set S of vertices of $C_n^{(k)}$ is a total dominating set if and only if $S \cap A \neq \phi$ and S is a vertex covering set of

$T_{[m]^{2k}}$. Therefore, it suffices to find the polynomial $\mathcal{C}^A\left(T_{[m]^{2k}}, x\right)$. From Lemma 2.3.11, we have,

$$\begin{aligned}\mathcal{C}^A\left(T_{[m]^{2k}}, x\right) &= x\left[\mathcal{C}(P_m, x)\right]^{2k} + x^{2k}\left[\mathcal{C}(P_{m-1}, x)\right]^{2k} \\ &- x\left[x\mathcal{C}(P_{m-2}, x)\right]^{2k} - x^{2k}\left[x\mathcal{C}(P_{m-3}, x)\right]^{2k}.\end{aligned}$$

Thus the proof is complete. □

Theorem 2.3.13. *If $n = 2m$, for some positive integer m and G_1, G_2 are the components of the open neighborhood hypergraph of $\mathcal{C}_n^{(k)}$, then*

$$\begin{aligned}\mathcal{C}(G_1, x) &= \left[\mathcal{C}(P_m, x)\right]^k - \left[x^2\mathcal{C}(P_{m-4}, x)\right]^k \text{ and} \\ \mathcal{C}(G_2, x) &= x\left[\mathcal{C}(P_{m-1}, x)\right]^k + x^{2k}\left[\mathcal{C}(P_{m-3}, x)\right]^k.\end{aligned}$$

Proof. For $j = 1, 2, \ldots, k$ let v be the vertex common to the cycles C_n^j. Let $\left(v, a_1^i, a_2^i, a_3^i, \ldots, a_{n-1}^i, v\right)$ be the cycle C_n^i. Since n is even, the graph $\mathcal{C}_n^{(k)}$ is bipartite. Let $X = \bigcup_{i=1}^{k}\{a_j^i : j \text{ is even }\} \cup \{v\}$ and $Y = \bigcup_{i=1}^{k}\{a_j^i : j \text{ is odd }\}$ be the bipartition. Let G_1 and G_2 are the components of $ONH(\mathcal{C}_n^{(k)})$ corresponding to X and Y respectively. Then $E(G_1) = N_{\mathcal{C}_n^{(k)}}(v) \cup E(H)$, where H is given in figure 2.13.

Therefore, a set S of vertices of G_1 is a vertex cover if and only if S is a vertex cover of H and $S \cap N_{\mathcal{C}_n^{(k)}}(v) \neq \phi$. Since $N_{\mathcal{C}_n^{(k)}}(v)$, denoted here by $N(v)$, is the set of all pendent vertices of H, it suffices to find the polynomial $\mathcal{C}^{N(v)}(H, x)$. Since $\mathcal{C}_{N(v)}(H, x)$ is the polynomial in x such that the coefficient of x^i is the number of vertex covering sets of H which does not intersect with $N(v)$, from Theorem

2.1.8 and 2.1.9, we have

$$\begin{aligned}\mathcal{C}(G_1, x) &= \mathcal{C}^{N(v)}(H, x) \\ &= \mathcal{C}(H, x) - \mathcal{C}_{N(v)}(H, x) \\ &= [\mathcal{C}(P_m, x)]^k - \left[x^2\mathcal{C}(P_{m-4}, x)\right]^k.\end{aligned}$$

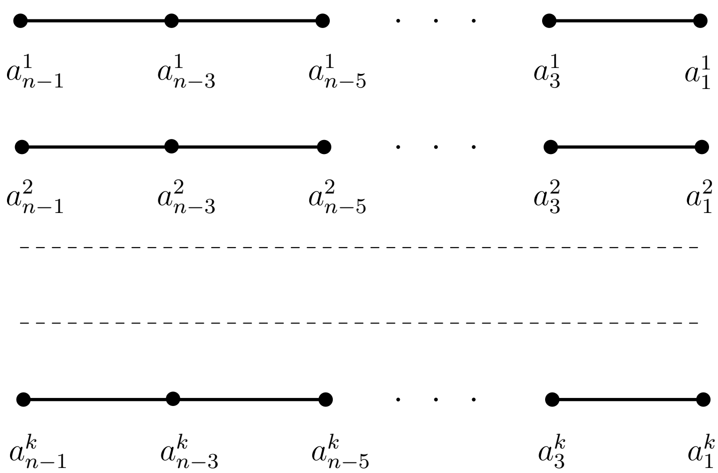

Figure 2.13: The graph H

Next, we find the vertex cover polynomial of the graph G_2. It can be represented as shown in figure 2.14. From Theorem 2.1.9, we have

$$\begin{aligned}\mathcal{C}(G_2, x) &= x\mathcal{C}(G_2 - v, x) + x^{2k}\mathcal{C}(G_2 - v - N_{G_2}(v), x) \\ &= x\left[\mathcal{C}(P_{m-1}, x)\right]^k + x^{2k}\left[\mathcal{C}(P_{m-3}, x)\right]^k.\end{aligned}$$

This completes the proof. □

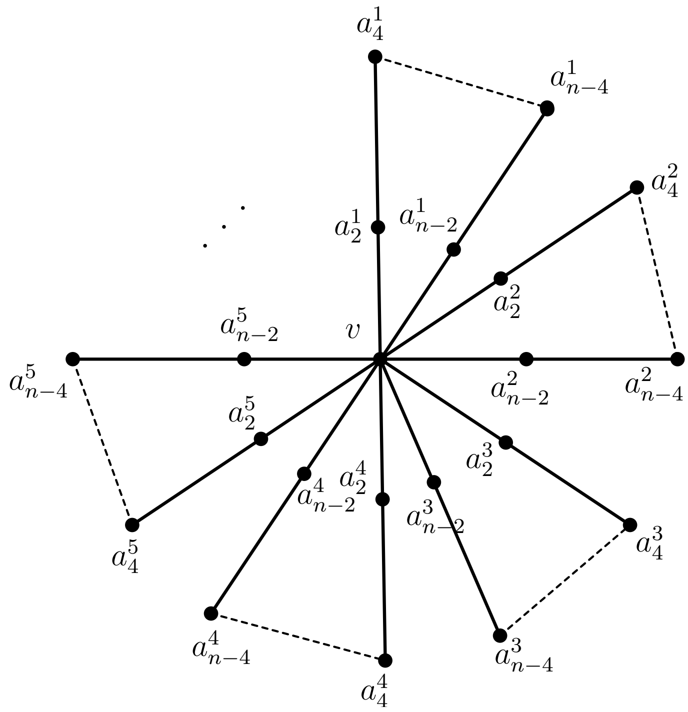

Figure 2.14: The graph G_2

Theorem 2.3.14. *If $n = 2m$ for some positive integer m, then*
$D_t(C_n^{(k)}, x) = \left([\mathcal{C}(P_m, x)]^k - \left[x^2\mathcal{C}(P_{m-4}, x)\right]^k\right)\left(x\,[\mathcal{C}(P_{m-1}, x)]^k + x^{2k}\,[\mathcal{C}(P_{m-3}, x)]^k\right).$

Proof. The proof follows immediately from Theorem 2.1.14 and 2.3.13. □

Next, we compute the total domination polynomial of $C_n^{2(k)}$, the n-gon book of k pages.

Theorem 2.3.15. *If $n = 2m$ for some positive integer m, then the total domination polynomial of $C_n^{2(k)}$ is, $D_t(C_n^{2(k)}, x) = \left[\mathcal{C}(T_{[m-1]^k}, x) - \left[x^2\mathcal{C}(P_{m-4}, x)\right]^k\right]^2.$*

Proof. Let u, v be the vertices common to the family of cycles in $C_n^{2(k)}$. Let $C_n^i = (u, a_1^i, a_2^i, \ldots, a_{n-2}^i, v, u)$. Let $A = N_{C_n^{2(k)}}(u) = \{v, a_1^1, a_1^2, a_1^3, \ldots, a_1^k\}$. Since

n is even, $C_n^{2(k)}$ is bipartite. Clearly, the components of the open neighborhood hypergraph of $C_n^{2(k)}$ are isomorphic. Let $X = \bigcup_{i=1}^{k} \{u, a_2^i, a_4^i, a_6^i, \ldots, a_{n-2}^i\}$ be one of the partite sets. Let H_X be the component of the open neighborhood hypergraph corresponding to X. Then $E(H_X) = A \cup E(H)$, where the graph H can be represented as shown in figure 2.15. Therefore, a set S is a vertex covering set of H_X if and only if $S \cap A \neq \phi$ and S is a vertex covering set of H. So we need to find the polynomial $\mathcal{C}^A(H, x)$. Note that

$$\begin{aligned}\mathcal{C}^A(H,x) &= \mathcal{C}(H,x) - \mathcal{C}_A(H,x)\\ &= \mathcal{C}(T_{[m-1]^k}, x) - \left[x^2\mathcal{C}(P_{m-4}, x)\right]^k.\end{aligned}$$

This completes the proof. □

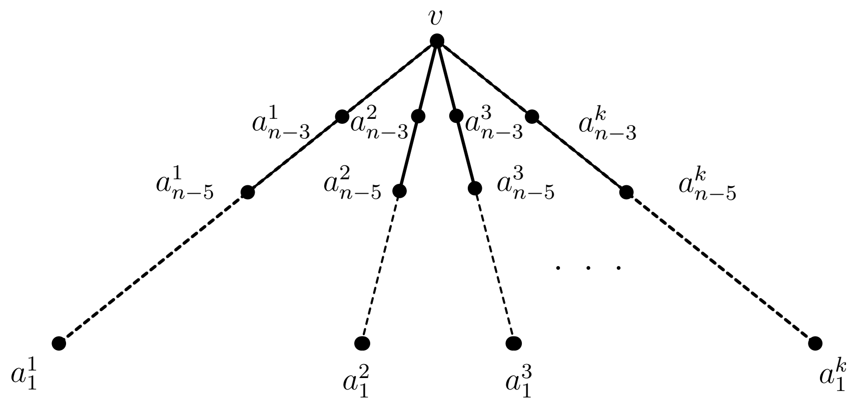

Figure 2.15: The graph H

Theorem 2.3.16. *If $n = 2m+1$ for some positive integer m, then the TD- Polynomial of $C_n^{2(k)}$ is* $x^2\left[\mathcal{C}(P_m, x)\mathcal{C}(P_{m-1}, x)\right]^k + 2x^{2k+1}\left[\mathcal{C}(P_{m-1}, x)\mathcal{C}(P_{m-2}, x)\right]^k + x^{4k}\left[\mathcal{C}(P_{m-2}, x)\mathcal{C}(P_{m-3}, x)\right]^k$.

Proof. Let C_n^i be the cycle $(u, a_1^i, a_2^i, a_3^i, \ldots, a_{n-2}^i, v, u)$. Since the open neighborhood hypergraph of an odd cycle is isomorphic to itself, we can represent $H_{C_n^{2(k)}}$, the open neighborhood hypergraph of $C_n^{2(k)}$ as shown in figure 2.16. Let $A = N[v]$ and $B = N[u]$. Then from Theorem 2.1.9 and 2.1.11 , we have

$$\begin{aligned}
D_t(C_n^{2(k)}, x) &= \mathcal{C}\left(H_{C_n^{2(k)}}, x\right) \\
&= x\mathcal{C}\left(H_{C_n^{2(k)}} - v, x\right) + x^{2k}\mathcal{C}\left(H_{C_n^{2(k)}} - A, x\right) \\
&= x\left[x\mathcal{C}\left(H_{C_n^{2(k)}} - v - u, x\right) + x^{2k}\mathcal{C}\left(H_{C_n^{2(k)}} - v - B, x\right)\right] \\
&+ x^{2k}\left[x\mathcal{C}\left(H_{C_n^{2(k)}} - u - A, x\right)\right] + x^{2k}\left[x^{2k}\mathcal{C}\left(H_{C_n^{2(k)}} - A - B, x\right)\right] \\
&= x^2\left[\mathcal{C}(P_m, x)\right]^k\left[\mathcal{C}(P_{m-1}, x)\right]^k + x^{2k+1}\left[\mathcal{C}(P_{m-1}, x)\right]^k\left[\mathcal{C}(P_{m-2}, x)\right]^k \\
&+ x^{2k+1}\left[\mathcal{C}(P_{m-1}, x)\right]^k\left[\mathcal{C}(P_{m-2}, x)\right]^k \\
&+ x^{4k}\left[\mathcal{C}(P_{m-2}, x)\right]^k\left[\mathcal{C}(P_{m-3}, x)\right]^k
\end{aligned}$$

This completes the proof. □

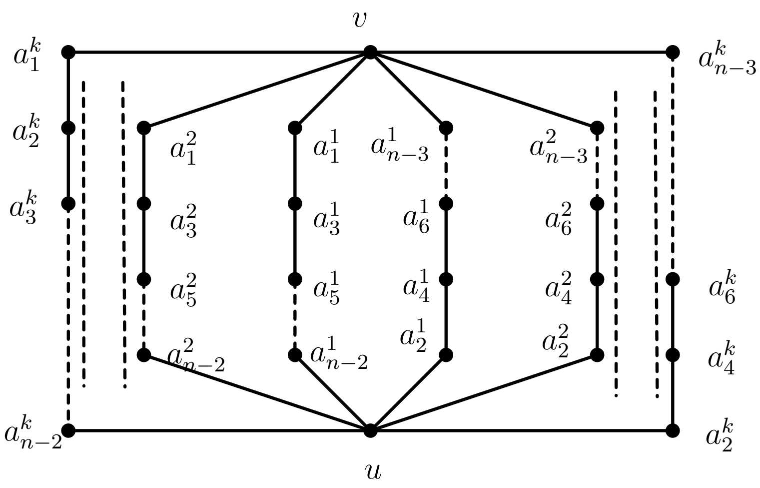

Figure 2.16: The Graph $H_{C_n^{2(k)}}$

Next, we find the TD-Polynomial of the theta graph $\theta(\underbrace{n, n, \ldots, n}_{k\ times})$ denoted by $\theta_{(n)^k}$.

Theorem 2.3.17. *Let $n = 2m + 2$ for some positive integer m, then the total domination polynomial of $\theta_{(n)^k}$ is*
$\left[x\left[\mathcal{C}(P_m, x)\right]^k + x^k\left[\mathcal{C}(P_{m-1}, x)\right]^k - x\left[x\mathcal{C}(P_{m-2}, x)\right]^k - x^k\left[x\mathcal{C}(P_{m-3}, x)\right]^k\right]^2$.

Proof. Let $(u, a_1^i, a_2^i, a_3^i, \ldots, a_{n-2}^i, v)$ be the path P_n^i in $\theta_{(n)^k}$. Clearly, the graph $\theta_{(n)^k}$ is bipartite and the components of the open neighborhood hypergraph of $\theta_{(n)^k}$ are isomorphic to each other. Let $X = \bigcup_{i=1}^{k}\{a_2^i, a_4^i, a_6^i, \ldots, a_{n-2}^i\} \cup \{u\}$ be one of the partite sets and H_X be the open neighborhood hypergraph corresponding to X. Let $A = N_{\theta_{(n)^k}}(u) = \{a_1^1, a_1^2, a_1^3, \ldots, a_1^k\}$. Then $E(H_X) = A \cup E(K)$, where the graph K is shown in figure 2.17.

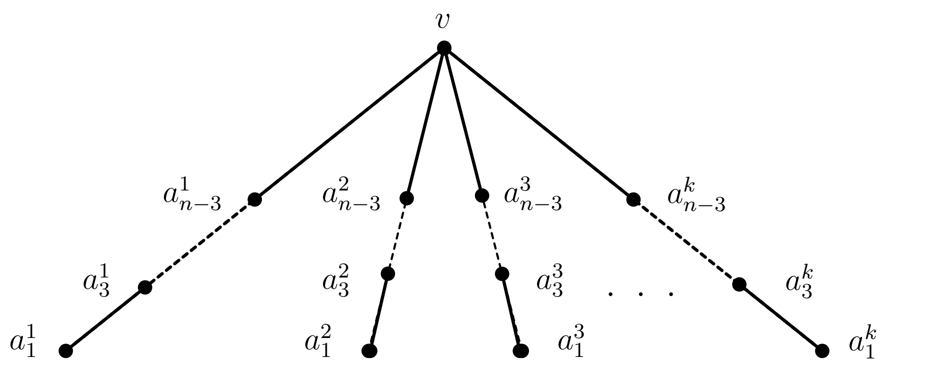

Figure 2.17: The graph K

So it suffices to find the polynomial $\mathcal{C}^A(K, x)$. Since K is isomorphic to $T_{[m]^k}$, from Lemma 2.3.11 we have,

$$\mathcal{C}(H_X, x) = \mathcal{C}^A(K, x)$$

$$
\begin{aligned}
&= \mathcal{C}^A\left(T_{[m]^k}, x\right) \\
&= x\left[\mathcal{C}(P_m, x)\right]^k + x^k\left[\mathcal{C}(P_{m-1}, x)\right]^k \\
&- x\left[x\mathcal{C}(P_{m-2}, x)\right]^k - x^k\left[x\mathcal{C}(P_{m-3}, x)\right]^k.
\end{aligned}
$$

Therefore, from Theorem 2.1.14, we have $D_t(\theta_{(n)^k}, x) = \left[\mathcal{C}(H_X, x)\right]^2$. □

Theorem 2.3.18. *Let $n = 2m+3$ for some positive integer m. If (X, Y) is the bipartition of $\theta_{(n)^k}$, then $D_t\left(\theta_{(n)^k}\right) = \mathcal{C}\left(H_X, x\right)\mathcal{C}\left(H_Y, x\right)$, where*
$\mathcal{C}\left(H_X, x\right) = x^2\left[\mathcal{C}(P_m, x)\right]^k + 2x^{k+1}\left[\mathcal{C}(P_{m-1}, x)\right]^k + x^{2k}\left[\mathcal{C}(P_{m-2}, x)\right]^k$ and
$\mathcal{C}\left(H_Y, x\right) = \left[\mathcal{C}(P_{m+1}, x)\right]^k - x^{2k}\left[\mathcal{C}(P_{m-3}, x)\right]^k$.

Proof. Let the i^{th} path P_n^i of $\theta_{(n)^k}$ be $(u, a_1^i, a_2^i, a_3^i, \ldots, a_{n-2}^i, v)$. Consider the bipartition $X = \bigcup_{i=1}^{k}\{a_1^i, a_3^i, \ldots, a_{n-2}^i\}$ and $Y = \bigcup_{i=1}^{k}\{a_2^i, a_4^i, \ldots, a_{n-3}^i\} \cup \{u, v\}$ of $\theta_{(n)^k}$. Let H_X and H_Y be the components of the open neighborhood hypergraph of $\theta_{(n)^k}$ corresponding to X and Y. Then H_X can be represented as shown in figure 2.18.

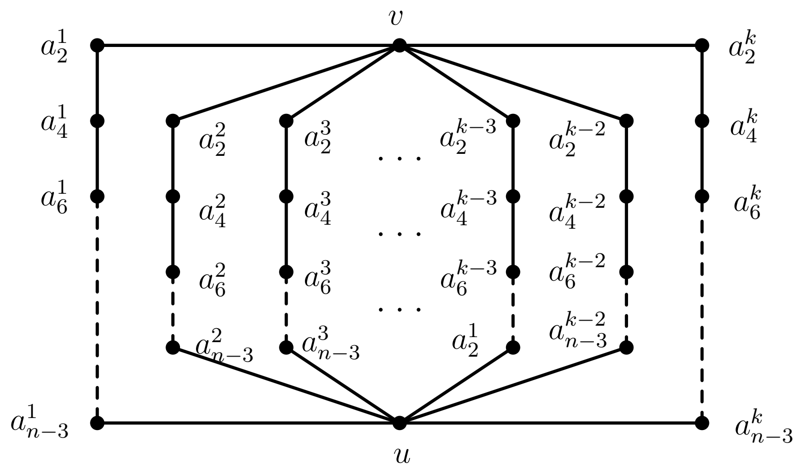

Figure 2.18: The Graph H_X

Then by Theorem 2.1.9, we have

$$\begin{aligned}
\mathcal{C}(H_X,x) &= x\mathcal{C}(H_X-u,x)+x^k\mathcal{C}(H_X-u-N(u),x)\\
&= x\left[x\mathcal{C}(H_X-u-v,x)+x^k\mathcal{C}(H_X-u-v-N(v),x)\right]\\
&+ x^k\left[x\mathcal{C}(H_X-u-v-N(u),x)\right]\\
&+ x^k\left[x^k\mathcal{C}(H_X-u-v-N(u)-N(v),x)\right]\\
&= x\left(x\left[\mathcal{C}(P_m,x)\right]^k+x^k\left[\mathcal{C}(P_{m-1},x)\right]^k\right)\\
&+ x^k\left(x\left[\mathcal{C}(P_{m-1},x)\right]^k+x^k\left[\mathcal{C}(P_{m-2},x)\right]^k\right)\\
&= x^2\left[\mathcal{C}(P_m,x)\right]^k+x^{k+1}\left[\mathcal{C}(P_{m-1},x)\right]^k\\
&+ x^{k+1}\left[\mathcal{C}(P_{m-1},x)\right]^k+x^{2k}\left[\mathcal{C}(P_{m-2},x)\right]^k\\
&= x^2\left[\mathcal{C}(P_m,x)\right]^k+2x^{k+1}\left[\mathcal{C}(P_{m-1},x)\right]^k+x^{2k}\left[\mathcal{C}(P_{m-2},x)\right]^k.
\end{aligned}$$

Next, we determine the vertex cover polynomial of H_Y.

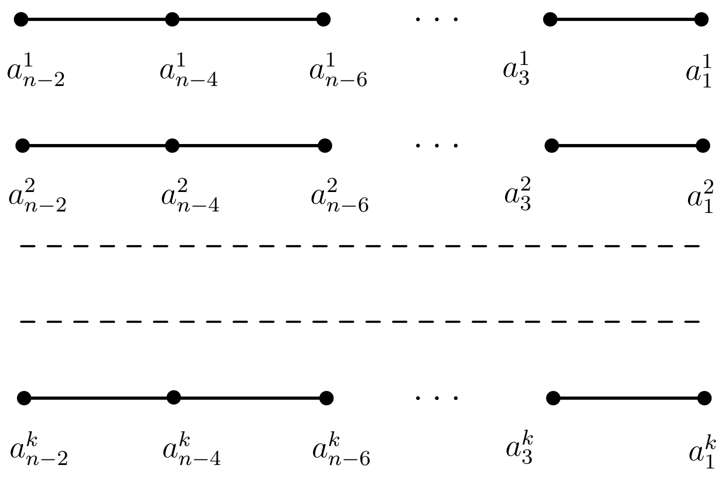

Figure 2.19: The Graph H

In the graph $\theta_{(n)^k}$, we have $N(u) = \{a^i_1 \colon i = 1,2,\ldots,k\}$ and $N(v) = \{a^i_{n-2} \colon i = 1,2,\ldots,k\}$. Then $E(H_Y) = N(u) \cup N(v) \cup E(H)$, where H is the graph shown in figure 2.19. Therefore,

$$\begin{aligned}\mathcal{C}(H_Y, x) &= \mathcal{C}(H, x) - \mathcal{C}_{N(u)\cup N(v)}(H, x) \\ &= [\mathcal{C}(P_{m+1}, x)]^k - x^2 [\mathcal{C}(P_{m-3}, x)]^k .\end{aligned}$$

Thus the proof follows. □

Next, we compute the total domination polynomial of $K_{n,n+1}^{(k)}$, the one point union of k copies of $K_{n,n+1}$.

Theorem 2.3.19. *The TD-Polynomial of $K_{n,n+1}^{(k)}$ is,*

$$D_t\left(K_{n,n+1}^{(k)}, x\right) = x\,[D(K_n, x)]^k + [D(K_n, x)]^{2k} .$$

Proof. Let v be the vertex common to the k copies of $K_{n,n+1}$ in $K_{n,n+1}^{(k)}$. Let $(A_i, B_i \cup \{v\})$ be the bipartition of the i^{th} copy of $K_{n,n+1}$, where $A_i = \{a_1^i, a_2^i, a_3^i, \ldots, a_n^i\}$ and $B_i = \{b_1^i, b_2^i, b_3^i, \ldots, b_n^i\}$. Then, for $1 \le i \le k$ and $1 \le t \le n$, $N(a_t^i) = B_i \cup \{v\}$, $N(b_t^i) = A_i$ and $N(v) = \bigcup_{j=1}^{k} A_j$.

Let S be set of vertices in $K_{n,n+1}^{(k)}$. Then we have two possibilities. Either $v \in S$ or $v \notin S$.

Case 1: Let $v \in S$. Since $N(v) = \bigcup_{j=1}^{k} A_j$, S is a total dominating set of $K_{n,n+1}^{(k)}$ if and only if $S \cap A_i \neq \phi$ for every i. Note that r vertices can be selected from A_i in $\binom{n}{r}$ ways. Therefore, in this case TD-Polynomial of the i^{th} copy of $K_{n,n+1}$ in $K_{n,n+1}^{(k)}$ is, $x\left[\binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right] = x\,[(1+x)^n - 1]$.

Since there are k copies of $K_{n,n+1}$, the TD-Polynomial is

$$\begin{aligned} x\left[(1+x)^n-1\right]^k &= x\left[D_t\left(K_n,x\right)+nx\right]^k \\ &= x\left[D\left(K_n,x\right)\right]^k. \end{aligned}$$

Case 2: Let $v \notin S$. In this case S is a total dominating set of $K_{n,n+1}^{(k)}$ if and only if it is a TD-set of $K_{n,n+1}^{(k)} - v$. Since $K_{n,n+1}^{(k)} - v$ is the union of k copies of $K_{n,n}$, the TD-Polynomial is $\left[\left[(1+x)^n-1\right]\left[(1+x)^n-1\right]\right]^k = \left[D\left(K_n,x\right)\right]^{2k}$.

Since the above cases are disjoint, $D_t\left(K_{n,n+1}^{(k)},x\right) = x\left[D(K_n,x)\right]^k + \left[D(K_n,x)\right]^{2k}$. This completes the proof. □

Chapter 3

Total Domination Polynomials of Ring Sum of some Graphs

3.1 Introduction

In this chapter, we study the total domination polynomials of ring sum of a graph G with the star graph $K_{1,m}$. The operation ring sum with a path and a star graph produces a number of graphs and the TD-Polynomial of each one is determined. The hypergraph terminology plays an important role in establishing relation between total domination polynomials of ring sum of graphs and vertex cover polynomials of paths. Moreover, the polynomial $D_t^v(G,x)$, in which a particular vertex v of G is present in every TD-set of G, is determined. We need the following.

Theorem 3.1.1. *(see [18]) Let $G = G_1 \cup G_2$ be the union of two graphs G_1 and G_2. Then $\mathcal{C}(G,x) = \mathcal{C}(G_1,x)\mathcal{C}(G_2,x)$.*

Theorem 3.1.2. *(see [18]) For the path graph P_n, where $n \geq 2$, we have*

$$\mathcal{C}(P_n,x) = \sum_{i=0}^{n} \binom{i+1}{n-i} x^i.$$

Theorem 3.1.3. *(see [18]) For the cycle graph C_n, where $n \geq 3$, we have*

$$\mathcal{C}(C_n,x) = \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i.$$

3.2 On the polynomial $D_t^v(G,x)$

The inclusion of a particular vertex in every total dominating set of a graph is important in the study of total domination polynomials. In this section, the polynomial $D_t^v(G,x)$ is determined for some graphs.

Definition 3.2.1. *(see [4]) Let G be a graph and v be a vertex of G. Let $\mathcal{D}_t^v(G,i)$ be the family of all total dominating sets of G of cardinality i containing the vertex v. If $d_t^v(G,i) = |\mathcal{D}_t^v(G,i)|$, the polynomial $D_t^v(G,x)$ is defined as $D_t^v(G,x) = \sum_{i=1}^{|V(G)|} d_t^v(G,i)x^i$.*

Definition 3.2.2. *Let G be a graph and v be a vertex of G. Let $d_{t_v}(G,i) = |\mathcal{D}_{t_v}(G,i)|$, where $\mathcal{D}_{t_v}(G,i) = \{S \subseteq V(G) \colon v \notin S,\ N(S) = V(G),\ \mid S \mid = i\}$. Then the polynomial $D_{t_v}(G,x)$ is defined as $D_{t_v}(G,x) = \sum_{i=1}^{|V(G)|} d_{t_v}(G,i)x^i$.*

Definition 3.2.3. *Let G be a graph and v be a vertex of G. Let $\mathcal{C}^v(G,i)$ be the family of all vertex covering sets of G of cardinality i containing the vertex v. If $c^v(G,i) = |\mathcal{C}^v(G,i)|$, the polynomial $\mathcal{C}^v(G,x)$ is defined as $\mathcal{C}^v(G,x) = \sum_{i=1}^{|V(G)|} c^v(G,i)x^i$.*

Theorem 3.2.4. *Let G be a graph and $v \in V(G)$. Then $D_t^v(G,x) = \mathcal{C}^v(H_G,x)$.*

Proof. Let S be a subset of $V(G)$. It is clear that a total dominating set of a graph G is a vertex covering set of the open neighborhood hypergraph, H_G of G and vice versa. Therefore, a set S is a total dominating set containing v if and only if S is a vertex covering set of H_G containing v. This completes the proof. □

Theorem 3.2.5. *Let G be a graph and $v \in V(G)$. Then $\mathcal{C}^v(G,x) = x\mathcal{C}(G-v,x)$.*

Proof. Let $S \subseteq V(G)$ be a vertex covering set of G of cardinality i containing the vertex v. Then $S \setminus \{v\}$ is a vertex covering set of $G - v$ of cardinality $i-1$. So for $i = 1,2,\ldots,|V(G)|$, $c(G,i) = c(G-v,i-1)$. This proves the result. □

Theorem 3.2.6. *If u is a vertex of the cycle graph C_{2n+1}, then*

$$D_t^u(C_{2n+1},x) = \sum_{i=0}^{2n} \binom{i+1}{2n-i} x^{i+1}.$$

Proof. Let $H_{C_{2n+1}}$ be the open neighborhood hypergraph of the cycle C_{2n+1}. Clearly, $H_{C_{2n+1}}$ is isomorphic to C_{2n+1}. Then from Theorems 3.2.4 and 3.2.5 we have, $D_t^u(C_{2n+1},x) = \mathcal{C}^u(H_{C_{2n+1}},x) = \mathcal{C}^u(C_{2n+1},x) = x\mathcal{C}(C_{2n+1}-u,x) = x\mathcal{C}(P_{2n},x)$. Then the result follows from Theorem 3.1.2. □

Theorem 3.2.7. *If u is a vertex of the cycle graph C_{2n}, then*

$$D_t^u(C_{2n},x) = x\mathcal{C}(C_n,x)\mathcal{C}(P_{n-1},x).$$

Proof. Let (X,Y) be the bipartition of C_{2n}. Assume that $u \in X$. Note that the components H_X, H_Y of $ONH(C_{2n})$ are cycles of length n. Then, from Theorems

3.1.1, 3.2.4 and 3.2.5 we have, $D_t^u(C_{2n},x) = \mathcal{C}^u(H_{C_{2n}},x) = \mathcal{C}^u(H_X,x)\mathcal{C}(H_Y,x) = x\mathcal{C}(H_X-u,x)\mathcal{C}(H_Y,x) = x\mathcal{C}(P_{n-1},x)\mathcal{C}(C_n,x)$. This completes the proof. □

For Theorems 3.2.8 and 3.2.10 we take the path graph as $P_n = (1,2,\ldots,n)$.

Theorem 3.2.8. *For the path $P_{2n} = (1,2,\ldots,2n)$, we have*

(i) $D_t^1(P_{2n},x) = x^3\mathcal{C}(P_{n-1},x)\mathcal{C}(P_{n-2},x)$,

(ii) For $1 \le r \le n-2$, $D_t^{2r+1}(P_{2n},x) = x^3\mathcal{C}(P_{n-1},x)\mathcal{C}(P_r,x)\mathcal{C}(P_{n-r-2},x)$.

Proof. Let $X = \{1,3,\ldots,2n-1\}$ and $Y = \{2,4,\ldots,2n\}$ be the bipartition of P_{2n}. Let H_X and H_Y (shown in figure 3.1) be the components of $ONH(P_{2n})$ corresponding to X and Y respectively.

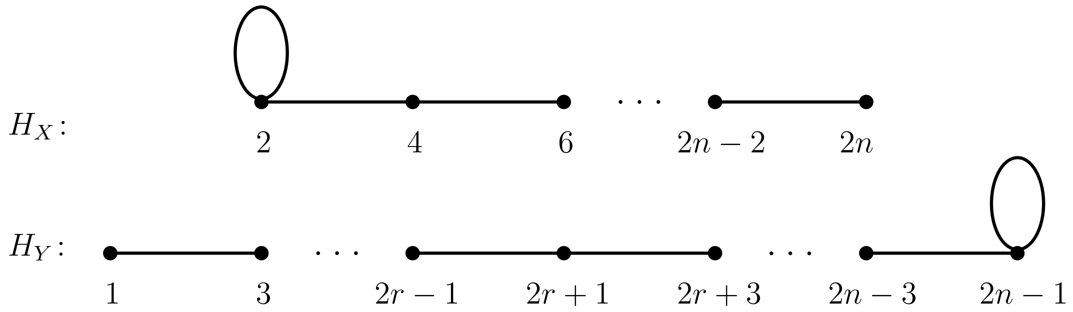

Figure 3.1: H_X and H_Y

(i) Using Theorems 3.1.1, 3.2.4 and 3.2.5 we have, $D_t^1(P_{2n},x) = \mathcal{C}^1(H_{P_{2n}},x)$ $= \mathcal{C}(H_X,x)\mathcal{C}^1(H_Y,x) = \mathcal{C}(H_X,x)\, x\, \mathcal{C}(H_Y-1,x) = x^3\mathcal{C}(P_{n-1},x)\mathcal{C}(P_{n-2},x)$.

(ii) From Theorems 3.1.1 and 3.2.5 we have, $\mathcal{C}(H_X,x) = x\mathcal{C}(P_{n-1},x)$ and $\mathcal{C}^{2r+1}(H_Y,x) = x\mathcal{C}(H_Y-(2r+1),x) = x^2\mathcal{C}(P_r,x)\mathcal{C}(P_{n-r-2},x)$. Applying Theorem 3.2.4, $D_t^{2r+1}(P_{2n},x) = \mathcal{C}^{2r+1}(H_{P_{2n}},x) = \mathcal{C}(H_X,x)\mathcal{C}^{2r+1}(H_Y,x)$.

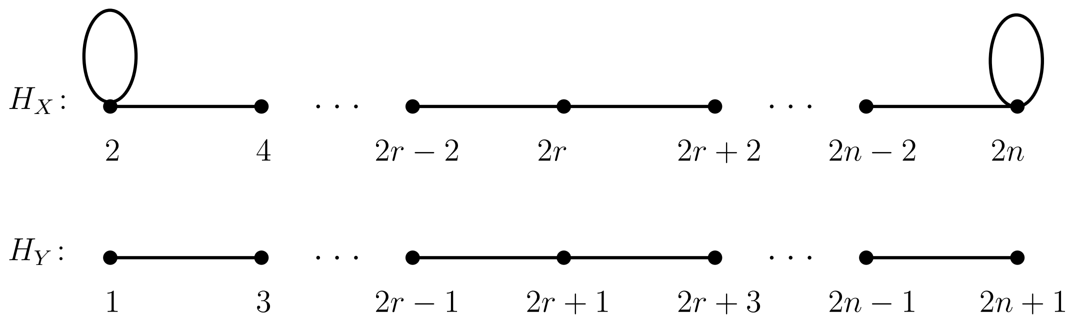

Figure 3.2: H_X and H_Y

Thus the result follows. □

Note 3.2.9. *Since $f\colon V(P_{2n}) \to V(P_{2n})$ defined by $f(k) = 2n-(k-1)$ is an isomorphism, we have $D_t^{2n}(P_{2n},x) = D_t^1(P_{2n},x)$ and $D_t^{2r}(P_{2n},x) = D_t^{2r+1}(P_{2n},x)$.*

Theorem 3.2.10. *For the path $P_{2n+1} = (1,2,\dots,2n+1)$, we have*

(i) $D_t^1(P_{2n+1},x) = x^3\mathcal{C}(P_n,x)\mathcal{C}(P_{n-2},x)$,

(ii) For $1 \le r \le n-1$, $D_t^{2r}(P_{2n+1},x) = x^3\mathcal{C}(P_{r-2},x)\mathcal{C}(P_{n-r-1},x)\mathcal{C}(P_{n+1},x)$,

(iii) For $1 \le r \le n-2$, $D_t^{2r+1}(P_{2n+1},x) = x^3\mathcal{C}(P_{n-2},x)\mathcal{C}(P_r,x)\mathcal{C}(P_{n-r},x)$.

Proof. Let $X = \{1,3,\dots,2n+1\}$ and $Y = \{2,4,\dots,2n\}$ be the bipartition of P_{2n+1}. Let H_X and H_Y (shown in figure 3.2) be the components of $ONH(P_{2n+1})$ corresponding to X and Y respectively. Then from Theorems 3.1.1, 3.2.4 and 3.2.5 we have,

(i) $\mathcal{C}(H_X,x) = x^2\mathcal{C}(P_{n-2},x)$ and $\mathcal{C}^1(H_Y,x) = x\mathcal{C}(H_Y-1,x) = x\mathcal{C}(P_n,x)$. Since $D_t^1(P_{2n+1},x) = \mathcal{C}^1(H_{P_{2n+1}},x) = \mathcal{C}(H_X,x)\mathcal{C}^1(H_Y,x)$, the proof follows.

(ii) $D_t^{2r}(P_{2n+1},x) = \mathcal{C}^{2r}(H_{P_{2n+1}},x) = \mathcal{C}^{2r}(H_X,x)\mathcal{C}(H_Y,x)$. Since $\mathcal{C}^{2r}(H_X,x)$ $= x\mathcal{C}(H_X-(2r),x) = x^3\mathcal{C}(P_{r-2},x)\mathcal{C}(P_{n-r-1},x)$ and $\mathcal{C}(H_Y,x) = \mathcal{C}(P_{n+1},x)$, the proof follows.

(iii) Proceeding as above, we get $D_t^{2r+1}(P_{2n+1},x) = \mathcal{C}(H_X,x)\mathcal{C}^{2r+1}(H_Y,x) = x^2\mathcal{C}(P_{n-2},x)x\mathcal{C}(H_Y-(2r+1),x) = x^3\mathcal{C}(P_{n-2},x)\mathcal{C}(P_r,x)\mathcal{C}(P_{n-r},x)$.

□

The following results can be easily derived from the definition of $D_t^v(G,x)$.

Theorem 3.2.11. *For any vertex v of K_n, $D_t^v(K_n,x) = x\left[(1+x)^{n-1}-1\right]$.*

Theorem 3.2.12. *For $v \in V(K_{n,n})$, $D_t^v(K_{n,n},x) = x(1+x)^{n-1}\left[(1+x)^n-1\right]$.*

Theorem 3.2.13. *If $K_{n+1}^{(k)}$ denotes the one point union of k copies of the complete graph K_{n+1}, then $D_t(K_{n+1}^{(k)},x) = x\left[(1+x)^{nk}-1\right] + \left[(1+x)^n-1-nx\right]^k$.*

Proof. Let u be the vertex common to the k copies of K_{n+1}. Let S be a total dominating set of $K_{n+1}^{(k)}$. Then we have two possibilities. Either $u \in S$ or $u \notin S$.

Case i: If $u \in S$, then for any vertex $v \neq u$ of $K_{n+1}^{(k)}$, the set $\{u,v\}$ is a total dominating set. Since there are nk vertices in $K_{n+1}^{(k)} - u$, the number of total dominating sets of $K_{n+1}^{(k)}$ containing the vertex u of cardinality i is $\binom{nk}{i-1}$. Therefore,

$$\begin{aligned} D_t^u(K_{n+1}^{(k)},x) &= x\left[\binom{nk}{1}x + \binom{nk}{2}x^2 + \ldots + \binom{nk}{nk}x^{nk}\right] \\ &= x\left[(1+x)^{nk}-1\right]. \end{aligned}$$

Case ii: Let $u \notin S$. Let $V_1, V_2, \ldots, V_k$ be the sets of vertices of the components of $K_{n+1}^{(k)} - u$. Then $\mid S \cap V_i \mid \geq 2$. In other words, a set containing at least two vertices from each and every component of $K_{n+1}^{(k)} - u$ forms a total dominating set. Since we can select i vertices from the set V_j in $\binom{n}{i}$ ways,

$$\begin{aligned} D_{t_u}(K_{n+1}^{(k)}, x) &= \left[\binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n\right]^k \\ &= \left[(1+x)^n - 1 - nx\right]^k. \end{aligned}$$

Then the proof follows from $D_t(K_{n+1}^{(k)}, x) = D_t^u(K_{n+1}^{(k)}, x) + D_{t_u}(K_{n+1}^{(k)}, x)$. □

3.3 Ring Sum of graphs

Definition 3.3.1. *(see [42]) Ring sum of two graphs H and K, denoted by $H \oplus K$, is the graph with vertex set $V(H) \cup V(K)$ and edge set $(E(H) \cup E(K)) - E(H \cap K)$.*

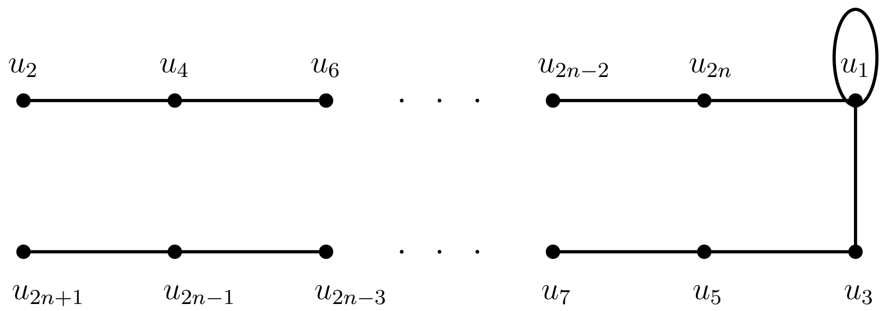

Figure 3.3: The graph H

Lemma 3.3.2. *For the graph H shown in figure 3.3, we have*

(i) $\mathcal{C}(H, x) = x\left[\mathcal{C}(P_n, x)\right]^2$,

(ii) $\mathcal{C}_{\{u_2, u_{2n+1}\}}(H, x) = x^3\mathcal{C}(P_{n-2}, x)^2$,

(iii) $\mathcal{C}^{\{u_2, u_{2n+1}\}}(H, x) = x\left[\mathcal{C}(P_n, x)\right]^2 - x^3\mathcal{C}(P_{n-2}, x)^2$.

Proof. **(i)** Since H has a loop at u_1, from Theorems 2.1.8 and 2.1.11 we have, $\mathcal{C}(H,x) = x\mathcal{C}(H-u_1,x) = x\left[\mathcal{C}(P_n,x)\right]^2.$

(ii) Let S be a vertex covering set of H. If $S\cap\{u_2,u_{2n+1}\} = \phi$, then $\{u_4,u_{2n-1}\} \subseteq S$. So it suffices to find the vertex cover polynomial of the graph $H - \{u_2,u_{2n+1}\}$ having loops at u_4 and u_{2n-1}. Therefore, from Theorems 2.1.8 and 2.1.11 we have, $\mathcal{C}_{\{u_2,u_{2n+1}\}}(H,x) = x^2\mathcal{C}(H-\{u_2,u_4,u_{2n+1},u_{2n-1}\},x) = x^3\left[\mathcal{C}(P_{n-2},x)\right]^2.$

(iii) Since, $\mathcal{C}^{\{u_2,u_{2n+1}\}}(H,x) = \mathcal{C}(H,x) - \mathcal{C}_{\{u_2,u_{2n+1}\}}(H,x)$, the proof follows from (i) and (ii).

□

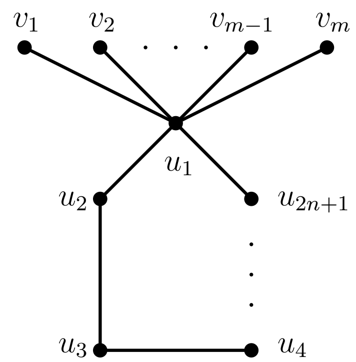

Figure 3.4: $C_{2n+1} \oplus K_{1,m}$

Theorem 3.3.3. *The TD-polynomial of the graph $C_{2n+1} \oplus K_{1,m}$ shown in figure 3.4 is,*

$$D_t(C_{2n+1} \oplus K_{1,m},x) = x(1+x)^m\left[\mathcal{C}(P_n,x)\right]^2 - x^3\mathcal{C}(P_{n-2},x)^2.$$

Proof. Let $H_{C_{2n+1}\oplus K_{1,m}}$ be the open neighborhood hypergraph of $C_{2n+1} \oplus K_{1,m}$.

Then, $D_t(C_{2n+1} \oplus K_{1,m}, x) = \mathcal{C}(H_{C_{2n+1} \oplus K_{1,m}}, x)$. Clearly the graph $C_{2n+1} \oplus K_{1,m}$ is not bipartite. Therefore, $H_{C_{2n+1} \oplus K_{1,m}}$ is connected and $E(H_{C_{2n+1} \oplus K_{1,m}}) = \{u_2, u_{2n+1}, v_1, v_2, \ldots, v_m\} \cup E(H)$ where H is the graph shown in figure 3.3. So it suffices to find the vertex covering sets of H having non-empty intersection with $\{u_2, u_{2n+1}, v_1, v_2, \ldots, v_m\}$. Let S be a vertex covering set of H. If $S \cap \{u_2, u_{2n+1}\} = \phi$, then S must contain at least one of the pendant vertices of $C_{2n+1} \oplus K_{1,m}$. Otherwise it is not necessary. Therefore, $D_t(C_{2n+1} \oplus K_{1,m}, x) = \mathcal{C}(H_{C_{2n+1} \oplus K_{1,m}}, x) = \mathcal{C}_{\{u_2, u_{2n+1}\}}(H, x)\left[(1+x)^m - 1\right] + \mathcal{C}^{\{u_2, u_{2n+1}\}}(H, x)\left[(1+x)^m\right]$. Then the proof follows from Lemma 3.3.2. □

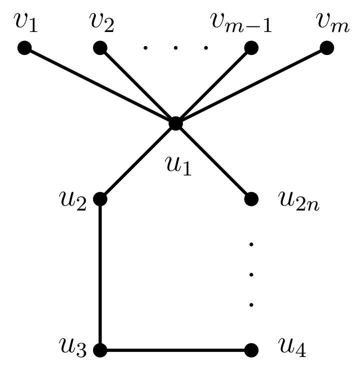

Figure 3.5: $C_{2n} \oplus K_{1,m}$

Theorem 3.3.4. *The TD-polynomial of the graph $C_{2n} \oplus K_{1,m}$ shown in figure 3.5 is,*

$$D_t(C_{2n} \oplus K_{1,m}, x) = \left[(1+x)^m \mathcal{C}(P_n, x) - x^2 \mathcal{C}(P_{n-4}, x)\right]\left[x\mathcal{C}(P_{n-1}, x)\right].$$

Proof. Let $V^{'} = \{v_1, v_2, \ldots, v_m\}$. Note that (X, Y) is a bipartition of $C_{2n} \oplus K_{1,m}$, where $X = \{u_1, u_3, \ldots, u_{2n-1}\}$ and $Y = \{u_2, u_4, \ldots, u_{2n}\} \cup V^{'}$. Let H_X and H_Y be the components of the ONH$(C_{2n} \oplus K_{1,m})$. Then $E(H_X) = E(P) \cup \{\{u_2, u_{2n}\} \cup V^{'}\}$

and $E(H_Y) = E(C) \cup \{\{u_1\}\}$, where P is the path $(u_2, u_4, u_6, \ldots, u_{2n-2}, u_{2n})$ and C is the cycle $(u_1, u_3, u_5, \ldots, u_{2n-1}, u_1)$. Let S be a vertex covering set of H_X. If $S \cap \{u_2, u_{2n}\} = \phi$, then S must contain at least one of the vertices of V'. Otherwise it is not necessary. Therefore,

$$
\begin{aligned}
\mathcal{C}(H_X, x) &= \mathcal{C}_{\{u_2,u_{2n}\}}(P, x)\left[(1+x)^m - 1\right] + \mathcal{C}^{\{u_2,u_{2n}\}}(P, x)\left[(1+x)^m\right]. \\
&= \mathcal{C}_{\{u_2,u_{2n}\}}(P, x)\left[(1+x)^m - 1\right] + \left[\mathcal{C}(P, x) - \mathcal{C}_{\{u_2,u_{2n}\}}(P, x)\right](1+x)^m. \\
&= \mathcal{C}(P, x)(1+x)^m - \mathcal{C}_{\{u_2,u_{2n}\}}(P, x) \\
&= (1+x)^m \mathcal{C}(P_n, x) - x^2 \mathcal{C}(P_{n-4}, x). \\
\mathcal{C}(H_Y, x) &= x\mathcal{C}(H_Y - u_1, x) \\
&= x\mathcal{C}(P_{n-1}, x).
\end{aligned}
$$

Since the TD-polynomial of a connected bipartite graph is the product of the vertex cover polynomials of the components of its open neighborhood hypergraph, the proof follows. □

Let $V' = \{v_1, v_2, \ldots, v_m\}$ be the set of all pendant vertices of the star graph $K_{1,m}$. For Theorems 3.3.5 and 3.3.6 we take the path graph P_n as $(1, 2, \ldots, n)$ and the vertex set of $K_{1,m}$ as $\{1\} \cup V'$.

Theorem 3.3.5. *The total domination polynomial of* $P_{2n+1} \oplus K_{1,m}$ *is*

$$D_t(P_{2n+1} \oplus K_{1,m}, x) = \left[x(1+x)^m \mathcal{C}(P_n, x) - x^2 \mathcal{C}(P_{n-3}, x)\right]\left[x\mathcal{C}(P_n, x)\right].$$

Proof. Let $X = \{1, 3, 5, \ldots, 2n-1, 2n+1\}$ and $Y = \{2, 4, 6, \ldots, 2n\} \cup V'$ be the bipartition of $P_{2n+1} \oplus K_{1,m}$. Let H_X and H_Y be the components of the

ONH($P_{2n+1} \oplus K_{1,m}$). Then $E(H_X) = E(P') \cup \{\{2, v_1, v_2, \ldots, v_m\}\}$, where P' is the graph shown in figure 3.6.

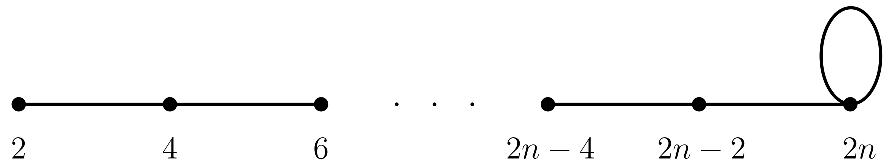

Figure 3.6: The Graph P'

Let S be a vertex covering set of H_X. Then $2n \in S$. If $2 \notin S$, then $S \cap V' \neq \phi$. Otherwise it is not necessary. Therefore,

$$\begin{aligned}
\mathcal{C}(H_X, x) &= \mathcal{C}^{\{2\}}(P', x)(1+x)^m + \mathcal{C}_{\{2\}}(P', x)\left[(1+x)^m - 1\right] \\
&= x^2\mathcal{C}(P_{n-2}, x)(1+x)^m + x^2\mathcal{C}(P_{n-3}, x)\left[(1+x)^m - 1\right] \\
&= x^2(1+x)^m\left[\mathcal{C}(P_{n-2}, x) + \mathcal{C}(P_{n-3}, x)\right] - x^2\mathcal{C}(P_{n-3}, x) \\
&= x(1+x)^m\mathcal{C}(P_{n-1}, x) - x^2\mathcal{C}(P_{n-3}, x).
\end{aligned}$$

The graph H_Y can be represented as shown in figure 3.7.

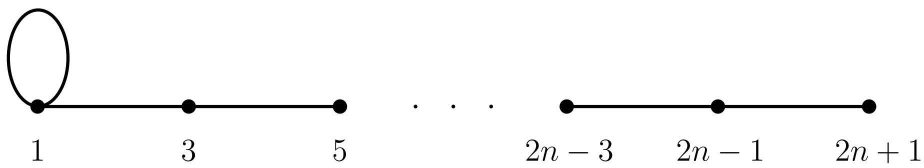

Figure 3.7: The Graph H_Y

Therefore, $\mathcal{C}(H_Y, x) = x\mathcal{C}(P_n, x)$. Since the product of $\mathcal{C}(H_X, x)$ and $\mathcal{C}(H_Y, x)$, gives the TD-Polynomial of $P_{2n+1} \oplus K_{1,m}$, the proof follows. $\square$

Next, we find the TD-Polynomial of $P_{2n} \oplus K_{1,m}$.

Theorem 3.3.6. *The total domination polynomial of $P_{2n} \oplus K_{1,m}$ is,*

$$D_t(P_{2n} \oplus K_{1,m}, x) = [(1+x)^m \mathcal{C}(P_n, x) - x\mathcal{C}(P_{n-2}, x)] \left[x^2 \mathcal{C}(P_{n-2}, x)\right].$$

Proof. Let $X = \{1, 3, 5, \ldots, 2n-1\}$ and $Y = \{2, 4, 6, \ldots, 2n\} \cup V'$ be the bipartition of $P_{2n} \oplus K_{1,m}$. Let H_X and H_Y be the components of the ONH($P_{2n} \oplus K_{1,m}$). Then $E(H_X) = E(P_n') \cup \{\{2, v_1, v_2, \ldots, v_m\}\}$, where P_n' is the path $(2, 4, 6 \ldots, 2n-2, 2n)$. Let S be a vertex covering set of H_X. If $2 \notin S$, then $S \cap V' \neq \phi$. Otherwise it is not necessary. Therefore,

$$\begin{aligned}
\mathcal{C}(H_X, x) &= \mathcal{C}^{\{2\}}(P_n', x)(1+x)^m + \mathcal{C}_{\{2\}}(P_n', x)\left[(1+x)^m - 1\right] \\
&= x\mathcal{C}(P_{n-1}, x)(1+x)^m + x\mathcal{C}(P_{n-2}, x)\left[(1+x)^m - 1\right] \\
&= x(1+x)^m \left[\mathcal{C}(P_{n-1}, x) + \mathcal{C}(P_{n-2}, x)\right] - x\mathcal{C}(P_{n-2}, x) \\
&= (1+x)^m \mathcal{C}(P_n, x) - x\mathcal{C}(P_{n-2}, x).
\end{aligned}$$

Note that the graph H_Y can be represented as in figure 3.8.

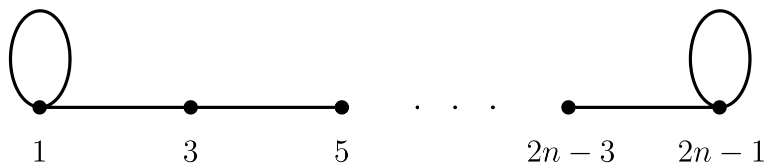

Figure 3.8: The Graph H_Y

Then, $\mathcal{C}(H_Y, x) = x^2 \mathcal{C}(P_{n-2}, x)$. Since $D_t(P_{2n} \oplus K_{1,m}, x) = \mathcal{C}(H_X, x)\mathcal{C}(H_Y, x)$, the proof follows. □

Theorem 3.3.7. *Let $P_{2n} = (1, 2, 3, \ldots, 2n)$ and $V' = \{v_1, v_2, \ldots, v_m\}$. Let $2r+1 \in \{3, 5, 7, \ldots, 2n-3\}$ be the root vertex of the star graph $K_{1,m}$ having the vertex set $\{2r+1\} \cup V'$. Then the TD-Polynomial of the ring sum $P_{2n} \oplus K_{1,m}$ is,*
$x^3\mathcal{C}(P_r, x)\mathcal{C}(P_{n-r-2}, x)\left[(1+x)^m\mathcal{C}(P_{r-1}, x)\mathcal{C}(P_{n-r}, x) - x^2\mathcal{C}(P_{r-3}, x)\mathcal{C}(P_{n-r-2}, x)\right].$

Proof. Let $X = \{1, 3, 5, \ldots, 2n-1\}$ and $Y = \{2, 4, 6, \ldots, 2n\} \cup V'$ be the bipartition of $P_{2n} \oplus K_{1,m}$ and H_X and H_Y be the components of $ONH(P_{2n} \oplus K_{1,m})$.

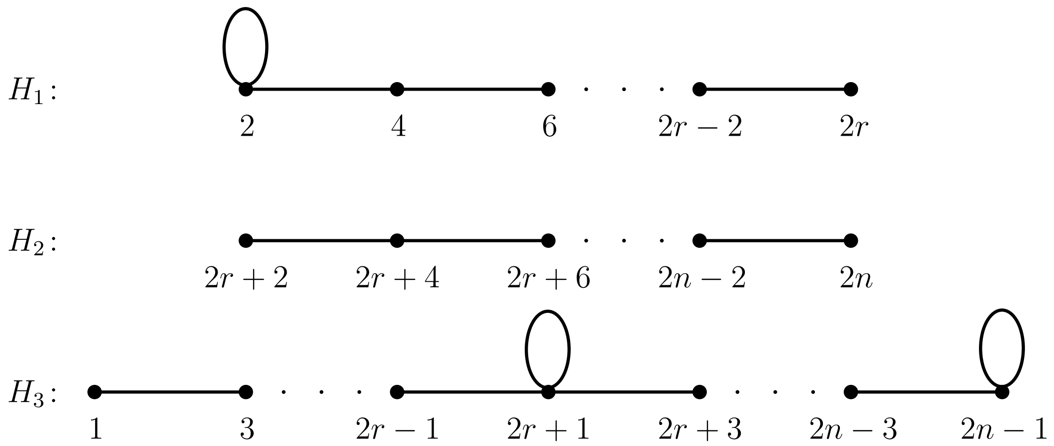

Figure 3.9: The graphs H_1, H_2 and H_3

Then $E(H_X) = E(H_1) \cup E(H_2) \cup \{\{2r, 2r+2, v_1, v_2, \ldots, v_m\}\}$ and $E(H_Y) = E(H_3)$, where the graphs H_1, H_2 and H_3 are shown in figure 3.9. Then $\mathcal{C}(H_Y, x) = \mathcal{C}(H_3, x) = x^2\mathcal{C}(P_r, x)\mathcal{C}(P_{n-r-2}, x)$. For a vertex covering set S of H_X, if $\{2r, 2r+2\} \cap S = \phi$, then $S \cap V' \neq \phi$. Therefore,

$$\begin{aligned}
\mathcal{C}(H_X, x) &= \mathcal{C}^{\{2r,2r+2\}}(H_1 \cup H_2, x)(1+x)^m + \mathcal{C}_{\{2r,2r+2\}}(H_1 \cup H_2, x)\left[(1+x)^m\text{-}1\right] \\
&= (1+x)^m\left[\mathcal{C}^{\{2r,2r+2\}}(H_1 \cup H_2, x) + \mathcal{C}_{\{2r,2r+2\}}(H_1 \cup H_2, x)\right] \\
&- \mathcal{C}_{\{2r,2r+2\}}(H_1 \cup H_2, x)
\end{aligned}$$

$$\begin{aligned}
&= (1+x)^m\mathcal{C}(H_1 \cup H_2, x) - \mathcal{C}_{\{2r,2r+2\}}(H_1 \cup H_2, x) \\
&= (1+x)^m\mathcal{C}(H_1, x)\mathcal{C}(H_2, x) - \mathcal{C}_{\{2r\}}(H_1, x)\mathcal{C}_{\{2r+2\}}(H_2, x) \\
&= x(1+x)^m\mathcal{C}(P_{r-1}, x)\mathcal{C}(P_{n-r}, x) - x^2\mathcal{C}(P_{r-3}, x)x\mathcal{C}(P_{n-r-2}, x) \\
&= x(1+x)^m\mathcal{C}(P_{r-1}, x)\mathcal{C}(P_{n-r}, x) - x^3\mathcal{C}(P_{r-3}, x)\mathcal{C}(P_{n-r-2}, x).
\end{aligned}$$

Since the product of $\mathcal{C}(H_X, x)$ and $\mathcal{C}(H_Y, x)$ gives the TD-Polynomial of $P_{2n} \oplus K_{1,m}$, the proof follows. □

Remark 3.3.8. *Since the graph $P_{2n} \oplus K_{1,m}$ in Theorem 3.3.7 is isomorphic to the graph G obtained by identifying the root vertex of $K_{1,m}$ with a vertex in $\{4, 6, 8, \ldots, 2n-2\}$ of the path P_{2n}, the TD-Polynomial of $P_{2n} \oplus K_{1,m}$ and that of G are same.*

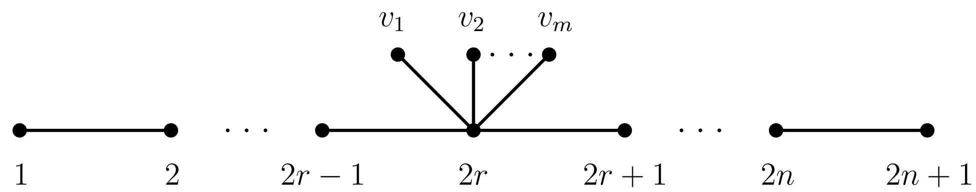

Figure 3.10: $P_{2n+1} \oplus K_{1,m}$

Theorem 3.3.9. *The TD-Polynomial of the graph $P_{2n+1} \oplus K_{1,m}$ shown in figure 3.10 is $\mathcal{C}(H_X, x)\mathcal{C}(H_Y, x)$, where $\mathcal{C}(H_X, x) = x^3\mathcal{C}(P_{r-2}, x)\mathcal{C}(P_{n-r-2}, x)$ and $\mathcal{C}(H_Y, x) = (1+x)^m\mathcal{C}(P_r, x)\mathcal{C}(P_{n-r+1}, x) - x^2\mathcal{C}(P_{r-2}, x)\mathcal{C}(P_{n-r-1}, x)$.*

Proof. Let $X = \{1, 3, 5, \ldots, 2n+1\} \cup V'$ and $Y = \{2, 4, 6, \ldots, 2n\}$ be the bipartition of $P_{2n+1} \oplus K_{1,m}$ and H_X, H_Y be the components of $ONH(P_{2n+1} \oplus K_{1,m})$. Then $E(H_X) = E(H_1)$ and $E(H_Y) = E(H_2) \cup E(H_3) \cup \{\{2r-1, 2r+1, v_1, v_2, \ldots, v_m\}\}$, where the graphs H_1, H_2 and H_3 are shown in figure 3.11.

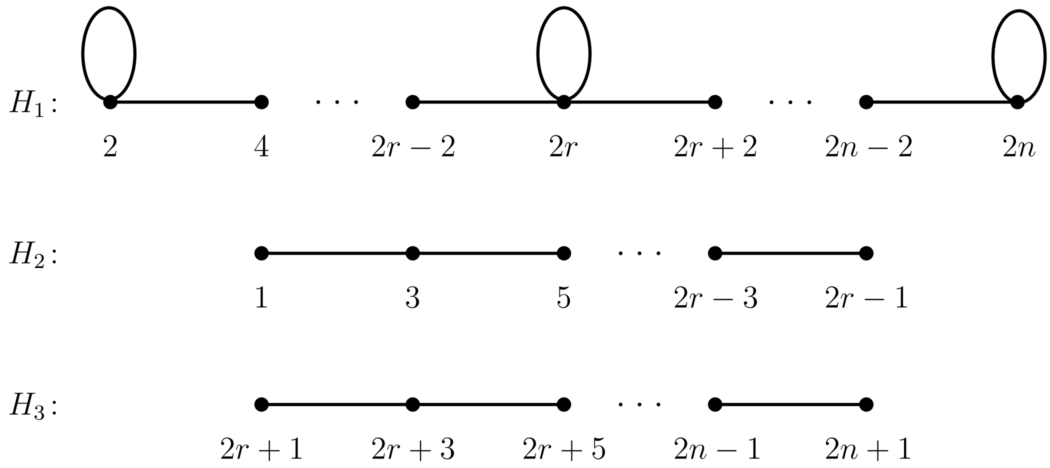

Figure 3.11: The graphs H_1, H_2 and H_3

Therefore, $\mathcal{C}(H_X, x) = \mathcal{C}(H_1, x) = x^3\mathcal{C}(P_{r-2}, x)\mathcal{C}(P_{n-r-2}, x)$. Let S be a vertex covering set of H_Y. If $S \cap \{2r-1, 2r+1\} = \phi$, then $S \cap V' \neq \phi$. So,

$$\begin{aligned}
\mathcal{C}(H_Y, x) &= \mathcal{C}^{\{2r-1,2r+1\}}(H_2 \cup H_3, x)(1+x)^m \\
&+ \mathcal{C}_{\{2r-1,2r+1\}}(H_2 \cup H_3, x)\left[(1+x)^m - 1\right] \\
&= \mathcal{C}(H_2 \cup H_3, x)(1+x)^m - \mathcal{C}_{\{2r-1,2r+1\}}(H_2 \cup H_3, x) \\
&= \mathcal{C}(H_2, x)\mathcal{C}(H_3, x)(1+x)^m - \mathcal{C}_{\{2r-1\}}(H_2, x)\mathcal{C}_{\{2r+1\}}(H_3, x) \\
&= \mathcal{C}(P_r, x)\mathcal{C}(P_{n-r+1}, x)(1+x)^m - \mathcal{C}(P_{r-2}, x)\mathcal{C}(P_{n-r-1}, x)
\end{aligned}$$

Since $D_t(P_{2n+1} \oplus K_{1,m}, x) = \mathcal{C}(H_X, x)\mathcal{C}(H_Y, x)$, the proof follows. □

Theorem 3.3.10. *The TD-Polynomial of the graph $P_{2n+1} \oplus K_{1,m}$ shown in figure 3.12 is $\mathcal{C}(H_X, x)\mathcal{C}(H_Y, x)$, where $\mathcal{C}(H_X, x) = x\mathcal{C}(P_r, x)\mathcal{C}(P_{n-r}, x)$ and $\mathcal{C}(H_Y, x) = x^2(1+x)^m\mathcal{C}(P_{r-1}, x)\mathcal{C}(P_{n-r-1}, x) - x^4\mathcal{C}(P_{r-3}, x)\mathcal{C}(P_{n-r-3}, x)$.*

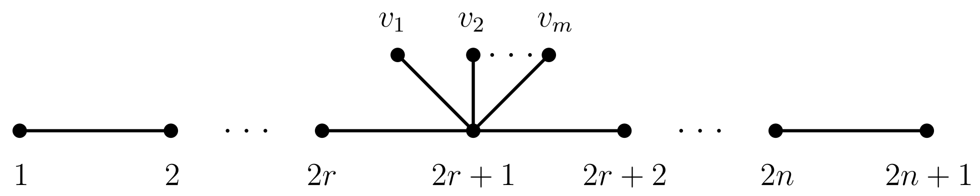

Figure 3.12: $P_{2n+1} \oplus K_{1,m}$

Proof. Let $X = \{1, 3, 5, \ldots, 2n+1\}$ and $Y = \{2, 4, 6, \ldots, 2n\} \cup V'$ be the bipartition of $P_{2n+1} \oplus K_{1,m}$ and H_X, H_Y be the components of $ONH(P_{2n+1} \oplus K_{1,m})$. Then $E(H_X) = E(H_1) \cup E(H_2) \cup \{\{2r, 2r+2, v_1, v_2, \ldots, v_m\}\}$ and $E(H_Y) = E(H_3)$, where the graphs H_1, H_2 and H_3 are shown in figure 3.13.

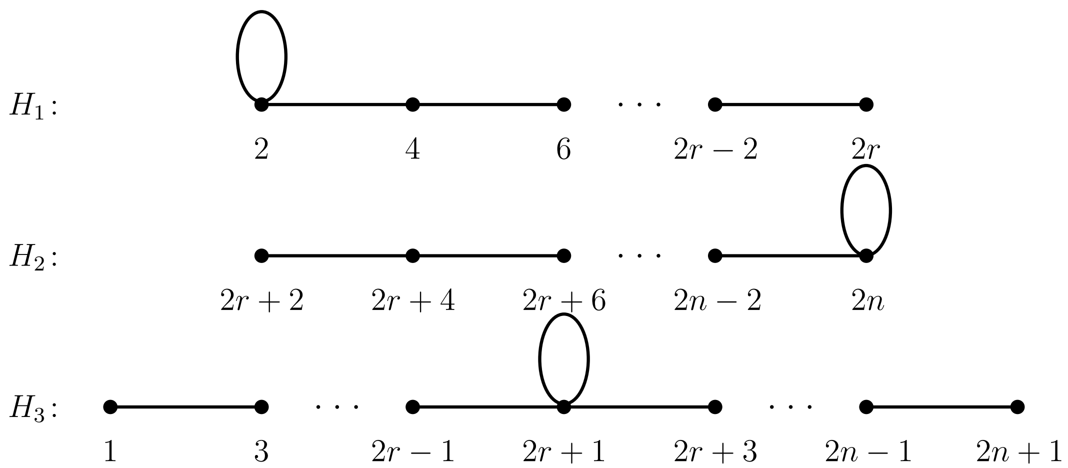

Figure 3.13: The graphs H_1, H_2 and H_3

So, $\mathcal{C}(H_Y, x) = \mathcal{C}(H_3, x) = x\mathcal{C}(P_r, x)\mathcal{C}(P_{n-r}, x)$. Let S be a vertex covering set of H_X. If $S \cap \{2r, 2r+2\} = \phi$, then $S \cap V' \neq \phi$. Therefore,

$$\begin{aligned}\mathcal{C}(H_X, x) &= \mathcal{C}^{\{2r,2r+2\}}(H_1 \cup H_2, x)(1+x)^m \\ &+ \mathcal{C}_{\{2r,2r+2\}}(H_2 \cup H_3, x)\,[(1+x)^m - 1]\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{C}(H_1 \cup H_2, x)(1+x)^m - \mathcal{C}_{\{2r,2r+2\}}(H_1 \cup H_2, x) \\
&= \mathcal{C}(H_1, x)\mathcal{C}(H_2, x)(1+x)^m - \mathcal{C}_{\{2r\}}(H_1, x)\mathcal{C}_{\{2r+2\}}(H_2, x) \\
&= x^2\mathcal{C}(P_{r-1}, x)\mathcal{C}(P_{n-r-1}, x)(1+x)^m - x^4\mathcal{C}(P_{r-3}, x)\mathcal{C}(P_{n-r-3}, x).
\end{aligned}
$$

Since $D_t(P_{2n+1} \oplus K_{1,m}, x) = \mathcal{C}(H_X, x)\mathcal{C}(H_Y, x)$, the proof follows. □

Chapter 4

TD-Polynomials of some graphs

4.1 Introduction

It is known that the concept of total domination in graphs can be converted to the concept of vertex cover in hypergaphs. In this chapter, we find the total domination polynomials of some graphs using the hypergraph terminology. The third section of this chapter deals with the total domination polynomials of total and middle graphs of some classes of graphs.

We need the following to proceed.

Definition 4.1.1. *(see [11]) A graph G is said to be an m-partite graph, if its vertex set can be partitioned into m subsets so that no edge has both ends in any one subset. A complete m-partite graph, denoted by $K_{n_1,n_2,\ldots,n_m}$, is a graph in which each vertex is joined to every vertex that is not in the same sub set. If $n_i = n$ for every i, then it is denoted by $K_{m[n]}$.*

[1] This chapter has been published in *Far East Journal of Mathematical Sciences* Volume 13, Number 10, (2017), 2277 – 2289.

Definition 4.1.2. *(see [7]) The* caterpillar graph $T(m_1, m_2, \ldots, m_n)$ *is obtained from a path* P_n *with* $n \geq 2$*, by attaching the central vertex of the star graph* K_{1,m_i} $(1 \leq i \leq n)$ *to the* i*-th vertex of the path* P_n*.*

Definition 4.1.3. *(see [14]) Let* Γ *be a finite group with identity* e*. Let* $S \subseteq \Gamma$ *such that* $e \notin S$ *and* $S = S^{-1}$*, that is,* S *is inverse closed. Then the* Cayley graph *[20]* $G = Cay(\Gamma, S)$*, is defined as a graph with vertex set* $V(G) = \Gamma$ *and edge set* $E(G) = \{ab \colon ab^{-1} \in S\}$*. Let* Γ *be a group and* $S \subset \Gamma$ *such that* $H = \overline{S}$ *is a subgroup of* Γ*. Then* $Cay(\Gamma, S)$ *is called* subgroup complementary Cayley graph *[14] denoted by* $\mathcal{SC}(\Gamma, H)$*.*

Definition 4.1.4. *The centipede* P_n^* *with* $2n$ *vertices is obtained by appending a single pendant edge to each vertex of a path* P_n*.*

Theorem 4.1.5. *(see [5]) If a graph* G *consists of* m *components* $G_1, G_2, \ldots, G_m$*, then* $D(G, x) = D(G_1, x) \ldots D(G_m, x)$*.*

Theorem 4.1.6. *(see [1]) For a complete graph* K_n*,* $D(K_n, x) = (1 + x)^n - 1$*.*

Theorem 4.1.7. *(see [47]) For a complete graph* K_n*,*

$$D_t(K_n, x) = (1 + x)^n - 1 - nx.$$

Theorem 4.1.8. *(see [14]) Let* Γ *be a finite group and* H *be a subgroup of* Γ *with* $o(H) = n$*. Then* $\mathcal{SC}(G, H) = K_{m[n]}$*, where* $m = [\Gamma \colon H]$*.*

4.2 Main results

In this section, the total domination polynomials of some graphs are determined easily using hypergraphs. Using the hypergraph terminology, we can easily prove Theorem 4.2.1 due to [45] and Corollary 4.2.3, Theorem 4.2.9 due to [12].

Theorem 4.2.1. *For a complete m- partite graph, $K_{n_1,n_2,\ldots,n_m}$,*

$$\begin{aligned} D_t(K_{n_1,n_2,\ldots,n_m},x) &= D_t(K_N,x) - \sum_{i=1}^{m} D_t(K_{n_i},x). \\ &= D(K_N,x) - \sum_{i=1}^{m} D(K_{n_i},x), \text{ where } N = \sum_{i=1}^{m} n_i. \end{aligned}$$

Proof. For $i = 1, 2, \ldots, m$, let A_i be the partite sets of $K_{n_1,n_2,\ldots,n_m}$ and let $A = \bigcup_{i=1}^{m} A_i$. Then for each vertex v, $N(v) = A \setminus A_i$ for some i. Therefore, a set S is a total dominating set if and only if $S \cap (A \setminus A_i) \neq \phi$ for every i. If $S \cap (A \setminus A_i) = \phi$ for some i, then $S \subseteq A_i$. If $|S| = k$, then for each i, there are $\binom{n_i}{k}$ subsets S such that $S \subseteq A_i$. Therefore, for each k, the number of non total dominating sets of the complete $m-$partite graph is $\sum_{i=1}^{m} \binom{n_i}{k}$. So $d_t(K_{n_1,n_2,\ldots,n_m},k) = d_t(K_N,k) - \sum_{i=1}^{m} d_t(K_{n_i},k)$. Therefore,

$$\begin{aligned} D_t(K_{n_1,n_2,\ldots,n_m},x) &= \sum_{k=2}^{N} d_t(K_N,k)x^k - \sum_{k=2}^{N} \left(\sum_{i=1}^{m} d_t(K_{n_i},k) \right) x^k \\ &= \sum_{k=2}^{N} \binom{N}{k} x^k - \sum_{i=1}^{m} \left(\sum_{k=2}^{N} \binom{n_i}{k} \right) x^k \\ &= D_t(K_N,x) - \sum_{i=1}^{m} D_t(K_{n_i},x) \\ &= D_t(K_N,x) + Nx - \sum_{i=1}^{m} (D_t(K_{n_i},x) + n_i x) \end{aligned}$$

$$= D(K_N, x) - \sum_{i=1}^{m} D(K_{n_i}, x).$$

This completes the proof. □

Corollary 4.2.2. *For the complete m-partite graph $K_{m[n]}$,*

$$\begin{aligned} D_t(K_{m[n]}, x) &= D(K_{mn}, x) - mD(K_n, x) \\ &= (1+x)^{mn} - m(1+x)^n + m - 1. \end{aligned}$$

Proof. The proof follows immediately from Theorem 4.2.1. □

Corollary 4.2.3. *For the complete bipartite graph $K_{m,n}$,*

$$\begin{aligned} D_t(K_{m,n}, x) &= D_t(K_{m+n}, x) - D_t(K_m, x) - D_t(K_n, x) \\ &= \left[(1+x)^{m+n} - 1 - (m+n)x\right] - \\ &\quad [(1+x)^m - 1 - mx] - [(1+x)^n - 1 - nx] \\ &= (1+x)^{m+n} - (1+x)^m - (1+x)^n + 1. \end{aligned}$$

Proof. The proof follows from Theorem 4.2.1. □

Corollary 4.2.4. *For the star graph $K_{1,n}$, $D_t(K_{1,n}, x) = x\left[(1+x)^n - 1\right].$*

Proof. The proof follows by substituting $m = 1$ in Corollary 4.2.3. □

Theorem 4.2.5. *For an $(n$-$1)$-regular bipartite graph G on $2n$ vertices,*

$$D_t(G, x) = [D_t(K_n, x)]^2 = [(1+x)^n - 1 - nx]^2.$$

Proof. Let $V = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$ and $E = \{a_i b_j : i \neq j\}$ be the vertex set and edge set of G. Let H_1 and H_2 be two complete graphs with vertex set $V = \{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ respectively. Then for $1 \leq i \leq n$, we have $N_G(a_i) = N_{H_2}(b_i)$ and $N_G(b_i) = N_{H_1}(a_i)$. Since G is bipartite, the open neighborhood hypergraph of G has two components. It can be observed that one component of $ONH(G)$ is isomorphic to $ONH(H_1)$ and the other is isomorphic to $ONH(H_2)$. Therefore, a set $S \subseteq V(G)$ is a total dominating set of G, if and only if S is a total dominating set of H_1 and H_2. Therefore, by Theorems 2.1.14 and 4.1.7 we have, $D_t(G,x) = D_t(H_1,x)D_t(H_2,x) = [D_t(K_n,x)]^2$. This completes the proof. □

Corollary 4.2.6. *If* $S = \{(1,1),(1,2),\ldots,(1,n\text{-}1)\}$, *then*

$$D_t\left(Cay(\mathbb{Z}_2\square\mathbb{Z}_n, S), x\right) = [D_t(K_n,x)]^2.$$

Proof. Since $Cay(\mathbb{Z}_2\square\mathbb{Z}_n, S)$ is an (n-1)-regular bipartite graph, the proof follows from Theorem 4.2.5. □

Theorem 4.2.7. *Let* Γ *be a group of order* n *and* $S \subseteq \Gamma$ *such that* $H = \overline{S}$ *is a subgroup of* Γ. *Then the total domination polynomial of the subgroup complementary cayley graph* $\mathcal{SC}(\Gamma, H) = Cay(\Gamma, S)$ *is*

$$D_t(Cay(\Gamma,S),x) = D_t(K_n,x) - mD_t(K_{|H|},x), \text{ where } m = [\Gamma : H].$$

Proof. The proof follows from Theorem 4.1.8 and Corollary 4.2.2. □

Theorem 4.2.8. *Let* H *be a subgroup of* $\mathbb{Z}_n$ *and* G *be a bipartite graph with vertex set* $V = \{a_0, a_1, a_2, \ldots, a_{n-1}\} \cup \{b_0, b_1, b_2, \ldots, b_{n-1}\}$ *such that* a_i *is not* *adjacent to b_j if and only if $i, j \in xH$ for some $x \in \mathbb{Z}_n$. Then,*

$$D_t(G, x) = [D_t(\mathcal{SC}(\mathbb{Z}_n, H), x)]^2 .$$

Proof. We construct two graphs H_1 and H_2 with vertex sets $V(H_1) = \{a_0, a_1, \ldots, a_{n-1}\}$ and $V(H_2) = \{b_0, b_1, \ldots, b_{n-1}\}$ such that a_i is not adjacent to a_j and b_i is not adjacent to b_j if and only if $i, j \in xH$ for some $x \in \mathbb{Z}_n$. Then the graphs H_1 and H_2 are isomorphic to $\mathcal{SC}(\mathbb{Z}_n, H)$. Therefore, $D_t(H_1, x) = D_t(H_2, x) = D_t(\mathcal{SC}(\mathbb{Z}_n, H)$. Since G is bipartite, the open neighborhood hypergraph of G has two components and they are isomorphic to the open neighborhood hypergraphs of H_1 and H_2 respectively. Therefore, $D_t(G, x) = D_t(H_1, x)D_t(H_2, x) = [D_t(\mathcal{SC}(\mathbb{Z}_n, H), x)]^2$. This completes the proof. □

Theorem 4.2.9. *Let G be connected graph with n vertices, then*

$$D_t(G \circ K_1, x) = x^n(1 + x)^n.$$

Proof. Let $V(G) = \{1, 2, 3, \ldots, n\}$ and $a_1, a_2, a_3, \ldots, a_n$ be the new vertices of $G \circ K_1$ such that $N(a_i) = \{i\}$ for $i = 1, 2, 3, \ldots, n$. So, a set S of vertices of $G \circ K_1$ is a total dominating set of $G \circ K_1$ if and only if $\{1, 2, 3, \ldots, n\} \subseteq S$. Therefore, $D_t(G \circ K_1, x) = x^n + \binom{n}{1}x^{n+1} + \binom{n}{2}x^{n+2} + \ldots + \binom{n}{n}x^{n+n} = x^n(1 + x)^n$. This completes the proof. □

Theorem 4.2.10. *Let $G(m_1, m_2, \ldots, m_n)$ is the graph obtained from a connected graph G with $n \geq 2$, by attaching the root vertex of the star graph K_{1,m_i}, $(1 \leq i \leq n)$ to the i-th vertex of the graph G. If $N = \sum_{i=1}^{n} m_i$, then the total domination polynomial of $G(m_1, m_2, \ldots, m_n)$ is $D_t(G(m_1, m_2, \ldots, m_n), x) = x^n(1 + x)^N$.*

Proof. If v is a pendant vertex in $G(m_1, m_2, \ldots, m_n)$ and u is the vertex adjacent to it, then $N(v) = \{u\}$. So if S is a TD-set of $G(m_1, m_2, \ldots, m_n)$, then $V(G) \subseteq S$. Since G is connected, $V(G)$ is TD-set of $G(m_1, m_2, \ldots, m_n)$. So a TD-set with $n+i$ elements can be selected in $\binom{N}{i}$ ways. Therefore, $D_t(G(m_1, m_2, \ldots, m_n), x) = x^n(1+x)^N$. This completes the proof. □

Corollary 4.2.11. *Let* $N = \sum_{i=1}^{n} m_i$, *then the TD-polynomial of the caterpillar graph* $T(m_1, m_2, \ldots, m_n)$ *is,* $D_t(T(m_1, m_2, \ldots, m_n), x) = x^n(1+x)^N$.

Proof. The caterpillar graph $T(m_1, m_2, \ldots, m_n)$ is obtained from a path P_n with $n \geq 2$, by attaching the central vertex of the star graph K_{1,m_i} $(1 \leq i \leq n)$ to the i-th vertex of the path P_n. So the proof follows immediately if we take G as P_n in Theorem 4.2.10. □

Corollary 4.2.12. *For a graph G with n vertices, the total domination polynomial of* $G \circ \overline{K}_m$ *is* $x^n (1+x)^{mn}$.

Proof. Note that the graph $G \circ \overline{K}_m$ is obtained from G and $\mid V(G) \mid$ copies of the star graph $K_{1,m}$ by identifying the root vertex of the i^{th} copy of $K_{1,m}$ with the i^{th} vertex of G. Therefore, the proof follows from Theorem 4.2.10. □

Corollary 4.2.13. *The total domination polynomial of the centipede P_n^* is*

$$D_t(P_n^*, x) = x^n(1+x)^n.$$

Proof. Observe that the centipede P_n^* is $P_n \circ \overline{K}_1$. Therefore, the proof follows from Corollary 4.2.12. □

Corollary 4.2.14. *If $B_{m,n}$ is the bi-star graph, then* $D_t(B_{m,n}, x) = x^2(1+x)^{m+n}$.

Proof. Let's label the vertices of $B_{m,n}$ as shown in figure 4.1.

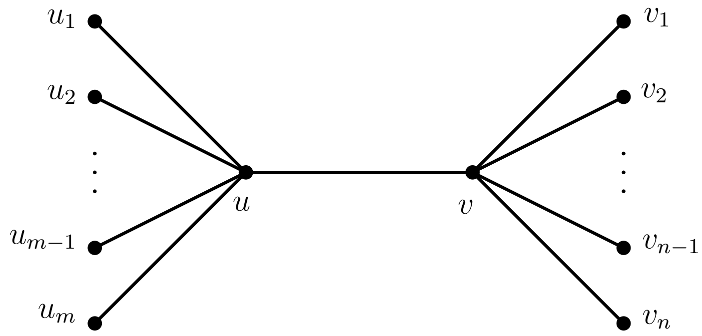

Figure 4.1: The Graph $B_{m,n}$

It can be observed that $B_{m,n}$ is obtained by identifying the vertex u of the graph K_2 with the root vertex of the star $K_{1,m}$ and the vertex v of K_2 with the root vertex of $K_{1,n}$. Therefore, the proof follows from Theorem 4.2.10. □

Theorem 4.2.15. *Let G and H be graphs of order m and n, respectively, then*

$$D_t(G \vee H, x) = [(1+x)^m - 1]\,[(1+x)^n - 1] + D_t(G, x) + D_t(H, x).$$

Proof. If $S \subseteq V(G) \cup V(H)$, such that $S \cap V(G) \neq \phi$ and $S \cap V(H) \neq \phi$, then S is a TD-set of $G \vee H$. Moreover if S is a TD-set of G or H, then S is a TD-set of $G \vee H$. Therefore, $D_t(G \vee H, x) = [(1+x)^m - 1]\,[(1+x)^n - 1] + D_t(G, x) + D_t(H, x)$. □

4.3 TD-Polynomials of total and middle graphs of some graphs

In this section, we obtain the total domination polynomials of *total graph* and *middle graph* of some graph classes.

Here we need the following.

Definition 4.3.1. *(see [28]) If G is a graph, the* total graph $T(G)$ *of G is the graph with vertex set $V(G) \cup E(G)$ in which two vertices are adjacent if they are either adjacent or incident in G. For a graph G, the* middle graph $M(G)$ *of G is the graph with vertex set $V(G) \cup E(G)$ in which two vertices are adjacent if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.*

Theorem 4.3.2. $D_t\left(T(K_{1,n}), x\right) = x\left[(1+x)^{2n} - 1\right] + x^n\left[(1+x)^n\right].$

Proof. Let$\{u\} \cup \{1, 2, 3, \ldots, n\}$ be the bipartition of the star graph $K_{1,n}$ and $E = \{e_1, e_2, e_3, \ldots, e_n\}$ be the edge set. Then the open neighborhoods of the vertices of $T(K_{1,n})$ are, for $1 \leq i \leq n$, $N(i) = \{u, e_i\}$, $N(e_i) = \{i, u\} \cup E \setminus \{e_i\}$ and $N(u) = \{1, 2, 3, \ldots, n, e_1, e_2, e_3, \ldots, e_n\}$

Let S be a total dominating set of $T(K_{1,n})$. We consider the following cases.

Case 1: Let $u \in S$.

For any $x \in V(T(K_{1,n}))$, the set $\{u, x\}$ is a TD-set of $T(K_{1,n})$. Therefore, the number of TD-sets of cardinality $i+1$ containing the vertex u is $\binom{2n}{i}$.

Case 2: Let $u \notin S$.

Since $N(i) = \{u, e_i\}$, for all i, S is a TD-set if and only if $E \subseteq S$. From the remaining n vertices $\{1, 2, 3, \ldots, n\}$, i vertices can be selected in $\binom{n}{i}$ ways.

Therefore,

$$D_t\left(T(K_{1,n}), x\right) = x\left[\binom{2n}{1}x + \binom{2n}{2}x^2 + \ldots + \binom{2n}{2n}x^{2n}\right]$$

$$
\begin{aligned}
&+ \quad x^n \left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right] \\
&= \quad x\left[(1+x)^{2n} - 1\right] + x^n\left[(1+x)^n\right].
\end{aligned}
$$

This completes the proof. □

Theorem 4.3.3. $D_t\left(M(K_{1,n}), x\right) = x^n(1+x)^{n+1}$.

Proof. Let$\{u\} \cup \{1, 2, 3, \ldots, n\}$ be the bipartition of the star graph $K_{1,n}$ and $E = \{e_1, e_2, e_3, \ldots, e_n\}$ be the edge set. Then the open neighborhoods of the vertices of $M(K_{1,n})$ are, $N(u) = \{e_1, e_2, \ldots, e_n\}$, $N(e_i) = \{i, u\} \cup E \setminus \{e_i\}$ and $N(i) = \{e_i\}$. So, a set S of vertices of $M(K_{1,n})$ is a TD-set if and only if $E \subseteq S$. From the remaining $n+1$ vertices, i vertices can be selected in $\binom{n+1}{i}$ ways. Therefore, $D_t\left(M(K_{1,n}), x\right) = x^n(1+x)^{n+1}$. Thus the proof is complete. □

Theorem 4.3.4. *If G is connected graph of order n and size l, then the total domination polynomial of the middle graph of $G(m_1, m_2, \ldots, m_n)$ is*

$$D_t(M(G(m_1, m_2, \ldots, m_n)), x) = x^N(1+x)^{N+n+l}, \text{ where } N = \sum_{i=1}^{n} m_i.$$

Proof. If v is a pendant vertex in $G(m_1, m_2, \ldots, m_n)$ and e, the edge incident with it, then $N_{M(G(m_1,m_2,\ldots,m_n))}(v) = \{e\}$. Let E be the set of all pendant edges of $G(m_1, m_2, \ldots, m_n)$. Therefore, if S is a TD-set of $M(G(m_1, m_2, \ldots, m_n))$, then $E \subseteq S$. Clearly E is a TD-set of $M(G(m_1, m_2, \ldots, m_n))$. So, from the remaining $N+n+l$ vertices in $M(G(m_1, m_2, \ldots, m_n))$, i vertices can be selected in $\binom{N+n+l}{i}$ ways. Therefore, $D_t(M(G(m_1, m_2, \ldots, m_n)), x) = x^N(1+x)^{N+n+l}$. Thus the proof is complete. □

Corollary 4.3.5. *The TD-polynomial of middle graph of the bi-star graph $B_{m,n}$ is $D_t\left(M(B_{m,n}), x\right) = x^{m+n}(1+x)^{m+n+3}$.*

Proof. The bi-star graph $B_{m,n}$ is obtained from K_2 by attaching the central vertices of the star graphs $K_{1,m}$ and $K_{1,n}$ to the first and second vertices of K_2 respectively. Therefore, the proof follows immediately from Theorem 4.3.4. □

Corollary 4.3.6. *Let G be a connected graph of order m and size l and if $n \geq 2$, then the total domination polynomial of the middle graph of $G \circ \overline{K}_n$ is* $D_t\left(M(G \circ \overline{K}_n), x\right) = x^{mn}(1+x)^{m+l+n}$.

Proof. The graph $G \circ \overline{K}_n$ is obtained from G by attaching the central vertices of n copies of the star graph $K_{1,n}$ to the vertices of G. So replacing m_i by n for all i in Theorem 4.3.4, we obtain the result. □

We take $N = \sum_{i=1}^{n} m_i$ for Corollary 4.3.7, 4.3.8 and 4.3.9.

Corollary 4.3.7. *The TD-polynomial of middle graph of the caterpillar graph is,* $D_t(M(T(m_1, m_2, \ldots, m_n)), x) = x^N(1+x)^{N+2n-1}$.

Proof. In Theorem 4.3.4, if we take G as P_n, we obtain the result. □

Corollary 4.3.8. *For the caterpillar graph $T(m_1, m_2, \ldots, m_n)$,*

$$D_t(M(T(m_1, m_2, \ldots, m_n)), x) = x^{N-n}(1+x)^{2n-1} D_t(T(m_1, m_2, \ldots, m_n), x).$$

Proof. The proof follows from 4.2.11 and 4.3.7. □

Corollary 4.3.9. *If G is the cycle C_n, then*

$$D_t(M(G(m_1, m_2, \ldots, m_n)), x) = x^{N-n}(1+x)^{2n} D_t(G(m_1, m_2, \ldots, m_n), x).$$

Proof. From Theorems 4.2.10 and 4.3.4 we have, $D_t(C_n(m_1, m_2, \ldots, m_n), x) = x^n(1+x)^N$ and $D_t(M(C_n(m_1, m_2, \ldots, m_n)), x) = x^N(1+x)^{N+2n}$. Hence the result follows. $\square$

Chapter 5

Total Domination Polynomials of Cartesian Products of some graphs

5.1 Introduction

In this chapter, we determine the total domination polynomials of Cartesian products of certain classes of graphs with K_2. We establish an interesting relation between domination polynomials and total domination polynomials of some graph classes. Moreover, we determine the total domination polynomials of the Cartesian product of some graphs with the cycle C_4. Further, the total domination number of some graphs are also determined. We need the following theorems for our further discussions.

Theorem 5.1.1. *(see [1]) If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then $D(G,x) = D(G_1,x)\ldots D(G_m,x)$.*

[1]A part of this chapter has been published in *Journal of Pure and Applied Mathematics: Advances and Applications*, Volume 16, Number 2, 2016, Pages 97-108.

Theorem 5.1.2. *(see [1]) For every* $n \geq 4$,

$$D(P_n, x) = x\left[D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x)\right],$$

with initial values $D(P_1, x) = x,\ D(P_2, x) = x^2 + 2x,\ D(P_3, x) = x^3 + 3x^2 + x.$

Theorem 5.1.3. *(see [1]) For every* $n \geq 4$,

$$D(C_n, x) = x\left[D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x)\right],$$

with initial values $D(C_1, x) = x,\ D(C_2, x) = x^2 + 2x,\ D(C_3, x) = x^3 + 3x^2 + 3x.$

Theorem 5.1.4. *(see [1])* $D(K_{m,n}, x) = ((1+x)^m - 1)((1+x)^n - 1) + x^m + x^n.$

Theorem 5.1.5. *(see [1]) If there exists* $1 \leq j \leq k$*, such that* $s_j \geq 3$*, then*

$$\begin{aligned} D(\theta_{s_1,s_2,\ldots,s_j,\ldots,s_k}, x) &= x[D(\theta_{s_1,s_2,\ldots,s_{j-1},\ldots,s_k}, x) + D(\theta_{s_1,s_2,\ldots,s_{j-2},\ldots,s_k}, x) \\ &+ D(\theta_{s_1,s_2,\ldots,s_{j-3},\ldots,s_k}, x)]. \end{aligned}$$

Theorem 5.1.6. *(see [30]) If* L_n *is the graph* $P_n \Box K_2$*, then domination polynomial of* L_n *satisfies the recurrence* $D(L_n, x) = x(x+2)D(L_{n-1}, x) + x(x+1)D(L_{n-2}, x) + x^2(x+1)D(L_{n-3}, x) - x^3 D(L_{n-5}, x)$ *with initial values,*

n	$P_n \Box K_2$
1	$x^2 + 2x$
2	$x^4 + 4x^3 + 6x^2$
3	$x^6 + 6x^5 + 15x^4 + 16x^3 + 3x^2$
4	$x^8 + 8x^7 + 28x^6 + 52x^5 + 48x^5 + 47x^4 + 2x^3$
5	$x^{10} + 10x^9 + 45x^8 + 116x^7 + 178x^6 + 148x^5 + 47x^4 + 2x^3$
6	$x^{12} + 12x^11 + 66x^{10} + 216x^9 + 453x^8 + 604x^7 + 470x^6 + 168x^5 + 17x^4$

5.2 On Cartesian products

In this section, we establish a relation between domination polynomials and total domination polynomials of some graphs. We prove that for any graph G, there exists a graph H such that the set of all open neighborhoods of vertices of $K_2\Box G$ is exactly the same as the set of all closed neighborhoods of vertices of H.

Theorem 5.2.1. *For a bipartite graph G, $D_t(K_2\Box G, x) = [D(G,x)]^2$.*

Proof. Let $X = \{x_1, x_2, x_3, \ldots, x_m\}$, $Y = \{y_1, y_2, y_3, \ldots, y_n\}$ be the bipartition of G. If $V(K_2) = \{a, b\}$, then the bipartition of $K_2\Box G$ is $\{(a, x_1), (a, x_2), \ldots, (a, x_m), (b, y_1), (b, y_2), \ldots, (b, y_n)\}, \{(b, x_1), (b, x_2), \ldots, (b, x_m), (a, y_1), (a, y_2), \ldots, (a, y_n)\}$. Then, for $i = 1, 2, \ldots, m$ and for $j = 1, 2, \ldots, n$, we have,

$$\begin{aligned}
N((a, x_i)) &= \{(a, y)/y \sim x_i \text{ in } G\} \cup \{(b, x_i)\} \\
N((b, x_i)) &= \{(b, y)/y \sim x_i \text{ in } G\} \cup \{(a, x_i)\} \\
N((a, y_j)) &= \{(a, x)/x \sim y_j \text{ in } G\} \cup \{(b, y_j)\} \\
N((b, y_j)) &= \{(b, x)/x \sim y_j \text{ in } G\} \cup \{(a, y_j)\}.
\end{aligned}$$

Let H_1 be a bipartite graph with partite sets $\{(b, x_1), (b, x_2), \ldots, (b, x_m)\}$, and $\{(a, y_1), (a, y_2), \ldots, (a, y_n)\}$, such that $(b, x_i) \sim (a, y_j)$ if and only if $(a, x_i) \sim (a, y_j)$ in $K_2\Box G$. Similarly, we construct another bipartite graph H_2 with partite sets $\{(a, x_1), (a, x_2), \ldots, (a, x_m)\}, \{(b, y_1), (b, y_2), \ldots, (b, y_n)\}$, such that $(a, x_i) \sim (b, y_j)$ if and only if $(b, x_i) \sim (b, y_j)$ in $K_2\Box G$. It can be observed that both H_1 and H_2 are isomorphic to G and a set N is an open neighborhood of a vertex in $K_2\Box G$ if and only if N is a closed neighborhood of a vertex in H_1 or H_2.

Therefore, a set $S \subseteq V(G)$ is a total dominating set of $K_2 \Box G$ if and only if it is a dominating set of $H_1 \cup H_2$. So, from Theorem 5.1.1 we have,

$$\begin{aligned} D_t(K_2 \Box G, x) &= D(G_1 \cup G_2, x) \\ &= D(G_1, x).D(G_2, x) \\ &= [D(G, x)]^2. \end{aligned}$$

This completes the proof. □

Corollary 5.2.2. *For a bipartite graph H, $\gamma_t(K_2 \Box H) = 2\ \gamma(H)$.*

Proof. By definition, $\gamma(H)$ is the smallest power of x in the domination polynomial and $\gamma_t(H)$ is the smallest power of x in the total domination polynomial of H. From Theorem 5.2.1, we have $D_t(K_2 \Box H, x) = [D(H, x)]^2$. So, the least power of x in $D_t(K_2 \Box H, x)$ is twice that of $D(H, x)$, which proves our result. □

Corollary 5.2.3. *From Theorem 5.2.1, we obtain the following results.*

1. $D_t(K_2 \Box P_n, x) = [D(P_n, x)]^2$.
2. $\gamma_t(K_2 \Box P_n) = 2.\gamma(P_n) = 2.\lceil \frac{n}{3} \rceil$
3. $D_t(K_2 \Box C_{2n}, x) = [D(C_{2n}, x)]^2$.
4. $\gamma_t(K_2 \Box C_{2n}) = 2.\gamma(C_{2n}) = 2.\lceil \frac{2n}{3} \rceil$
5. $D_t(K_2 \Box K_{m,n}, x) = [D(K_{m,n}, x)]^2$.
6. $\gamma_t(K_2 \Box K_{m,n}) = 4$.
7. $D_t(K_2 \Box B_{m,n}, x) = [D(B_{m,n}, x)]^2$, *where $B_{m,n}$ is the bi-star graph.*

8. *If T is a tree, then $D_t(K_2\Box T, x) = [D(T, x)]^2$.*

Proof. The proof is straight forward. □

Theorem 5.2.4. *If G is a non bipartite graph with n vertices, then there exists a bipartite graph H with $2n$ vertices such that $D_t(K_2\Box G, x) = D(H, x)$.*

Proof. Let $V(G) = \{1, 2, 3, \ldots, n\}$ and $V(K_2) = \{a, b\}$. Let H be a bipartite graph with vertex set $\{(a, 1), (a, 2), \ldots, (a, n)\} \cup \{(b, 1), (b, 2), \ldots, (b, n)\}$ such that (a, i) is adjacent to (b, j) if and only if i is adjacent to j in G. Then for $1 \leq i, j \leq n$, $N_{K_2\Box G}((a, i)) = N_H[(b, i)]$ and $N_{K_2\Box G}((b, j)) = N_H[(a, j)]$. Therefore, a TD-set of $K_2\Box G$ dominates H and if S dominates H, then it is a TD-set of $K_2\Box G$. This completes the proof. □

Lemma 5.2.5. *A set S is a dominating set of a cycle C_n if and only if it has non empty intersection with the set of vertices of each and every path of length two in C_n.*

Proof. Let S be a dominating set and $P = uvw$ be a path of length two in C_n. If $S \cap \{u, v, w\} = \phi$, then the vertex v is not adjacent to any vertex in S and so S cannot be a dominating set of C_n. The converse is obvious. □

Theorem 5.2.6. *If n is odd, then, $D_t(K_2\Box C_n, x) = D(C_{2n}, x)$.*

Proof. Let the vertices of $K_2\Box C_n$ be labeled as shown in figure 5.1. Let H_G be the open neighborhood hypergraph of $K_2\Box C_n$. Let C_{2n} be the cycle with vertex set $\{(r, 1), (r, 2), \ldots, (r, n), (t, 1), (t, 2), \ldots, (t, n)\}$ such that (r, i) is adjacent to (t, j) if and only if i is adjacent to j in C_n. It can be observed that if $v \in V(K_2\Box C_n)$, then the vertices of $N(v)$ form a path of length two in C_{2n}. Also the set of vertices of any path of length two in C_{2n} is an edge in H_G. Therefore, the proof follows from Theorem 2.1.14 and Lemma 5.2.5. $\square$

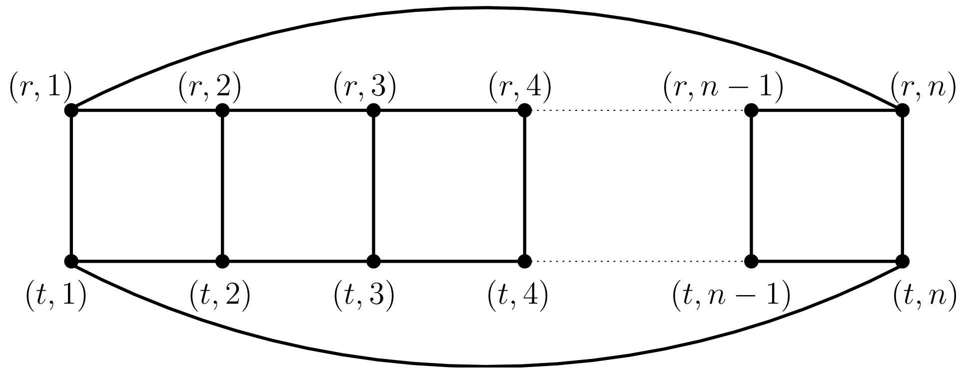

Figure 5.1: The graph $K_2 \square C_n$

5.3 TD-Polynomials of Cayley graphs

In this section, we find the total domination polynomials of some cubic cayley graphs.

Theorem 5.3.1. *Let* $G = Cay(\mathbb{Z}_n, S)$, *where* $S = \{a, b, b^{-1}\}$ *is a generating set of* $\mathbb{Z}_n$ *such that* $a^{-1} = a$ *and* $a \notin < b >$. *Then,*

$$D_t(G, x) = \begin{cases} \left[D(C_{\frac{n}{2}}, x)\right]^2, & \text{if } n/2 \text{ is even} \\ D(C_n, x), & \text{otherwise.} \end{cases}$$

Proof. Here $G \cong K_2 \square C_{\frac{n}{2}}$. So the proof follows from Corollary 5.2.3 and Theorem 5.2.6. $\square$

Theorem 5.3.2. *Let $G = Cay(\mathbb{Z}_n, S)$, where $S = \{a, b, b^{-1}\}$ such that $a^{-1} = a$ and b is a generator of $\mathbb{Z}_n$. Then,*

$$D_t(G,x) = \begin{cases} D(C_n, x), & \text{if } n \text{ is a multiple of 4} \\ \left[D(C_{\frac{n}{2}}, x)\right]^2, & \text{otherwise} \end{cases}$$

Proof. Since b is a generator of $\mathbb{Z}_n$, the graph $G = Cay(\mathbb{Z}_n, S)$ can be labeled as shown in figure 5.2.

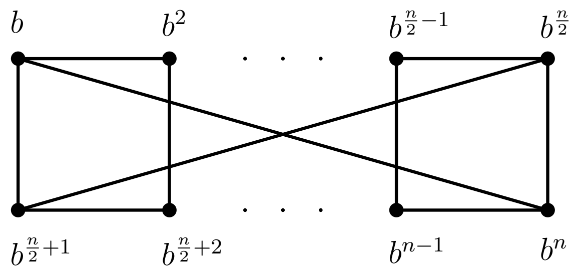

Figure 5.2: The graph $G = Cay(\mathbb{Z}_n, S)$

Case 1: When n is a multiple of 4.

Let C_n be the cycle $(b^n, b^{\frac{n}{2}+1}, b^2, b^{\frac{n}{2}+3}, b^4, \ldots, b^{n-2}, b^{\frac{n}{2}-1}, b^n)$. It can be observed that if $v \in V(G)$, then the vertices of $N(v)$ form a path of length two in C_n and any path of length two in C_n is an open neighborhood of a vertex in G. Therefore, by Lemma 5.2.5, $D_t(G, x) = D(C_n, x)$.

Case 2: When n is not a multiple of 4.

In this case the graph $G = Cay(\mathbb{Z}_n, S)$ is bipartite with bipartition $X = \{b^k \colon k \text{ is odd}\}$ and $Y = \{b^s \colon \text{ s is even}\}$. Let the components of the open neighborhood hypergraph of G are $H_X = (Y, \{N(x) \colon x \in X\})$ and $H_Y = (X, \{N(x) \colon x \in Y\})$. Since H_X is isomorphic to H_Y, by Theorem 2.1.14,

$D_t(G,x) = [\mathcal{C}(H_X,x)]^2$. Also $E(H_X) = \{N(b), N(b^{\frac{n}{2}+2}), N(b^3), \ldots, N(b^{\frac{n}{2}})\}$ $= \{\{b^n, b^{\frac{n}{2}+1}, b^2\}, \{b^{\frac{n}{2}+1}, b^2, b^{\frac{n}{2}+3}\}, \{b^2, b^{\frac{n}{2}+3}, b^4\}, \ldots, \{b^{\frac{n}{2}-1}, b^n, b^{\frac{n}{2}+1}\}\}$. Let $C_{\frac{n}{2}}$ be the cycle, $(b^n, b^{\frac{n}{2}+1}, b^2, b^{\frac{n}{2}+3}, b^4, \ldots, b^{\frac{n}{2}-1}, b^n)$, with vertex set Y. It is obvious that if $v \in Y$, then the vertices of $N_G(v)$ form a path of length two in $C_{\frac{n}{2}}$. Also any path of length two in $C_{\frac{n}{2}}$ is an open neighborhood of a vertex in Y. Therefore, by Lemma 5.2.5, $\mathcal{C}(H_X,x) = D(C_{\frac{n}{2}},x)$.

This completes the proof. □

5.4 TD-Polynomials of regular graphs

In this section, we investigate how the domination polynomials of some regular graphs and total domination polynomials of their Cartesian product with K_2 are related.

Lemma 5.4.1. *If G is an $(m-1)$-regular bipartite graph, then*

$$\begin{aligned} D(G,x) &= mx^2\left[1+\binom{2m-2}{1}x+\ldots+\binom{2m-2}{m-3}x^{m-1}\right] \\ &+ \left[m\binom{2m-2}{m-2}+2\right]x^m \\ &+ \binom{2m}{m+1}x^{m+1}+\binom{2m}{m+2}x^{m+2}+\ldots+\binom{2m}{2m}x^{2m}. \end{aligned}$$

Proof. Let $X = \{a_1, a_2, \ldots, a_m\}$ and $Y = \{b_1, b_2, \ldots, b_m\}$ be the bipartition of G. Assume that a_i is not adjacent to b_i for all i. Let S be a set of vertices of G. If $\{a_r, b_r\} \subseteq S$ for some r, then S is a dominating set of G. Since there are m pairs, one pair can be selected in m ways. So when $k < m$, the coefficient of x^k in $D(G,x)$ is $m\binom{2m-2}{k-2}$. When $k = m$, since X and Y are also dominating sets of G, the coefficient of x^m is $m\binom{2m-2}{m-2}+2$. When $k > m$, any subset of the vertices of G contains a pair a_i, b_i. So the coefficient of x^k in $D(G,x)$ is $\binom{2m}{k}$. This completes the proof. □

Theorem 5.4.2. $D_t(K_2\square K_n, x) = D(G,x)$, *where G is an $(n-1)$-regular bipartite graph on $2n$ vertices.*

Proof. Let the vertex sets of K_2 and K_n be $\{a,b\}$ and $\{1,2,3,\ldots,n\}$ respectively. If $A = \{(a,1),(a,2),\ldots,(a,n)\}$ and $B = \{(b,1),(b,2),\ldots,(b,n)\}$, then $V(K_2\square K_n) = A \cup B$. Also $N((a,i)) = \{(b,i)\} \cup A \setminus \{(a,i)\}$ and $N((b,i)) = \{(a,i)\} \cup B \setminus \{(b,i)\}$. We construct an ($n$-1)-regular bipartite graph G in which the open neighborhoods $N((a,i))$ and $N((b,i))$ are represented as star graphs with root vertices (b,i) and (a,i) respectively.

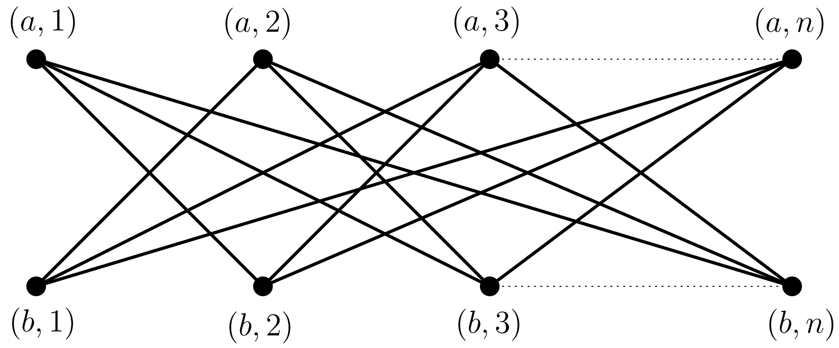

Figure 5.3: The graph G

Let S be a dominating set of G. Then for all i, either $(a,i) \in S$ or (a,i) is adjacent to some vertex in S. If $(a,i) \in S$ for all i, then $S \cap N((b,i)) \neq \phi$ for all i. If $(a,r) \notin S$ for some r, then there exists $(b,s) \in S$ for some $s \neq r$. Therefore, $S \cap N_{K_2\square K_n}((b,i)) \neq \phi$ for all i. Similarly we can prove that $S \cap$

$N_{K_2\square K_n}((a,i)) \neq \phi$ for all i. Therefore, S is a vertex covering set of the open neighborhood hypergraph $H_{K_2\square K_n}$. Conversely, let S is a vertex covering set of $H_{K_2\square K_n}$. We prove that S is a dominating set of G. Consider a vertex (a,i). If $(a,i) \in S$, then there is nothing to prove. If $(a,i) \notin S$, then $(b,j) \in S$ for some $j \neq i$. So (a,i) is adjacent to (b,j) in G. Similarly we can prove the case of (b,j) also. Therefore, S is a dominating set of G. Thus the result follows from Theorem 2.1.14. □

5.5 TD-Polynomials of friendship graphs

In this section, we determine the total domination polynomials of Cartesian product of friendship graphs with K_2.

Here we need the following.

Definition 5.5.1. *(see [1]) The friendship graph F_n with $2n+1$ vertices and $3n$ edges, is the graph formed by the join of K_1 with n copies of K_2.*

Theorem 5.5.2. $D_t(K_2\square F_n, x) = D(\theta_{\underbrace{3,3,\ldots,3}_{(2n\ times)}}, x)$.

Proof. Let $V(F_n) = \{u,1,2,3,\ldots,n\}$ such that $N(u) = \{1,2,3,\ldots,n\}$. For an even vertex x, $N(x) = \{u, x-1\}$ and for an odd vertex x, $N(x) = \{u, x+1\}$. Let $V(K_2) = \{a,b\}$. Then in $K_2\square F_n$,

$$\begin{aligned} N((a,u)) &= \{(b,u),(a,1),(a,2),\ldots,(a,n)\} \\ N((b,u)) &= \{(a,u),(b,1),(b,2),\ldots,(b,n)\} \\ \text{For even } x,\ N((a,x)) &= \{(a,u),(b,x),(a,x-1)\} \end{aligned}$$

$$\begin{aligned} N((b,x)) &= \{(b,u),(a,x),(b,x-1)\} \\ \text{For odd } x,\ N((a,x)) &= \{(a,u),(b,x),(a,x+1)\} \\ N((b,x)) &= \{(b,u),(a,x),(b,x+1)\} \end{aligned}$$

Since the *Friendship* graph F_n is not bipartite, by Theorem 5.2.4 there exists a bipartite graph H such that $D_t(K_2\Box F_n, x) = D(H, x)$. In H, we have,

$$\begin{aligned} N((a,u)) &= \{(b,1),(b,2),\ldots,(b,n)\} \\ N((b,u)) &= \{(a,1),(a,2),\ldots,(a,n)\} \\ \text{If } x \text{ is even, then } N((a,x)) &= \{(b,u),(b,x-1)\} \text{ and} \\ \text{If } x \text{ is odd, then } N((a,x)) &= \{(b,u),(b,x+1)\} \end{aligned}$$

Note that H is the *theta graph* $\theta_{\underbrace{3,3,\ldots,3}_{(2n\ times)}}$. It can be observed that for each vertex s of $K_2\Box F_n$ there exists a vertex t in H such that $N_{K_2\Box F_n}(s) = N_H[t]$. Therefore, $D_t(K_2\Box F_n, x) = D(H, x)$. This completes the proof. □

The construction of H in the case of F_2 is shown in figure 5.4.

5.6 Total domination polynomials of $C_4\Box G$

In this section, we determine the total domination polynomials of Cartesian product of certain classes of graphs with the cycle C_4.

Theorem 5.6.1. *For a bipartite graph G,*

$$D_t(C_4\Box G, x) = [D(K_2\Box G, x)]^2 .$$

Proof. Since the graph $K_2 \square K_2 \square G$ is isomorphic to $C_4 \square G$, the result follows from Theorem 2.1.14 and 5.2.1. □

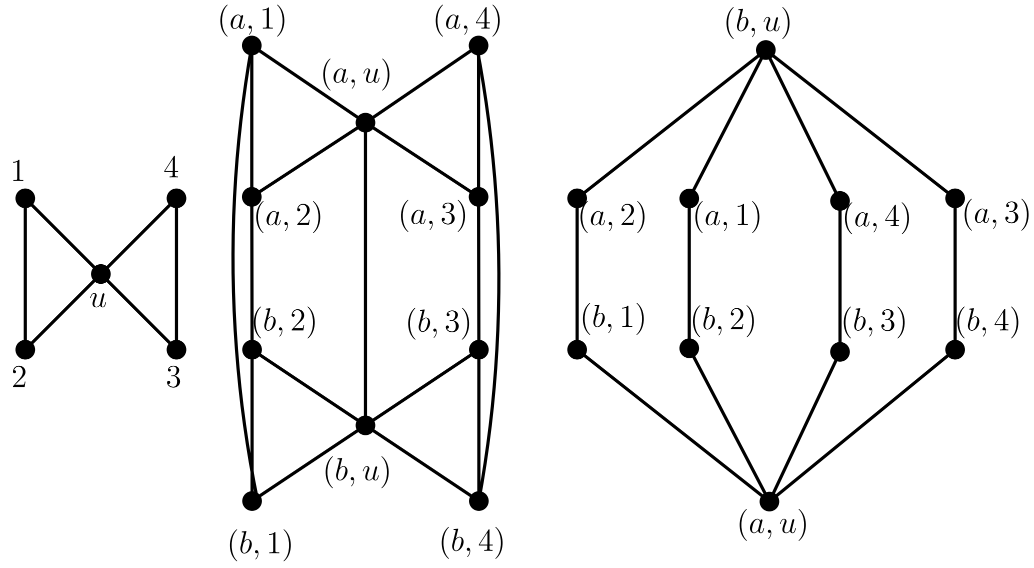

Figure 5.4: The graphs $F_2, K_2 \square F_2$ and H

Corollary 5.6.2. *From Theorem 5.6.1 we obtain the following results.*

(i) $D_t(C_4 \square P_n, x) = [D(L_n, x)]^2$,

(ii) $\gamma_t(C_4 \square P_n) = 2 \left\lceil \frac{n+1}{2} \right\rceil$,

(iii) $D_t(C_4 \square C_{2n}, x) = [D(K_2 \square C_{2n}, x)]^2$,

(iv) $\gamma_t(C_4 \square C_{2n}) = \begin{cases} 2n, & \text{if } n \text{ is even} \\ 2n+2, & \text{otherwise.} \end{cases}$

(v) $D_t(C_4 \square K_{1,n}, x) = [D(K_2 \square K_{1,n}, x)]^2$,

(vi) $\gamma_t(C_4 \square K_{1,n}) = 4$.

Chapter 6

TD-Polynomials of Splitting Graphs

6.1 Introduction

In this chapter, we are concerned with total domination polynomials of splitting graphs of order k of a graph G. Moreover, we introduce the terminology of iterated splitting graph $S^i(G)$ of a graph G and determine its total domination polynomial.

We need the following results.

Theorem 6.1.1. *(see [18]) For the path graph P_n, where $n > 1$, we have*
$\mathcal{C}(P_n, x) = \sum_{i=0}^{n} \binom{i+1}{n-i} x^i.$

Theorem 6.1.2. *(see [18]) For the cycle graph C_n, where $n \geq 3$, we have*
$\mathcal{C}(C_n, x) = \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i.$

Definition 6.1.3. *(see [43]) The splitting graph of a graph G is defined as, for*

[1] A part of this chapter has been published in *Advances and Applications in Discrete Mathematics*, Volume 18, Number 3, 2017, Pages 331-343.

each vertex v of G, take a new vertex $v^{'}$ and join $v^{'}$ to all vertices of G adjacent to v. The graph $spl(G)$ thus obtained is called the splitting graph of G.

Definition 6.1.4. *The splitting graph of order k of a graph G, denoted by $spl^k(G)$ is defined as for each vertex v of G, take k new vertices $v_1, v_2, \ldots, v_k$ and join each of these vertices to all vertices of G adjacent to v.*

Lemma 6.1.5. *If G is bipartite, then $spl^k(G)$ is bipartite.*

Proof. Let (X, Y) be the bipartition of G and $X^{'}$, $Y^{'}$ be collection of new vertices of $spl^k(G)$ corresponding to the vertices of X and Y respectively. Then $X \cup X^{'}$ and $Y \cup Y^{'}$ are the partite sets of $spl^k(G)$. This proves the result. □

6.2 On splitting graphs

It is noted that for a graph G, the TD- polynomials of splitting graph of a graph G is closely related to the total domination polynomial of G. In this section, the total domination polynomial of splitting graph of G is determined.

Theorem 6.2.1. *For a connected graph G with n vertices,*

$$D_t\left(spl^k(G), x\right) = D_t(G, x)(1 + x)^{nk}.$$

Proof. For a vertex v in G, let $v^1, v^2, \ldots, v^k$ be the new vertices in $spl^k(G)$. Then for $i = 1, 2, \ldots, k$, $N_{spl^k(G)}(v^i) = N_G(v)$. Also $N_{spl^k(G)}(v) \supseteq N_G(v)$. Therefore, if S is a total dominating set of G, then S is a total dominating set of $spl^k(G)$. From the construction of $spl^k(G)$, it can be observed that if K is a total dominating set of $spl^k(G)$, then $K \cap V(G)$ is a total dominating set of G. Since G has n vertices,

$spl^k(G)$ has $n(k+1)$ vertices. So the nk new vertices need not be present in a total dominating set of $spl^k(G)$. Note that from nk vertices r vertices can be selected in $\binom{nk}{r}$ ways. Therefore, $D_t\left(spl^k(G), x\right) = D_t(G, x)\left[1 + \binom{nk}{1} + \binom{nk}{2} + \ldots + \binom{nk}{nk}\right]$. This proves the result. □

Corollary 6.2.2. *For the path P_{2n}, where $n \geq 1$, we have*

$$D_t\left(spl^k(P_{2n}), x\right) = \left[\sum_{i=1}^{n} \binom{i+1}{n-(i+1)} x^i\right]^2 (1+x)^{2nk}.$$

Proof. From Theorem 2.2.5 we have, $\left[\sum_{i=1}^{n} \binom{i+1}{n-(i+1)} x^i\right]^2$. Since $spl^k(P_{2n})$ has $2nk$ new vertices, the proof follows from Theorem 6.2.1. □

Corollary 6.2.3. *For the path P_{2n+1}, where $n \geq 1$, we have*

$$D_t\left(spl^k(P_{2n+1}), x\right) = x^2\left[\sum_{i=0}^{n+1} \binom{i+1}{n+1-i} x^i\right]\left[\sum_{i=0}^{n-2} \binom{i+1}{n-(2+i)} x^i\right](1+x)^{(2n+1)k}.$$

Proof. From Theorem 2.2.7 we have,
$D_t(P_{2n+1}, x) = x^2\left[\sum_{i=0}^{n+1} \binom{i+1}{n+1-i} x^i\right]\left[\sum_{i=0}^{n-2} \binom{i+1}{n-(2+i)} x^i\right]$. Thus the proof follows from Theorem 6.2.1. □

Corollary 6.2.4. *For the cycle graph C_{2n}, where $n \geq 2$, we have*

$$D_t\left(spl^k(C_{2n}), x\right) = \left[\sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i\right]^2 (1+x)^{2nk}.$$

Proof. From Theorem 2.2.8 we have, $D_t(C_{2n}, x) = \left[\sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i\right]^2$. Therefore, the proof follows from Theorem 6.2.1. □

Corollary 6.2.5. *If n is odd, then*

$$D_t\left(spl^k(C_n)\,, x\right) = \left[\sum_{i=1}^{n} \frac{n}{i}\binom{i}{n-i} x^i\right] (1+x)^{nk}.$$

Proof. If n is odd, from Theorem 2.2.10, we have, $D_t(C_n, x) = \sum_{i=1}^{n} \frac{n}{i}\binom{i}{n-i} x^i$. Then the proof follows from Theorem 6.2.1. □

Corollary 6.2.6. *Using Theorem 6.2.1 we can infer the following results.*

(*i*) $D_t(spl^k(K_{m,n}), x) = \left[(1+x)^{m+n} - (1+x)^m - (1+x)^n + 1\right] [1+x]^{(m+n)k}$,

(*ii*) $D_t(spl^k(K_{1,n}), x) = x\left[(1+x)^n - 1\right] [1+x]^{(1+n)k}$,

(*iii*) $D_t(spl^k(K_n), x) = \left[(1+x)^n - 1 - x\right] [1+x]^{nk}$.

Proof. The proof is straight forward. □

6.3 On iterated splitting graph

In this section, we introduce the terminology of iterated splitting graph of a graph G and derive its total domination polynomial.

Definition 6.3.1. *The iterated splitting graph $S^i(G)$ of a graph G is defined as $S^i(G) = S^1\left(S^{i-1}(G)\right)$, where $i = 2, 3, \ldots, k$ and $S^1(G)$ denotes the splitting graph $spl(G)$ of G.*

Lemma 6.3.2. *If G is bipartite, then $S^k(G)$ is bipartite.*

Proof. The proof is similar to that of Lemma 6.1.5. □

Theorem 6.3.3. *For a connected graph G with n vertices, the total domination polynomial of the iterated splitting graph of G is*

$$D_t\left(S^k(G)\right) = D_t(G,x)(1+x)^{n2^{k-1}}.$$

Proof. Let G be a connected graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Let $v_1^k, v_2^k, \ldots,$ v_n^k be the new vertices in the iterated splitting graph $S^k(G)$ such that $N_{S^k(G)}\left(v_i^k\right)$ $= N_G(v_i)$. From the construction of $S^k(G)$, for $1 \le j \le k-1$, we have, $N_{S^k(G)}\left(v_i^j\right) \supseteq N_{S^k(G)}\left(v_i^k\right) = N_G(v_i)$. Therefore, a vertex covering set of G is a vertex covering set of $S^k(G)$. Also if S is a vertex covering set of $S^k(G)$, then $S \cap V(G)$ is a vertex covering set of G. Since the iterated splitting graph $S^k(G)$ has $n2^k$ vertices, the total domination polynomial of $S^k(G)$ is $D_t(G,x)(1+x)^{n2^{k-1}}$. This completes the proof. □

Corollary 6.3.4. *For the path P_{2n}, where $n \ge 1$, we have*

$$D_t\left(S^k(P_{2n}),x\right) = \left[\sum_{i=1}^{n}\binom{i+1}{n-(i+1)}x^i\right]^2 (1+x)^{n2^k}.$$

Proof. From Theorem 2.2.5 we have, $\left[\sum_{i=1}^{n}\binom{i+1}{n-(i+1)}x^i\right]^2$. Since the iterated splitting graph $S^k(P_{2n})$ has $n2^{k+1}$ vertices, from Theorem 6.3.3 we have,

$$\begin{aligned} D_t\left(S^k(P_{2n}),x\right) &= D_t\left(P_{2n},x\right)(1+x)^{2n2^{k-1}} \\ &= \left[\sum_{i=1}^{n}\binom{i+1}{n-(i+1)}x^i\right]^2 (1+x)^{n2^k}. \end{aligned}$$

This proves the result. □

Corollary 6.3.5. *For the path P_{2n+1}, where $n \geq 1$, we have*

$$D_t\left(S^k(P_{2n+1}),x\right) = x^2\left[\sum_{i=0}^{n+1}\binom{i+1}{n+1i}x^i\right]\left[\sum_{i=0}^{n-2}\binom{i+1}{n-(2+i)}x^i\right](1+x)^{(2n+1)2^{k-1}}.$$

Proof. From Theorem 2.2.7 we have,
$D_t(P_{2n+1},x) = x^2\left[\sum_{i=0}^{n+1}\binom{i+1}{n+1\text{-}i}x^i\right]\left[\sum_{i=0}^{n-2}\binom{i+1}{n-(2+i)}x^i\right]$. Thus the proof follows from Theorem 6.3.3. □

Corollary 6.3.6. *For the cycle graph C_{2n}, where $n \geq 2$, we have*

$$D_t\left(S^k(C_{2n}),x\right) = \left[\sum_{i=1}^{n}\frac{n}{i}\binom{i}{n-i}x^i\right]^2(1+x)^{n2^k}.$$

Proof. From Theorem 2.2.8 we have, $D_t(C_{2n},x) = \left[\sum_{i=1}^{n}\frac{n}{i}\binom{i}{n-i}x^i\right]^2$. Therefore, the proof follows from Theorem 6.3.3. □

Corollary 6.3.7. *If n is odd, then*

$$D_t\left(S^k(C_n),x\right) = \left[\sum_{i=1}^{n}\frac{n}{i}\binom{i}{n-i}x^i\right](1+x)^{n2^{k-1}}.$$

Proof. If n is odd, from Theorem 2.2.10, we have, $D_t(C_n,x) = \sum_{i=1}^{n}\frac{n}{i}\binom{i}{n-i}x^i$. Then the proof follows from Theorem 6.3.3. □

Corollary 6.3.8. *Using Theorem 6.3.3 we can infer the following results.*

(*i*) $D_t(S^k(K_{m,n}),x) = \left[(1+x)^{m+n}-(1+x)^m-(1+x)^n+1\right][1+x]^{(m+n)2^{k-1}}$,

(*ii*) $D_t(S^k(K_{1,n}),x) = x\left[(1+x)^n-1\right][1+x]^{(1+n)2^{k-1}}$,

(*iii*) $D_t(S^k(K_n),x) = \left[(1+x)^n-1-x\right][1+x]^{n2^{k-1}}$.

Proof. **(i)** We have, $D_t(K_{m,n}, x) = [(1+x)^m - 1]\,[(1+x)^n - 1]$. Since, $S^k(K_{m,n})$ has $(m+n)2^k$ vertices, the proof follows from Theorem 6.3.3.

(ii) The proof follows immediately by substituting $m = 1$ in (i).

(iii) The TD-Polynomial of K_n is $[(1+x)^n - 1 - x]$. Since $\mid V(K_n) \mid = n$, $S^k(K_n)$ has $n2^k$ vertices. Thus the proof follows from Theorem 6.3.3.

□

Chapter 7

Global Bipartite domination in Graphs

7.1 Introduction

In this chapter the concepts of the *global bipartite domination number,* $\gamma_{gb}(G)$ and *global bipartite total domination number,* $\gamma_{gbt}(G)$ of a connected bipartite graph G are introduced. We study some of the general properties of γ_{gb} and γ_{gbt}. Moreover, we determine the global bipartite domination number and global bipartite total domination number of certain classes of graphs. Connected spanning subgraphs of $K_{m,n}$ with global bipartite domination number and global bipartite total domination number $m+n$ or $m+n-1$ are characterized.

[1] A part of this chapter has been published in *Malaya Journal of Mathematik.* Volume 4, Number 3, 2016, Pages 438-442.

7.2 Global bipartite domination

Definition 7.2.1. *Let G be a connected bipartite graph with bipartition (X, Y), with $|X| = m$ and $|Y| = n$. The relative complement of G in $K_{m,n}$, denoted by $\widehat{G}$ is the graph obtained by deleting all edges of G from $K_{m,n}$ (i.e., $K_{m,n} - E(G)$). A global bipartite dominating set (GBDS) of G is a set S of vertices of G such that it dominates G and its relative complement $\widehat{G}$. The global bipartite domination number, $\gamma_{gb}(G)$ is the minimum cardinality of a global bipartite dominating set of G.*

Example 7.2.2. *For the graph given in figure 7.1, $S = \{1, 2, 3\}$ is the minimum global bipartite dominating set of G. So $\gamma_{gb}(G) = 3$.*

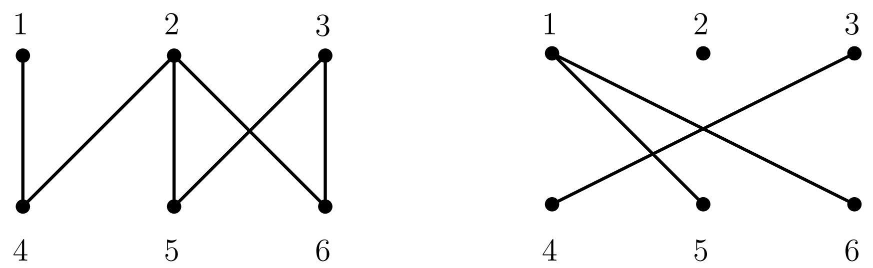

Figure 7.1: The graphs G and $\widehat{G}$

It can be observed that global bipartite domination is defined for connected bipartite graphs only.

Theorem 7.2.3. *For any connected spanning subgraph G of $K_{m,n}$, $\gamma(G) \leq \gamma_{gb}(G) \leq m + n$.*

Proof. A global bipartite dominating set of G is a dominating set of G and so

$\gamma(G) \leq \gamma_{gb}(G)$. The set of all vertices of G is clearly a GBDS of G so, $\gamma_{gb}(G) \leq m + n$. Therefore, $\gamma(G) \leq \gamma_{gb}(G) \leq m + n$. □

Remark 7.2.4. *The bounds in Theorem 7.2.3 are sharp. For the complete bipartite graph $K_{m,n}$, $\gamma_{gb}(K_{m,n}) = m+n$. For the path graph P_4, $\gamma(P_4) = \gamma_{gb}(P_4) = 2$. So $K_{m,n}$ has the largest possible GBD number. Also the bounds in Theorem 7.2.3 are strict. For the graph $K_{2,3} - e$, $\gamma(K_{2,3} - e) = 2$ and $\gamma_{gb}(K_{2,3} - e) = 4$.*

Theorem 7.2.5. *For a spanning subgraph G of $K_{m,n}$, if G and $\widehat{G}$ does not contain any isolated vertices, then $\gamma_{gb}(G) \leq min\{m, n\}$.*

Proof. Let (X, Y) be the bipartition of G with $|X| = m \leq |Y| = n$. Since G and $\widehat{G}$ does not contain isolated vertices, X is a G.B.D.S. of G. Therefore, $\gamma_{gb}(G) \leq m$. □

Theorem 7.2.6. *For any two positive integers m and n, $\gamma_{gb}(K_{m,n}) = m + n$.*

Proof. Let G be a complete bipartite graph with partitions X and Y. Then $uv \in E(G)$ for every $u \in X$ and $v \in Y$. Let $\widehat{G}$ denotes the relative complement of G in $K_{m,n}$. Then $\widehat{G}$ contains $m+n$ isolated vertices. Hence every global bipartite dominating set of G must contain all vertices of $\widehat{G}$ and so $\gamma_{gb}(G) \geqslant m + n$. Now $V(G)$ is a global bipartite dominating set of G. Hence $\gamma_{gb}(G) = m + n$. □

Theorem 7.2.7. *For a spanning subgraph G of $K_{m,n}$, a vertex v is in every global bipartite dominating set of G if and only if v is an isolated vertex in $\widehat{G}$.*

Proof. If $|V(G)| \leq 3$, the proof is trivial. So let $|V(G)| > 3$. If v is an isolated vertex in $\widehat{G}$, then v is in every global bipartite dominating set of G. Conversely if v is not an isolated vertex in $\widehat{G}$, then there exist at least two vertices u and w

such that u is adjacent to v in G and w is adjacent to v in $\widehat{G}$. So $V(G) \setminus \{v\}$ is a global bipartite dominating set of G. This completes the proof. □

Theorem 7.2.8. *Let G be a connected bipartite graph with partite sets X and Y. Let $S = V_1 \cup V_2$ be a GBDS of G, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. Then if $V_1 = \phi$, then $V_2 = Y$ and if $V_2 = \phi$, then $V_1 = X$.*

Proof. Let $S = V_1 \cup V_2$ be a global bipartite dominating set of G such that $V_1 \subseteq X$ and $V_2 \subseteq Y$. If $V_1 = \phi$, then $S \subseteq Y$. Since G is bipartite, the vertices in Y are not adjacent. Therefore, S is a GBDS of G only if $S \supseteq Y$. Therefore, $S = V_2 = Y$. Similarly, we can prove that if $V_2 = \phi$ then $V_1 = X$. □

Theorem 7.2.9. *Let (X, Y) be the bipartition of a connected graph G. Then X is a GBDS of G if and only if $|N(y)| < |X|, \ \forall y \in Y$.*

Proof. Let X be a GBDS of G. If possible assume that there exists a vertex $y \in Y$ such that $|N(y)| = |X|$. Then y is an isolated vertex in $\widehat{G}$, contradicting the fact that X is a GBDS of G. Conversely, since G is connected, X is dominating set of G. So it is sufficient to show that X dominates $\widehat{G}$ also. Let $y \in Y$, then $N(y)$ is a proper subset of X. So y is adjacent to at least one vertex of X in $\widehat{G}$. This completes the proof. □

Theorem 7.2.10. *Let G be a connected sub graph of $K_{m,n}$. Then $\gamma_{gb}(G) = m + n - 1$ if and only if $G \cong K_{m,n} - e$.*

Proof. Let $G \cong K_{m,n} - e$. where $e = uv \in E(K_{m,n})$. So $uv \notin E(G)$ and hence $uv \in E(\widehat{G})$. Since $\widehat{G}$ contains $m + n - 2$ isolated vertices, every global bipartite dominating set of G contains either u or v and all vertices of $V(G) \setminus \{u, v\}$.

Thus, $\gamma_{gb}(G) \geq m+n-1$. Since $V(G)-\{u\}$ is a GBDS of G, it follows that $\gamma_{gb}(G) \leq m+n-1$ Therefore, we obtain $\gamma_{gb}(G) = m+n-1$. Conversely assume that $\gamma_{gb}(G) = m+n-1$. To prove $G \cong K_{m,n}-e$. We observe that $\gamma_{gb}(K_{m,n}) = m+n$ and $\gamma_{gb}(K_{m,n}-e) = m+n-1$. Let G be a proper subgraph of $K_{m,n}-e$ containing $m+n$ vertices. Then $\widehat{G}$ contains at most $m+n-3$ isolated vertices. In that case $\widehat{G}$ contains a path uvw. Then $V(G)-\{u,w\}$ is a GBDS of G. So $\gamma_{gb}(G) \leq m+n-2$. This completes the proof. □

Theorem 7.2.11. *Let G be a graph with bipartition (X,Y). If G has a γ-set $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ then S is a γ_{gb}-set of G if and only if $\bigcap_{x\in V_1} N(x) \subseteq V_2$ and $\bigcap_{y\in V_2} N(y) \subseteq V_1$.*

Proof. Let $\bigcap_{x\in V_1} N(x) \subseteq V_2$ and $\bigcap_{y\in V_2} N(y) \subseteq V_1$. Since S is a γ- set of G, it suffices to show that S dominates the relative compliment of G. Let $u \in X$. If $u \in \bigcap_{y\in V_2} N(y)$, then $u \in V_1$. If $u \notin \bigcap_{y\in V_2} N(y)$ then u is adjacent to at least one vertex of V_2 in $\widehat{G}$. Similarly, we can prove that if $v \in Y$ then $v \in V_2$ or v is adjacent to at least one vertex of V_1 in $\widehat{G}$. Conversely, let S dominates $\widehat{G}$. Let x be an arbitrary vertex in X. If $x \in \bigcap_{y\in V_2} N(y)$, then in $\widehat{G}$, x is not adjacent to any vertex of V_2. Since S dominates $\widehat{G}$, we can deduce that $x \in V_1$. If $x \notin \bigcap_{y\in V_2} N(y)$, then x is adjacent to at least one element of V_2 in $\widehat{G}$. Hence the proof. □

Corollary 7.2.12. *Let G be a connected bipartite graph with n vertices, $n \geq 4$. Then $\gamma_{gb}(G \circ K_1) = n$, where $G \circ K_1$ denotes the corona of the graphs G and K_1.*

Proof. If $G \cong K_{1,n}$, the proof is trivial. Otherwise, let (X,Y) be the bipartition of $G \circ K_1$. Let $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$, be the set of all pendant vertices of $G \circ K_1$. Clearly S is γ-set of $G \circ K_1$. Also $\bigcap_{x \in V_1} N(x) = \phi$ and $\bigcap_{y \in V_2} N(y) = \phi$. Therefore, the proof follows immediately from Theorem 7.2.11. □

Corollary 7.2.13. *For $n \geq 10$, $\gamma_{gb}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. Let $(1, 2, 3, \ldots, n)$ be the path P_n. Then $X = \{x \colon x \text{ is even}, x \leq n\}, Y = \{y \colon y \text{ is odd}, \ y \leq n\}$ is the bipartition of P_n. Let $S_1 = \{i \colon i \equiv 1(mod\ 3), i \leq n\}$ and $S_2 = \{i \colon i + 1 \equiv 0(mod\ 3), i \leq n\}$. Then either S_1 or S_2 is a γ-set of P_n. Also for $i = 1, 2$, $\bigcap_{x \in S_i \cap X} N(x) = \phi$ and $\bigcap_{y \in S_i \cap Y} N(y) = \phi$. Thus the proof follows from Theorem 7.2.11. □

The global bipartite domination number, $\gamma_{gb}(P_n)$ for $1 < n < 10$ is given in Table 7.1.

Table 7.1: $\gamma_{gb}(P_n)$ for $n < 10$

P_n	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$\gamma_{gb}(P_n)$	2	3	3	3	3	3	4	4

Corollary 7.2.14. *For an even integer $n \geq 10$, $\gamma_{gb}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

Proof. The proof is exactly similar to Corollary 7.2.13. □

Theorem 7.2.15. *For any two positive integers a and b with $a < b$, there exists a graph G such that $\gamma(G) = a$ and $\gamma_{gb}(G) = b$.*

Proof. Consider the graph $K_{b-a,a}$, with partite sets $W = \{w_1, w_2, \ldots, w_{b-a}\}$ and $U = \{u_1, u_2, \ldots, u_a\}$. Let G be the graph obtained from $K_{b-a,a}$ by adding new vertices $v_1, v_2, \ldots, v_a$ and join v_i with u_i for $i = 1, 2, \ldots, a$. Let S be a dominating set of G. Since for each i, v_i is adjacent to u_i only, $|S| \geq a$. Now U is a dominating set of G. So $|S| \leq a$. Hence $\gamma(G) = a$. In $\widehat{G}$, the vertices $w_1, w_2, \ldots, w_{b-a}$ are isolated. So W is a subset of every γ_{gb}-set of G. Therefore, the set $\{u_1, u_2, \ldots, u_a, w_1, w_2, \ldots, w_{b-a}\}$ is a γ_{gb}-set of G. Hence $\gamma_{gb}(G) = b$. This completes the proof. □

A graph G with $\gamma(G) = 2$ and $\gamma_{gb}(G) = 6$ is given in figure 7.2

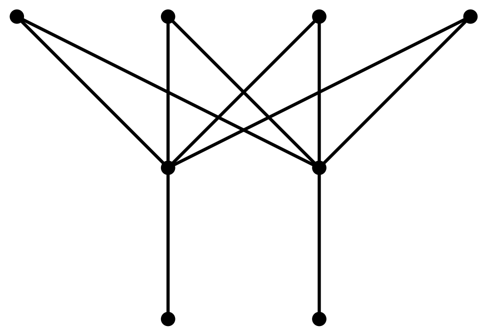

Figure 7.2: Graph G with $\gamma = 2$ and $\gamma_{gb} = 6$

Lemma 7.2.16. *If G is an r-regular connected bipartite graph with bipartition (X, Y) then $|X| = |Y|$.*

Proof. In an r-regular connected bipartite graph G with bipartition (X, Y), each edge contributes exactly one to the degree sums $r|X|$ and $r|Y|$. Therefore, $r|X| = r|Y| = |E|$ and so $|X| = |Y|$. This completes the proof. □

Theorem 7.2.17. *If G is an $(n-1)$-regular bipartite graph with $2n$ vertices, then $\gamma_{gb}(G) = n$.*

Proof. Let (X, Y) be the bipartition of G. Since G has $2n$ vertices, from Lemma 7.2.16, we have $|X| = |Y| = n$. Since G is $(n-1)$ regular, $\widehat{G}$ has n components and all of them are paths with two vertices. So $\gamma(\widehat{G}) = n$. Then by Theorem 7.2.11, we can find a γ-set of $\widehat{G}$ such that it dominates G also. Therefore, $\gamma_{gb}(G) = n$. □

Theorem 7.2.18. *Let G be a healthy spider with $2n+1$ vertices, then $\gamma_{gb}(G) = n+1$.*

Proof. Let S be a γ-set of G, then $|S| = n$ and $u \notin S$ (see figure 7.3). So S dominates all vertices except u in $\widehat{G}$. So $S \cup \{u\}$ is a γ_{gb}-set of G. This completes the proof. □

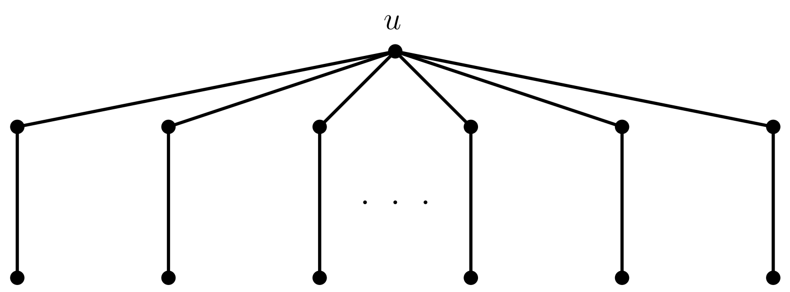

Figure 7.3: Healthy Spider

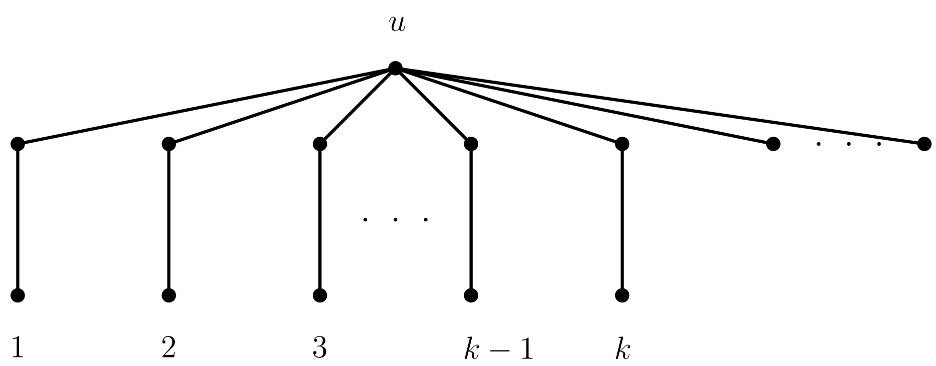

Figure 7.4: Wounded Spider

Theorem 7.2.19. *If G is a wounded spider with $n+k+1$ vertices, then $\gamma_{gb}(G) = k+1$.*

Proof. Observe that $\gamma(G) = k+1$. Also, the set $S = \{1, 2, 3, \ldots, k, u\}$ is a γ_{gb}-set of G (see figure 7.4).

Thus the proof follows. □

Theorem 7.2.20. $\gamma_{gb}(B_n) = 4$, *where B_n is the book graph on $2n+2$ vertices.*

Proof. Let the vertices of B_n be labeled as shown in figure 7.4. Then $X = \{v, u_1, u_2, \ldots, u_n\}$, $Y = \{u, v_1, v_2, \ldots, v_n\}$ is the bipartition of B_n. Clearly the set $\{u, v\}$ is the γ-set of B_n. Also $\{u, v, u_1, v_1\}$ is a γ-set of $\widehat{B_n}$. Therefore, $\gamma_{gb}(B_n) = 4$. □

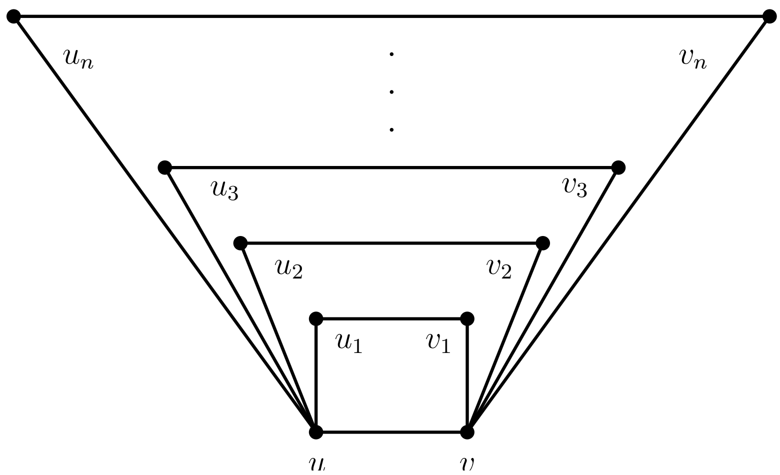

Figure 7.5: Book Graph

Theorem 7.2.21. *For $n \geq 3$, $\gamma_{gb}(S(K_n)) = n$, where $S(K_n)$ is the subdivision of the complete graph K_n.*

Proof. Let X be the set of all old vertices and Y be the set of all new vertices of $S(K_n)$. Then (X, Y) is a bipartition of $S(K_n)$. In $S(K_n)$, the degree of each vertex in X is $n-1$ and the degree of each vertex in Y is 2. We construct a γ-set of $S(K_n)$ as follows. Let $S \subseteq X$ such that $|S| = n-2$. Then S dominates all but one vertex u in Y. Also $N(u) = \{x, y\}$ and $X - S = \{x, y\}$. So $S \cup \{u\}$ is a γ-set of $S(K_n)$. Note that any γ-set of $S(K_n)$ contains exactly $n-2$ vertices from X and one vertex from Y. Since $S \cup \{u\}$ does not dominate x and y in $\widehat{G}$, this set is not a γ_{gb}-set. So a γ_{gb}-set of $S(K_n)$ contains at least n vertices. Clearly the set X is a global bipartite dominating set of $S(K_n)$. Therefore, $\gamma_{gb}(S(K_n)) = n$. □

Remark 7.2.22. $\gamma_{gb}(S(K_2)) = 3$.

Proof. Since $S(K_2) = P_3$, the proof follows. □

7.3 Global bipartite total domination

In this section, we introduce the concept of *global bipartite total domination* in graphs. We study some of its general properties and determine the global bipartite total domination number of certain classes of graphs.

Definition 7.3.1. *Let G be a connected bipartite graph with bipartition (X, Y), with $|X| = m$ and $|Y| = n$. Let $\widehat{G}$ denotes the relative complement of G in $K_{m,n}$. Let S be a total dominating set of G. If S dominates $\widehat{G}$, then S is called a global bipartite total dominating set (GBTDS) of G. The global bipartite total domination number, $\gamma_{gbt}(G)$ is the minimum cardinality of a global bipartite total dominating set of G.*

Example 7.3.2. *For the graph G given in figure 7.6, $S_1 = \{1, 3, 4, 5\}$, $S_2 = \{1, 3, 4, 6\}$ and $S_3 = \{2, 3, 4, 6\}$ are the minimum global bipartite total dominating sets of G, so that $\gamma_{gbt}(G) = 4$.*

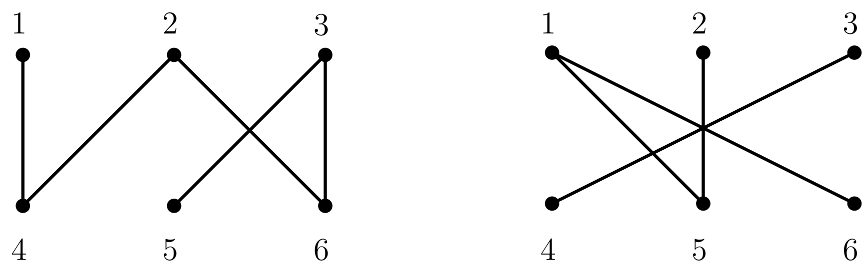

Figure 7.6: A graph G and $\widehat{G}$

Theorem 7.3.3. *For a connected spanning subgraph G of $K_{m,n}$,*

$$\gamma_t(G) \leq \gamma_{gbt}(G) \leq m + n.$$

Proof. Since every γ_{gbt}-set is a total dominating set we have, $\gamma_{gbt}(G) \geq \gamma_t(G)$. Also $V(G)$ is a GBTDS of G. Thus we have the result. □

Remark 7.3.4. *The bounds of Theorem 7.3.3 are sharp. For the graph K_2, $\gamma_t(K_2) = \gamma_{gbt}(K_2) = 2$ and for $K_{m,n}$, $\gamma_{gbt}(K_{m,n}) = m + n$. Also the bounds of Theorem 7.3.3 are strict. For the graph G given in figure 7.6, $\gamma_{gbt}(G) = 4$.*

Theorem 7.3.5. *Let G be a connected bipartite graph with partite sets X, Y. Let $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$, be a total dominating set of G. Then S is a GBTDS of G if and only if $\bigcap_{x \in V_1} N(x) \subseteq V_2$ and $\bigcap_{y \in V_2} N(y) \subseteq V_1$.*

Proof. Since every total dominating set is a dominating set, the proof follows from Theorem 7.2.11. □

Theorem 7.3.6. *For any two positive integers a, b with $a < b$, there exists a graph G with $\gamma_t(G) = a$ and $\gamma_{gbt}(G) = b$.*

Proof. Consider the complete bipartite graph $K_{a-1,\ b-a+1}$, with partite sets $U = \{u_1, u_2, \ldots, u_{a-1}\}$ and $W = \{w_1, w_2, \ldots, w_{b-a+1}\}$. Let G be the graph obtained from $K_{a-1,\ b-a+1}$ by adding new vertices $v_1, v_2, \ldots, v_{a-1}$ and joining v_i with u_i for $i = 1, 2, \ldots, a-1$. Then $\{w_1, u_1, u_2, \ldots, u_{a-1}\}$ is a γ_t set of G. Since the vertices $w_1, w_2, \ldots, w_{b-a+1}$ are isolated in $\widehat{G}$, the set W is a subset of every GBTDS of G. Therefore, $\{w_1, w_2, \ldots, w_{b-a+1}, u_1, u_2, \ldots, u_{a-1}\}$ is a γ_{gbt}-set of G. □

For the graph G given in figure 7.7, $\gamma_t(G) = 4$ and $\gamma_{gbt}(G) = 7$.

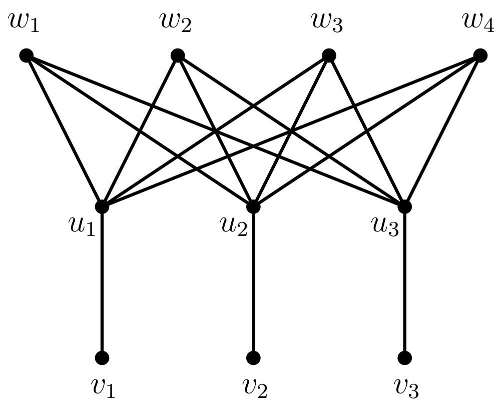

Figure 7.7: Graph G with $\gamma_t = 4$ and $\gamma_{gbt} = 7$

Theorem 7.3.7. *Let G be a connected spanning subgraph of $K_{m,n}$. Then $\gamma_{gbt}(G) = m + n - 1$ if and only if G is isomorphic to $K_{m,n} - e$.*

Proof. Let G be isomorphic to $K_{m,n} - e$. Then $\widehat{G}$ consists of $m + n - 2$ isolated vertices and an edge $e = uv$. So if S is a GBTDS of G, then S contains all isolated vertices of $\widehat{G}$ and one of u or v. Therefore, $\gamma_{gbt}(G) \geq m+n-1$. Since $V(G) - \{v\}$

is a GBTDS of G, we have $\gamma_{gbt}(G) \leq m+n-1$. Therefore, $\gamma_{gbt}(G) = m+n-1$. Conversely, let $\gamma_{gbt}(G) = m+n-1$. We proved that $\gamma_{gbt}(K_{m,n}) = m+n$ and $\gamma_{gbt}(K_{m,n}-e) = m+n-1$. If G is a proper subgraph of $K_{m,n}-e$, then $\widehat{G}$ has at most $m+n-3$ isolated vertices. In that case $\widehat{G}$ has a path uvw. Therefore, $\gamma_{gbt}(G) \leq m+n-2$. This completes the proof. □

Chapter 8

Global Bipartite Domination Polynomial

8.1 Introduction

In this chapter, we introduce the concept of the *global bipartite domination polynomial* of a connected bipartite graph and study some of its general properties. We establish some relationships between domination polynomial and global bipartite domination polynomial of certain classes of graphs.

8.2 Main results

Definition 8.2.1. *Let $\mathcal{D}_{gb}(G,i)$ be the family of global bipartite dominating sets of a simple connected bipartite graph G with cardinality i and let $d_{gb}(G,i) = |\mathcal{D}_{gb}(G,i)|$. Then the global bipartite domination polynomial $D_{gb}(G,x)$ of G is*

[1]This chapter has been published in *Palestine Journal of Mathematics*, Volume 7, Number 1, 2018, Pages 227-233.

defined as $D_{gb}(G,x) = \sum_{i=\gamma_{gb}(G)}^{n} d_{gb}(G,i)x^i$.

Theorem 8.2.2. *For a connected bipartite graph G, if $\widehat{G}$ is also connected, then $D_{gb}(G,x) = D_{gb}(\widehat{G},x)$.*

Proof. Let S be a global bipartite dominating set of G. Since $\widehat{G}$ is connected, S is a global bipartite dominating set of $\widehat{G}$ also. Similarly, every GBDS of $\widehat{G}$ is a GBDS of G. This completes the proof. □

Theorem 8.2.3. *For any two positive integers m and n,*

(i) $D_{gb}(K_{m,n},x) = x^{m+n}$,

(ii) *If $K_{m,n} - e$ is connected, then* $D_{gb}(K_{m,n} - e, x) = x^{m+n-1}(x+2)$.

Proof. **(i)** Obviously $\gamma_{gb}(K_{m,n}) = m+n$. Therefore, $D_{gb}(K_{m,n},x) = x^{m+n}$.

(ii) If $e = uv$, then $V(K_{m,n}) \setminus \{u\}$ and $V(K_{m,n}) \setminus \{v\}$ are the only γ_{gb}-sets of $K_{m,n}-e$. Therefore, $\gamma_{gb}(K_{m,n}-e) = m+n-1$ and $d_{gb}(K_{m,n}-e, m+n-1) = 2$.

Since $d_{gb}(K_{m,n} - e, m+n) = 1$, the proof follows. □

Next, we compute the global bipartite domination polynomial of bi-star graph $B_{m,n}$, obtained from the graph K_2 with vertices u and v by attaching m pendant edges to u and n pendant edges to v.

Theorem 8.2.4. *The global bipartite domination polynomial of bi-star graph is*

$$D_{gb}(B_{m,n},x) = x^2 \left[x^m + x^n + [(1+x)^m - 1]\,[(1+x)^n - 1]\right].$$

Proof. Clearly, $\gamma_{gb}(B_{m,n}) = 4$. Let U and V be the set of all pendant vertices at u and v respectively. Since the vertices u and v are isolated in $\widehat{B}_{m,n}$, every GBDS of $B_{m,n}$ contains u and v. Let S be a subset of vertices of $B_{m,n}$ such that $\{u, v\} \subseteq S$. If $S \cap U \neq \phi$ and $S \cap V \neq \phi$, then S is a GBDS of $B_{m,n}$. Also the sets $U \cup \{u, v\}$ and $V \cup \{u, v\}$ are G.B.D.S of $B_{m,n}$. This completes the proof. $\square$

The next theorem follows immediately from the definition of global bipartite domination polynomial.

Theorem 8.2.5. *For any connected spanning subgraph G of $K_{m,n}$,*

(*i*) $d_{gb}(G, m + n) = 1$,

(*ii*) $d_{gb}(G, i) = 0$ *if and only if* $i < \gamma_{gb}(G)$ *or* $i > m + n$,

(*iii*) $D_{gb}(G, x)$ *has no constant term,*

(*iv*) $D_{gb}(G, x)$ *is a strictly increasing function in* $[0, \infty)$,

(*v*) *If H is an induced subgraph of G, then* $deg(D_{gb}(G, x)) \geq deg(D_{gb}(H, x))$,

(*vi*) *Zero is a root of $D_{gb}(G, x)$ with multiplicity* $\gamma_{gb}(G)$.

Theorem 8.2.6. *If G is an $(n-1)$-regular connected bipartite graph with $2n$ vertices, then*

$$D_{gb}(G, x) = [x\,(x + 2)]^n - 2nx^n.$$

Proof. Since G is $(n-1)$ regular, each component of $\widehat{G}$ is P_2. Therefore, a G.B.D.S of G contains at least one vertex from each component of $\widehat{G}$. So $\gamma_{gb}(G) = n$ and for $1 \leq i \leq n$, $d_{gb}(G, n + i) = \binom{n}{i} 2^{n-i}$. It follows from Theorem 7.2.11 that

$d_{gb}(G,n) = 2^n - 2n$. Then,

$$\begin{aligned} D_{gb}(G,x) &= \binom{n}{0}2^n x^n + \binom{n}{1}2^{n-1}x^{n+1} + \ldots + \binom{n}{n}2^{n-n}x^{n+n} - 2nx^n \\ &= x^n (x+2)^n - 2nx^n. \end{aligned}$$

This completes the proof. □

8.3 Global bipartite domination polynomials of paths

In this section, we shall study the relation between domination polynomials and global bipartite domination polynomials of paths.

We need the following:

Lemma 8.3.1. *For a path P_n with bipartition (X,Y), let $S = V_1 \cup V_2$ where $V_1 \subseteq X$ and $V_2 \subseteq Y$ be a dominating set. If $|V_i| > 2$, for $i = 1,2$, then S is a G.B.D.S. of P_n.*

Proof. In P_n if $|V_i| > 2$, then $\bigcap_{v \in V_i} N(v) = \phi$. Then by Theorem 7.2.11, S is a G.B.D.S of P_n. □

Let G be a connected bipartite graph with partite sets X and Y. Let $S = V_1 \cup V_2$ be a GBDS of G such that $V_1 \subseteq X$ and $V_2 \subseteq Y$. Then by Theorem 7.2.8 we have, if $S \cap X = \phi$, then $S = Y$ and if $S \cap Y = \phi$, then $S = X$. So for $n \geq 12$, to find $d(P_n, i) - d_{gb}(P_n, i)$ it suffices to consider the dominating sets $S = V_1 \cup V_2$ of P_n with $1 \leq |V_1| \leq 2$ or $1 \leq |V_2| \leq 2$. To prove theorems 8.3.2 to 8.3.5, the partite sets of P_{2n} is taken as $X = \{1, 3, 5, \ldots, 2n-1\}$ and $Y = \{2, 4, 6, \ldots, 2n\}$ and $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ is taken as a dominating set. Using the following theorems we can find the number of dominating sets which are not global bipartite dominating sets.

Theorem 8.3.2. *For* $|V_1| = 1$, *we have*

$$(i)\ d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 2,$$

$$(ii)\ d(P_{2n}, n-1) - d_{gb}(P_{2n}, n-1) = n - 2.$$

Proof. Since a vertex in X is adjacent to at most two vertices in Y, $n - 2 \leq |V_2| \leq n$. If $|V_2| = n$, then $S = V_1 \cup V_2$ is a G.B.D.S and the proof is complete. So $|V_2| = n - 2$ or $n - 1$. We consider the following cases:

Case 1: $V_1 = \{1\}$.

Here $V_2 = \{4, 6, 8, \ldots, 2n\}$. Since $N(1) = \{2\} \nsubseteq V_2$, S is not a G.B.D.S.

Case 2: $V_1 = \{3\}$.

Here also $|V_2| = n - 1$ and $V_2 = \{2, 6, 8, \ldots, 2n\}$. Since $N(3) = \{2, 4\} \nsubseteq V_2$, S is not a G.B.D.S.

Case 3: $V_1 = \{i\}, i \neq 1, 3$.

Then for each i, $V_1 \cup (Y \setminus \{i-1, i+1\})$, $V_1 \cup (Y \setminus \{i-1\})$ and $V_1 \cup (Y \setminus \{i+1\})$ are dominating sets of P_{2n}. Since $N(i) = \{i-1, i+1\} \nsubseteq V_2$, these are not G.B.D.S of P_{2n}.

In cases 1 and 2 we have two dominating sets of order n. In case 3 we have $2(n-2)$ dominating sets of order n and $n - 2$ dominating sets of order $n - 1$.

Therefore, the result follows. □

Theorem 8.3.3. *For* $|V_2| = 1$, *we have*

$$(i)\ d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 2,$$

$$(ii)\ d(P_{2n}, n-1) - d_{gb}(P_{2n}, n-1) = n - 2.$$

Proof. The proof is exactly similar to that of Theorem 8.3.2. □

Theorem 8.3.4. *For* $|V_1| = 2$, *we have*

$$(i)\ d(P_{2n}, n-1) - d_{gb}(P_{2n}, n-1) = n - 3,$$

$$(ii)\ d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 4,$$

$$(iii)\ d(P_{2n}, n+1) - d_{gb}(P_{2n}, n+1) = n - 1.$$

Proof. Since $|V_1| = 2$, we have $n - 3 \leq |V_2| \leq n$. If $|V_2| = n$, then $S = V_1 \cup V_2$ is a G.B.D.S. So it suffices to consider the cases $|V_2| = n - 3, n - 2$ and $n - 1$.

Case 1: $V_1 = \{1, 3\}$.

Subcase 1: $|V_2| = n - 2$.

Then $V_2 = \{6, 8, \ldots, 2n\}$. Since $N(1) \cup N(3) = \{2\} \nsubseteq V_2, S$ is not a G.B.D.S of P_{2n}.

Subcase 2: $|V_2| = n - 1$.

Then $V_2 = \{4, 6, 8, \ldots, 2n\}$. Since $N(1) \cup N(3) = \{2\} \nsubseteq V_2$, the dominating set S is not a G.B.D.S.

Case 2: $V_1 = \{3, 5\}$.

As in case 1 we get two dominating sets which are not G.B.D.S of P_{2n}.

Case 3: $V_1 = \{i, i+2\}, i \neq 1, 3$.

Subcase 1: $|V_2| = n - 3$.

Then $V_2 = Y \setminus \{i-1, i+1, i+3\}$.

Subcase 2: $|V_2| = n - 2$.

In this case we have the possibilities, $V_2 = Y \setminus \{i-1, i+1\}$ and $V_2 = Y \setminus \{i+1, i+3\}$.

Subcase 3: $|V_2| = n - 1$.

Then $V_2 = Y - \{i+1\}$.

In sub case 1, 2 and 3, $S = V_1 \cup V_2$ is a dominating set but since $N(i) \cap N(i+1) = \{i+1\} \nsubseteq V_2,$ S is not a G.B.D.S of P_{2n}.

In cases 1 and 2 we have two dominating sets of order n and $n+1$. In case 3 we have $n-3$ dominating sets of order $n-1, 2(n-3)$ dominating sets of order n and $n-3$ dominating sets of order $n+1$. Hence the result follows. □

Theorem 8.3.5. *For* $|V_2| = 2$, *we have*

$$(i)\ d(P_{2n}, n-1) - d_{gb}(P_{2n}, n-1) = n-3,$$

$$(ii)\ d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n-4,$$

$$(iii)\ d(P_{2n}, n+1) - d_{gb}(P_{2n}, n+1) = n-1.$$

Proof. The proof is exactly similar to that of Theorem 8.3.4. □

Theorem 8.3.6. *For* $n \geq 6$,

$$D(P_{2n}, x) - D_{gb}(P_{2n}, x) = (4n-10)x^{n-1} + (8n-12)x^n + (2n-2)x^{n+1}.$$

Proof. It follows from Theorems 8.3.2, 8.3.3, 8.3.4 and 8.3.5. □

Next, we find the relationship between domination polynomials and global bipartite domination polynomials of P_{2n+1}. To prove theorems 8.3.7 to 8.3.10, we take $X = \{1, 3, 5, \ldots, 2n+1\}$ and $Y = \{2, 4, 6, \ldots, 2n\}$ as the bipartition of P_{2n+1} and $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ as a dominating set of P_{2n+1}.

Theorem 8.3.7. *For* $|V_1| = 1$, *we have*

$$(i)\ d(P_{2n+1}, n-1) - d_{gb}(P_{2n+1}, n-1) = n-3,$$

$$(ii)\ d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = 2n-2.$$

Proof. **Case 1:** $V_1 = \{1\}$. Let $V_2 = Y \setminus \{2\}$. Since $N(1) = \{2\}, S = V_1 \cup V_2$ is not a G.B.D.S.

The case $V_1 = \{2n+1\}$ is similar.

Case 2: $V_1 = \{3\}$. Let $V_2 = Y \setminus \{4\}$. Since $N(3) = \{2, 4\}, S = V_1 \cup V_2$ is not a G.B.D.S.

The case $V_1 = \{2n-1\}$ is similar.

Case 3: $V_1 = \{i\}$, $i \notin \{1, 3, 2n-1, 2n+1\}$. In this case we have the possibilities, $V_2 = Y \setminus \{i-1, i+1\}$ or $V_2 = Y \setminus \{i-1\}$ and $V_2 = Y \setminus \{i+1\}$. Since $N(i) = \{i-1, i+1\}, S = V_1 \cup V_2$ is not a G.B.D.S.

In cases 1 and 2 we have four dominating sets of order n and in case 3 there are $n-3$ dominating sets of order $n-1$ and $2(n-3)$ dominating sets of order n. This completes the proof. □

Theorem 8.3.8. *For* $|V_2| = 1$, *we have*

$$(i)\ d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = n,$$

$$(ii)\ d(P_{2n+1}, n+1) - d_{gb}(P_{2n+1}, n+1) = 2n.$$

Proof. Let $V_2 = \{i\}, i \in Y \Rightarrow N(i) = \{i-1, i+1\}$. Then V_1 can be $X \setminus \{i-1\}$ or $X \setminus \{i+1\}$ or $X \setminus \{i-1, i+1\}$. Since i can be selected in n ways , we have $2n$ dominating sets of order $n+1$ and n dominating sets of order n. Since $N(i) = \{i-1, i+1\},\ S = V_1 \cup V_2$ is not a G.B.D.S. of P_{2n+1}. Hence the result follows. □

Theorem 8.3.9. *For* $|V_1| = 2$, *we have*

$$(i)\ d(P_{2n+1}, n-1) - d_{gb}(P_{2n+1}, n-1) = n-4,$$

$$(ii)\ d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = 2n-4,$$

$$(iii)\ d(P_{2n+1}, n+1) - d_{gb}(P_{2n+1}, n+1) = n.$$

Proof. **Case 1:** $V_1 = \{1, 3\}$. Then V_2 can be $Y \setminus \{2\}$ or $Y \setminus \{2, 3\}$. Since $N(1) \cap N(3) = \{2\},\ S = V_1 \cup V_2$, is not a G.B.D.S.

The case $V_1 = \{2n-1, 2n+1\}$ is similar.

Case 2: $V_1 = \{3, 5\}$. Then V_2 can be $Y \setminus \{4\}$ or $Y \setminus \{4, 5\}$. Since $N(3) \cap N(5) = \{4\},\ S = V_1 \cup V_2$, is not a G.B.D.S.

The case $V_1 = \{2n-3, 2n-1\}$ is similar.

Case 3: $V_1 = \{i, i+2\}, i \notin \{1, 3, 2n-3, 2n-1\}$. Then V_2 can be $Y \setminus \{i-1, i+1, i+3\}$ or $Y \setminus \{i-1, i+1\}$ or $Y \setminus \{i+1, i+3\}$. Since $N(i) \cap N(i+2) =$

$\{i+1\}$, $S = V_1 \cup V_2$, is not a G.B.D.S.

In cases 1 and 2 we have four dominating sets of order n and $n+1$. In case 3 there are $n-4$ dominating sets of order $n-1$ and $n+1$ and $2(n-4)$ dominating sets of order n. Thus the result follows. □

Theorem 8.3.10. *For* $|V_2| = 2$, *we have*

$$(i)\ d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = n-1,$$

$$(ii)\ d(P_{2n+1}, n+1) - d_{gb}(P_{2n+1}, n+1) = 2n-2,$$

$$(iii)\ d(P_{2n+1}, n+2) - d_{gb}(P_{2n+1}, n+2) = n-1.$$

Proof. Let $V_2 = \{i, i+2\}$, $i \in Y \Rightarrow N(i) \cap N(i+2) = \{i+1\}$. Then V_1 can be $X \setminus \{i-1, i+1, i+3\}$ or $X \setminus \{i-1, i+1\}$ or $X \setminus \{i+1, i+3\}$. Since V_2 can be selected in $n-1$ ways , we have $n-1$ dominating sets of order n and $2(n-1)$ dominating sets of order$n+1$ and $n-1$ dominating sets of order $n+2$. Since $N(i) \cap N(i+2) = \{i+1\}$, $S = V_1 \cup V_2$ is not a G.B.D.S. of P_{2n+1}. This proves the result. □

Theorem 8.3.11. *For* $n \geq 6$,

$$D(P_{2n+1}, x) - D_{gb}(P_{2n+1}, x) = (2n-7)x^{n-1} + (6n-7)x^n + (5n-2)x^{n+1} + (n-1)x^{n+2}.$$

Proof. It follows from Theorems 8.3.7, 8.3.8, 8.3.9 and 8.3.10. □

8.4 Global bipartite domination polynomials of cycles

In this section, we find the relation between the domination polynomials and the global bipartite domination polynomials of cycles. To prove theorems 8.4.1 to 8.4.5, let $X = \{1, 3, 5, \ldots, 2n-1\}$ and $Y = \{2, 4, 6, \ldots, 2n\}$ be the bipartition of C_{2n} and $S = V_1 \cup V_2$ where $V_1 \subseteq X$ and $V_2 \subseteq Y$ be a dominating set of C_{2n}.

Theorem 8.4.1. *For* $|V_1| = 1$, *we have*

$$(i)\ d(C_{2n}, n-1) - d_{gb}(C_{2n+1}, n-1) = n,$$

$$(ii)\ d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2n.$$

Proof. Let $V_1 = \{i\},\ i \in X$. Then $N(i) = \{i-1, i+1\}$ (if $i = 1,$ then we take $i-1 = 2n$.) Then V_2 can be $Y \setminus \{i-1, i+1\}$ or $Y \setminus \{i-1\}$ or $Y \setminus \{i+1\}$. Since i can be selected in n ways, we have n dominating sets of order $n-1$ and $2n$ dominating sets of order n. Since $N(i) = \{i-1, i+1\},\ S = V_1 \cup V_2$ is not a G.B.D.S. of C_{2n}. Hence the result follows. □

Theorem 8.4.2. *For* $|V_2| = 1$, *we have*

$$(i)\ d(C_{2n}, n-1) - d_{gb}(C_{2n+1}, n-1) = n,$$

$$(ii)\ d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2n.$$

Proof. The proof is exactly similar to Theorem 8.4.1. □

Theorem 8.4.3. *For* $|V_1| = 2$, *we have*

$$(i)\ d(C_{2n}, n-1) - d_{gb}(C_{2n}, n-1) = n-1,$$

$$(ii)\ d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2(n-1),$$

$$(iii)\ d(C_{2n}, n+1) - d_{gb}(C_{2n}, n+1) = n-1.$$

Proof. Let $V_1 = \{i, i+2\},\ i \in X$. Then $N(i) \cap N(i+2) = \{i+1\}$(if $i = 2n-1$, then we take $i+2 = 1$ and $i+3 = 2$.) Then V_2 can be $Y \setminus \{i-1, i+1, i+3\}$ or $Y - \{i-1, i+1\}$ or $Y \setminus \{i+1, i+3\}$ or $Y \setminus \{i+1\}$. Since V_1 can be selected in $n-1$ ways , we have $(n-1)$ dominating sets of order $n-1$,$2(n-1)$ dominating sets of order$n$ and $n-1$ dominating sets of order $n+1$. Since $N(i) \cap N(i+2) = \{i+1\}$, $S = V_1 \cup V_2$ is not a G.B.D.S. of C_{2n}. Hence the result follows. □

Theorem 8.4.4. *For* $|V_2| = 2$, *we have*

$$(i)\ d(C_{2n}, n-1) - d_{gb}(C_{2n}, n-1) = n-1,$$

$$(ii)\ d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2(n-1),$$

$$(iii)\ d(C_{2n}, n+1) - d_{gb}(C_{2n}, n+1) = n-1.$$

Proof. The proof is exactly similar to Theorem 8.4.3. □

Theorem 8.4.5. *For* $n \geq 6$,

$$D(C_{2n}, x) - D_{gb}(C_{2n}, x) = (4n-2)x^{n-1} + (8n-4)x^n + (2n-2)x^{n+1}.$$

Proof. It follows from Theorems 8.4.1, 8.4.2, 8.4.3 and 8.4.4. □

Further scope for research

1. Characterize non isomorphic graphs having same total domination polynomial.

2. Determine the total domination polynomial of an arbitrary Cayley graph.

3. Determine the total domination polynomial of Cartesian product of arbitrary graphs.

4. Determine the polynomials $D_{t_v}(G, x)$ and $D_t^v(G, x)$ of arbitrary graphs.

5. Determine the total domination polynomial of ring sum of arbitrary graphs.

6. Characterize graphs G for which $\gamma(G) = \gamma_{gb}(G)$.

7. Characterize graphs G for which $\gamma_t(G) = \gamma_{gbt}(G)$.

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APPENDIX

List of Publications

1. Latheeshkumar A.R., Anil Kumar. V, A Note on Global Bipartite Domination in Graphs, *Malaya Journal of Mathematik*, Vol. 4, No. 3, (2016), $438 - 442$.

2. Latheeshkumar A.R., Anil Kumar. V, Total Domination Polynomials of Some Graphs, *Journal of Pure and Applied Mathematics: Advances and Applications*, Vol. 16, No. 2, (2016), $97 - 108$.

3. Latheeshkumar A.R., Anil Kumar. V, Total Domination Polynomials of Some splitting Graphs, *Advances and applications in Discrete Mathematics. Pushpa Publishing House,* Vol. 18, No. 3, (2017), $331 - 343$.

4. Latheeshkumar A.R., Anil Kumar. V, TD-Polynomials of Paths and Cycles - A New Approach, *Global Journal of Pure and Applied Mathematics* Vol.13, No. 10, (2017), $7315 - 7319$.

5. Latheeshkumar A.R., Anil Kumar. V, On Global Bipartite Domination Polynomials, *Palestine Journal of Mathematics*, Vol. 7, No. 1, 2018, $227 - 233$.

6. Latheeshkumar A.R., Anil Kumar. V, On Total Domination Polynomials, *Far East Journal of Mathematics* Vol.102, No. 10, (2017), $2277 - 2289$.

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