CONTRIBUTIONS TO STOCHASTIC MODELLING IN RELIABILITY

Thesis submitted to the

University of Calicut

for the award of the degree of

Doctor of Philosophy

in

Statistics

under the Faculty of Science

by

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under the guidance of

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December 2018

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I hereby certify that the work reported in this thesis entitled **CONTRIBUTIONS TO STOCHASTIC MODELLING IN RELIABILITY** that is being submitted by Smt. **Vidhya G. Nair** for the award of Doctor of Philosophy, to the University of Calicut, is based on the bonafide research work carried out by her under my supervision and guidance in the Department of Statistics, University of Calicut. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma of any other university or institution. Also certify that the contents of the thesis have been checked using anti-plagiarism data base and no unacceptable similarity was found through the software check.

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I hereby declare that this thesis entitled **CONTRIBUTIONS TO STOCHASTIC MODELLING IN RELIABILITY** submitted to the University of Calicut for the award of the degree of **Doctor of Philosophy in Statistics** under the Faculty of Science is an independent work done by me under the guidance and supervision of **Dr. M. Manoharan**, Professor, Department of Statistics, University of Calicut.

I also declare that this thesis contains no material which has been accepted for the award of any other degree or diploma of any university or institution and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference made in the text of the thesis.

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ACKNOWLEDGEMENT

Words are not enough to express my immeasurable gratitude to those who influenced my life during my research period. First of all, I express my great indebtedness to my research guide ***Dr. M. Manoharan*** *for giving me the opportunity to research under his guidance. He is a mentor par excellence supporter, an embodiment of human values. Without his advice, encouragement, support and motivation at various stages, I would not have completed this work.*

Besides my supervisor, I am extremely thankful to my teacher ***Dr. N. Raju*** *for his genuine motivation and constant advices on my research work. I wish to extend my sincere thanks to* ***Dr. C. Chandran****-Head of the Department of Statistics and* ***Dr. K. Jayakumar*** *for their exemplary advice and help. I remember with gratitude the* ***librarians and office staffs*** *of the Department of Statistics for providing me the necessary facilities and help in time of my need. I fervently express my gratitude to all my* ***research scholar friends*** *for their valuable support and cooperation.*

I acknowledge gratefully ***Smt. Rema. A****, Principal and my dear* ***colleagues*** *of G.H.S.S Karaparamba, Kozhikode for their helping hand and moral support. I extend my sincere gratitude to my colleague and dear friend smt.* ***D. Jayasree*** *for having made this thesis presentable in language and style. I wish to thank all my loving* ***friends*** *who had bestowed me their love, affection and encouragement in ups and downs of my life during this research period.*

I am very thankful to ***authorities*** *of K.S.E.B. (Kerala State Electricity Board Ltd.) for furnishing me the necessary details and helping me to collect the data from K.H.E.P at Kakkayam. I also wish to express my sincere thanks to* ***Executive engineer and staffs****, K.H.E.P, Kakkayam who provided me valid and useful information for my research work.*

I am indebted my parents Sri. ***Gopalakrishnan*** *and Smt.* ***Sarojini*** *and my* ***relatives*** *for their love and support. My husband* ***Muralikrishnan*** *has been my constant source of inspiration throughout these years. He has always been there during my tenure as a research scholar boosting my morale and helping me wade through the difficulties that came on my way. My appreciation is limitless for my beloved son*

Niranjan Krishna *and my little boy* ***Darshan Krishna*** *who had sacrificed their affectionate moments with me. I have no word to describe the tolerance my children displayed during my research work.*

Once again I express my heartfelt thanks to all who supported me in all the disciplines during the completion of this work. Finally I thank the spiritual strength-the personification and embodiment of goodness that guided me throughout my research work.

Vidhya G. Nair

CONTRIBUTIONS TO STOCHASTIC MODELLING IN RELIABILITY

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Introduction

The word reliability is used to denote the efficiency of a person or a mechanic equipment performing its intended function in the social, political, economical and practical field. The concept of reliability of technical system has been applied not more than seventy years. Component or system is designed to substantiate certain principles and aims. A reliable equipment is the one which works for a given stipulated time period under given environmental conditions without interruptions. It is mandatory to have a high degrees of reliability. Day by day there is an increase in the complexity level of technological system and its products. Therefore the reliable performance of system is really a challenge for the designers and engineers. Reliability is a critical measure for the system performance of power generations, spacecrafts, telecommunication networks, oil and gas pipelines, nuclear reactors and control systems. In binary system reliability theory the system and its components have two possible states-'working' or 'failed'. There are limitations in binary system models. For real life engineering system and its components, there are many states between complete working and total failure. This leads to multi state reliability modeling. Multi state system reliability models permits more than two levels of performance for a system and its components. Multi state reliability model is an actual representation of engineering system and it is far more complex than binary reliability model. The assessment of system performance measures plays a very important role in system reliability theory. Reliability can be considered both as discipline and measure. When reliability stands for discipline it is considered as development and application of techniques increasing the system effectiveness by reducing the frequency of failures and high maintenance cost whereas reliability is taken as measure, it means quality of an equipment in quantitative terms. Reliability means probability that a unit or system can perform its intended function adequately over a specified period of time under stated operational conditions. In mathematical terms the reliability of a component or a system is $P(T > t)$ where T is the life time of the random variable

Multi state system (MSS) reliability assessment methods are based on five different approaches namely : Structure function approach, Stochastic process approach (mainly Markov, Semi-Markov), Universal Generating Function (UGF) approach, Monte-Carlo simulation approach and Recursive algorithm approach. Structure function approach, which is the first method used for binary reliability system, does not have the ability to investigate the dynamic behavior of multi state system. Stochastic process methods are widely used for MSS reliability analysis. This approach had been successfully used for reliability assessment of multi state power systems and some communication system before that the theoretical frame work of MSS was defined. As the number of system state increases with an increase in the number of system elements, stochastic process method is difficult to apply because it can be easily applied for small MSS. This problem gets intensified in semi Markov technique because system of integral equations are solved in it while system of differential equations are solved in Markov process. Another technique, namely universal generating function (UGF) technique, was introduced to reduce the computational burden of stochastic process approach. This technique helps us to find out the performance distribution of whole MSS using rapid algebraic procedure based on the performance distribution of components. The main disadvantage of Monte-Carlo simulation approach is that excess time and high expense are involved in the development and execution of a model, even though almost every real world MSS can be represented by the Monte-Carlo simulation for the reliability assessment. Recursive algorithm approach for reliability evaluation has been developed in recent years. Extension of UGF technique which is called Lz transform technique is widely used for evaluation of dynamic behavior of MSS now a days.

1.2 Review of Literature

The classical works of Barlow and Proshan (1965 and 1975) presented the basic concepts and further developments of binary reliability theory. Many researchers had addressed the reliability problems in binary setup complementing the work of Barlow and Proshan. The ageing properties (increasing failure rate, increasing failure rate average, decreasing failure rate, decreasing failure rate average etc.) had been studied by many authors [refer Brayson and Siddiqui (1969) and Deshpande et al. (1986)]. Evaluation in multi state system is complicated than evaluation in binary system. The development of MSS reliability analysis started in the second half of 1970s. Murchland (1975), El-Neweihi et al. (1978), Barlow and Wu (1978), Ross (1979) and Griffith (1980) gave structural and statistical foundation for the finite state MSS. In their work they defined series MSS, parallel MSS, reliability bounds, redundancy at series and parallel level, stochastic performance of MSS and component importance measures in MSS. The notion of minimal cut set and minimal path set and coherence and element relevancy were introduced in the context of MSS. Natvig (1982) and Hudson and Kapur (1982) generalized this results in their works. The coherence definition was generalized and different types of coherence was studied by Griffith (1980). Up to date development in MSS theory can be seen in Hudson and Kapur (1982), Block and Savits (1982, 1984), El-Neweihi and Proschan (1984), Aven (1985, 1988), Ebrahimi (1991), Abouammoh and Al-kadi (1991, 1995), Brunelle and Kapur(1999). MSS reliability measure was studied systematically using stochastic process by Aven (1993). Korczak (1997) analyzed MSS behavior using semi Markov process technique. MSS reliability optimization problems was formulated and solved firstly by El- Neweihi et al. (1988). Application of stochastic methods were presented in electric power system by Endrenyi (1979) and Billinton and Allan (1996). Natvig (1986,1993), Lindqvist (1987) used random process method for finding MSS reliability bounds. Component in the binary state reliability analysis were demonstrated by Birnbaum (1969) and Barlow and Proshan (1975). El-Neweihi et al. (1978), Griffith (1980), Bueno (1989) and Abouammoh and Al-kadi (1991) extended the idea of component importance in binary reliability system to multi state system. The joint importance measures for multi state systems have been discussed by Chacko and Manoharan (2011 a, 2011 b).

The basic concepts of MSS, tools for MSS reliability assessment and optimization problems were discussed by Lisnianski and Levitin (2003). With the application of reliability functions to the reliability evaluation of large system of multi state system with degrading components were emphasized by Kolowrocki (2004). A comprehensive introduction to system reliability theory along with failure models, qualitative system analysis and reliability importance were discussed by Rausand and Hoyland (2004). A thorough exposition of system reliability theory has been presented by them. Universal generating function was first introduced and its mathematical basis was discussed by Ushakov (1986). More detailed mathematical foundations were exposed by Gnedenko and Ushakov (1995) and Ushakov (2000). The method was applied to the reliability analysis by Lisnianski et al. (1996). The combined method using random process and UGF were suggested by Lisnianski and Levitin (2003). UGF plays a crucial role in the steady state analysis of MSS. In the series of works Levitin et al. (1998), Levitin and Lisnianski (1998), Lisnianski et al. (2000) various operators providing the evaluation of the whole MSS performance distribution based on performance distributions of system components were described for MSS with series, parallel, series-parallel and bridge structure. For extending UGF technique application to dynamic reliability model, a particular mathematical technique Lz transform was proposed by Lisnianski (2012). Lz transform technique has been proved for short term evaluation of the power generating system by Lisnianski and Ben-Haim (2013). A detailed study of semi Markov process and its application in reliability theory were demonstrated by Limnios and Oprisan (2001). A periodic maintenance system was presented and examined by Xu et al. (2008). In this thesis above techniques are applied to a real case study of reliability analysis of a power generating system.

Markov regenerative process has been used to evaluating reliability and availability of a multi state system. Reliability and availability of power plants and fault tree systems can be found using this process by Wereley and Walker (1988), Fricks et al. (1997), Perman et al.(1997). A detailed study of the properties of Phase type (PH) distribution and its application in stochastic modeling was learned by Neuts (1981). Reliability modeling using Phase type distribution was demonstrated by Neuts and Meier (1981). In this work these techniques are effectively used in the reliability analysis especially for availability context with numerical examples.

Some of the basic concepts of stochastic process are described briefly in the following sections.

1.3 Continuous Time Markov chain

Stochastic process, continuous time Markov chain, regenerative process and renewal theory, semi Markov process was demonstrated by Cinlar (1975) and Ross (1996). According to Cinlar, stochastic process with state space Ω is a collection $\{X(t); t \in T\}$ of random variables. $X(t)$ is defined on the same probability space and taking values in Ω. The set T is called parameter set.

Consider a continuous time stochastic process $\{X(t); t \geq 0\}$ is a continuous time Markov chain if for all $s, t \geq 0$ and non negative integers $k, l, x(u), 0 \leq u \leq s$,

$$\begin{aligned} P\{X(t+s) = l/X(s) = k, X(u) = x(u), 0 \leq u < s\} \\ = P\{X(t+s) = l/X(s) = k\}. \end{aligned}$$

In other words, a continuous time Markov chain is a stochastic process having the Markovian property that the conditional distribution of the future state at time $(t+s)$, given the present state at s and all past states depend only on the present state and is independent of the past [refer Ross (1996)].

1.4 Regenerative process and Renewal theory

Cinlar (1975) defined regenerative process as follows: Consider a stochastic process $Z = \{Z(t); t \geq 0\}$ with state space Ω. Suppose that every time a certain phenomenon occurs, the future of the process Z after that time becomes a probabilistic replica of the future after time zero. Such times which is usually random are called regeneration times of Z and the process is said to be regenerative.

Let Z be a regenerative process with a discrete state space and suppose the probability $f(t) = P\{Z(t) = i\}$ for some fixed state i. The process Z regenerates itself at S_1 and the future process $Z^{'}$ defined by $Z^{'}(u) = Z(S_1 + u)$ has the same probability law as Z itself. If $S_1 = s \leq t$ then $Z(t) = Z^{'}(t - s)$ and then$P\{Z(t) = i/S_1\} = P\{Z^{'}(t-s) = i\} = f(t-s)$ on $\{S_1 = s \leq t\}$. Define $g(t) = P\{Z(t) = i, S_1 > t\}$. Then

$$f(t) = g(t) + \int_0^t F(ds) f(t-s).$$

This equation is known as a renewal equation. The renewal theory has been proved to be a powerful tool for studying regenerative processes and several other stochastic models.

1.5 Semi Markov Process

The stochastic process $X(t) = \{X_n, T_n; n \in N\}$ is said to be a Markov renewal process with state space Ω provided that $P\{X_{n+1} = j, T_{n+1} - T_n \leq t/X_0, ..., X_n; T_0, ..., T_n\}$ $= P\{X_{n+1} = j, T_{n+1} - T_n \leq t/X_n\}$ for all $n \in N, j \in \Omega$ and $t \in R_+$. $X(t)$ is always assumed to be time homogeneous. For any $i, j \in \Omega, t \in R_+$,
$P\{X_{n+1} = j, T_{n+1} - T_n \leq t/X_n = i\} = Q(i, j, t), \quad i, j \in \Omega, \quad t \in R_+$ which is called semi Markov kernel over Ω is independent of n. That is $X(t)$ be a Markov renewal process with state space Ω and semi Markov kernel Q. $L = sup_n T_n$, L is the life time of $X(t)$. The process $Y = \{Y(t); t \geq 0\}$ defined by

$$Y(t) = \begin{cases} X_n & \text{if } T_n \leq t < T_{n+1}, \\ \\ \Delta & \text{if } t \geq L \end{cases}$$

where Δ is point not in Ω. This continuous time parameter process $Y(t)$ is termed as semi Markov process associated with $X(t)$ [refer Cinlar (1975)].

A brief description of the concepts of binary and multi state system are given in the ensuing two sections.

1.6 Binary State System

The theory of binary state system is a unified basis for mathematical and statistical theory of reliability, see Barlow and Proshan (1965, 1975). In the classical binary reliability theory, the system and its components are assumed to be in one of the two states: functioning or failed. The state of component i ($i = 1, 2, 3, ...n$) can be represented by a binary indicator variable x_i where

$$x_i = \begin{cases} 1 & \text{if } i^{th} \text{ component is functioning} \\ 0 & \text{if } i^{th} \text{ component is failed} \end{cases}$$

.

$\mathbf{x} = (x_1, x_2, ...x_n)$ is called state vector.

More over we consider that by knowing the states of all the n components, we can evaluate whether the system is functioning or not. Similarly the state of the system can be defined by a binary function $\phi(x) = \phi(x_1, x_2, ..., x_n)$ where

$$\phi(x) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is failed} \end{cases}$$

.

The function $\phi(x)$ is called structure function of the system. Consider a system with n different components . A series structure is one that will function if and only if all its n components function. The structure function is given by $\phi(x) = \prod_i^n x_i = min(x_1, x_2, ..., x_n)$. A parallel structure is one that will function if at least one of its n components function. The structure function is given by $\phi(x) = \bigsqcup_i^n x_i = max(x_1, x_2, ..., x_n)$. A k out of n structure is one that will function if and only if at least k out of n components function. The structure function is given by

$$\phi(x) = \begin{cases} 1 & \text{if } \sum_i^k x_i \geq k \\ \\ 0 & \text{if } \sum_i^k x_i < k \end{cases}$$

.

A component is relevant to the system performance if an improvement in component performance increases the system performance. A system of components is coherent if its structure function ϕ is increasing and each component is relevant. Esary and Proschan (1962) contributed the properties of coherent structures in reliability theory.

Consider a system of n components, which are statistically independent. Suppose that the state x_i of the i^{th} component is random with $P[x_i = 1] = p_i = E(x_i)$, $i = 1, 2, 3, ..., n$. p_i is called the reliability of component i. The reliability of binary system is given by

$P[\phi(x) = 1] = h = E(\phi(x))$. We can represent system reliability as a function of component reliabilities $h = h(p)$ where $p = (p_1, p_2, ...p_n)$.

$h = h(p)$ is referred as the reliability function of the structure ϕ. If the components are not independent system reliability may not a function of p alone. In this case $h(p)$ will not be used. The reliability function is $h(p) = \prod_i^n p_i$ for series structure function $\phi(x) = \prod_i^n x_i$ and for parallel structure function $\phi(x) = \coprod_i^n x_i$ has the reliability function $h(p) = 1 - \prod_i^n (1 - p_i)$. The reliability function of k-out of-n system is

$$h(p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

.

1.7 Multi State System

In many real life situations, however, systems and components have actually diverse range of levels of performance, varying from perfect functioning to complete failure. In this circumstance the dichotomized model is an over simplification of a real situation. So models of multi state system and multi state components are much more practical in explaining the performance of these systems in respect of the performance of their components. Hirsch et al. (1968) perceived the idea of multi state system in 1968 itself. Earlier we mentioned that more researchers entered in this area in 1970s and in 1980s. They focussed on multi state system reliability. Efforts of Lisnianski and Levitin (2003) were remarkable in summarizing multi state reliability theory. Previously we indicated that multi state system reliability evaluation is based on five different approaches.

The Strucure function approach is an extension of binary models to multi state case. Application of the Boolean methods to determine the multi state system Structure and reliability measures in practice are founded by Boedidheimer and Kapur (1994). Aven (1985) proffered an algorithm which is based on state space decomposition. Natvig and Streller (1984) first applied the stochastic process approach to evaluate multi state system reliability. Modern theory of stochastic process contributes an improved probabilistic framework [Aven (1999)]. This allows us to conceive formulae not only for general failure models but also for computing various performance measures. It also helps us to determine the optimal replacement policies in complex situation. The idea of bringing together the Markov processes and coherent structure function was proposed by Xue and Yang (1995). Brunelle and Kapoor (1999) comprehensively studied multi state system reliability evaluation using stochastic process. The universal generating function method in power system reliability analysis was firstly adapted by Linsianski et al. (1996). Various operators are contributed by Levitin and Lisnianski (1999) for the determination of the entire multi state system performance distribution based on the performance distribution of component. Lisnianski and Ding (2009) put forward a method that extends the classical reliability block diagram method to a repairable multi state system. This method is based on combined random processes and the universal generating function technique. It diminishes intensely the number of states in the multi state model. Levitin et al. (2011) introduced an algorithm for assessing performance distribution of complete series- parallel multi state system with propagated failures and imperfect problems. The proposed algorithm is based on the universal generating functions and generalized reliability block diagram method. For the reliability evaluation of most of the real world multi state system, Monte-Carlo simulation can be applied. Billinton and Li (1991) introduced a hybrid approach using Monte-Carlo simulation. They used an enumeration technique for the reliability evaluation of large scale composite generation transmission systems, including multi state representation of generating units. Zio et al. (2007) put forward a Monte-Carlo simulation technique which allows modeling the complex dynamics of multi state components based on operational dependencies with the system overall state. Zio et al. (2004) introduced another Monte-carlo simulation approach. This is used to estimate all the importance measures of the components at a given performance level in a multi state series parallel system. The fifth approach for multi state system reliability can be evaluated by Recursive algorithm, which was proposed by Zuo and Tian (2006). This is for the reliability evaluation of generalized multi state systems k out of n. Li and Zuo (2008) contributed two models of multi state weighed k out of n system models in which recursive algorithm is used for reliability evaluation.

Development of optimization algorithm to solve different application problems in multi state reliability theory is a milestone in multi state reliability research. The gradient method was modified and applied by Vaurio (1984) to find the minimal cost configuration of a multi state series - parallel power system structure. Nourelfath and Ait-Kadi (2007) developed a redundancy optimization model under reliability constraints, the minimal cost configuration of a multi state series parallel system, which is based on a specified maintenance policy. Tian et al. (2009) introduced a method for determining the optimal version and numbers of components and the optimal set of technical organizational actions for each subsystem of a multi state series parallel system, so as to minimize the system cost while satisfying the system availability constraint. Li et al. (2010) developed a heterogenous redundancy optimization method for multi state series parallel systems based on common cause failures.

A maintenance policy for system with multi state components was presented by Gurler and Kaya (2002). The joint redundancy and maintenance replacement schedule optimization problem generalized to multi state system were constructed by Levitin and Lisnianski (1998). Nourelfath and Ait-kadi (2004) enhanced the redundancy optimization problem of multi state systems to a more general case where maintenance resources are limited. Zuo et al. (2006) developed replacement repair policy for multi state deteriorating products under warranty. Two stage preventive maintenance (PM) policy for multi state degradation system under periodic inspection was proposed by Huang and Yuan (2010). Wu et al. (2010) described a finite life cycle multi state system that is based on both degradation and Poisson failures. He proposed strategies for optimizing maintenance thresholds. A different model was introduced by Issac et al. (2010). This is for evaluating availability, production rate and reliability function of multi state degraded systems subjected to minimal repairs and imperfect preventive maintenance.

Multi state coherent system (MCS) theory was developed by El-Neweihi et al. (1978). $\mathbf{x} = (x_1, x_2, ..., x_n)$ be the vector of states of components $i = 1, 2, ..., n$. $S = \{1, 2, ..., M\}$ be the set of possible states of both components and systems. $(j_i, \mathbf{x}) = (x_1, ..., x_{i-1}, j, x_{i+1}, ..., x_n)$, where $j = 0, 1, ..., M$. $(\cdot_i, \mathbf{x}) = (x_1, ..., x_{i-1}, \cdot, x_{i+1}, ..., x_n)$. $x \vee y$ denotes $max(x, y)$. $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, ..., x_n \vee y_n)$. $x \wedge y$ denotes $min(x, y)$.

$\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, ..., x_n \wedge y_n)$.

Structural Properties of MCS

Definition 1.7.1. *A system of n components is said to be multi state coherent system (MCS) if its structure function satisfies*

1. ϕ is increasing

2. For level j of component i, there exist a vector $(.i, \boldsymbol{x})$ such that $\phi(j_i, \boldsymbol{x}) = j$ while $\phi(l_i, \boldsymbol{x}) \neq j$ for $l \neq j, i = 1, 2, ..., n$ and $j = 0, 1, 2, ...M$.

3. $\phi(j) = j$ for $j = 0, 1, 2, ...M$.

This definition is due to Neweihi et.al (1978). The condition (2) mentioned as the relevance condition.

Theorem 1.7.1. *Let ϕ be a structure function of a MCS , then*

1. $\phi(\boldsymbol{x} \vee \boldsymbol{y}) \geq \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$, and

2. $\phi(\boldsymbol{x} \wedge \boldsymbol{y}) \geq \phi(\boldsymbol{x}) \wedge \phi(\boldsymbol{y})$.

equality in (1) and (2) holds for parallel and series system respectively.

The universal generating function (UGF) approach which is a powerful technique for reliability evaluation of MSS has an important role in this work. The basic concepts of UGF has been discussed in the next section.

1.7.1 Universal Generating Function (UGF)

Universal Generating Function (u function) technique was founded by Ushakov (1986). UGF technique is essentially based on moment generating functions and it is a mathematical concept for random variables. It is assumed that functioning of each component of the system j is characterized by random discrete performance G_j. The performance of the whole multi state system is a well defined function of the performance of its individual components.

The u function of an independent discrete random variable X is defined as a polynomial

$$u(z) = \sum_{k=1}^{K} p_k z^{x_k}.$$

where the variable X has K possible values and p_k is the probability that X is equal to x_k. The polynomial $u^j(z)$ defined as probability distribution of component j (probability mass function of random value G_j). It gives all the probable states of the component by relating the probabilities of each state to the performance of the component in that state. If probability distribution of the component j is defined by $g^j = \{g_i^j\}, 1 \leq i \leq k_j$ and $p^j = \{p_i^j\}, 1 \leq i \leq k_j$, then

$$u^j(z) = \sum_{i=1}^{k_j} p_i^j z^{g_i^j}.$$

composition operators are introduced for getting the u-function of a system . These operators decide the u-function for components connected in parallel and in series using simple algebraic operations over the individual u-functions of basic components. The composition operators $\otimes_{par}$ and $\otimes_{ser}$ characterize the parallel and series connections of two component system. Applying composition operators for two components in a system we get the following

$$
\begin{aligned}
u(z) = u^j(z) \otimes_{par} u^l(z) &= \sum_{i=1}^{k_j} p_i^j z^{g_i^j} \otimes_{par} \sum_{h=1}^{k_l} p_h^l z^{g_h^l} \\
&= \sum_{i=1}^{k_j} \sum_{h=1}^{k_l} p_i^j p_h^l z^{\{g_i^j + g_h^l\}}
\end{aligned}
$$

$$or$$

$$
= \sum_{i=1}^{k_j} \sum_{h=1}^{k_l} p_i^j p_h^l z^{max\{g_i^j, g_h^l\}}
$$

and

$$
\begin{aligned}
u(z) = u^j(z) \otimes_{ser} u^l(z) &= \sum_{i=1}^{k_j} p_i^j z^{g_i^j} \otimes_{ser} \sum_{h=1}^{k_l} p_h^l z^{g_h^l} \\
&= \sum_{i=1}^{k_j} \sum_{h=1}^{k_l} p_i^j p_h^l z^{min\{g_i^j, g_h^l\}}
\end{aligned}
$$

The Ushakov's composition operators $\Omega_{\phi(p)}$ (for parallel connections) and $\Omega_{\phi(s)}$ (for series connections) or their combinations can be applied over the u function of individual component for obtaining u function of entire MSS with any number of components. Lisnianski and Levitin (2003) demonstrated that UGF is very effective for the reliability evaluation of different types of multi state systems.

The UGF Tecnique can be used for random variables and so we can apply this method only for evaluation of steady state behaviour of MSS. Hence it led to the concept of Lz transform which is introduced by Lisnianski (2012).

1.7.2 Lz Transform: Definition, Existence and Uniqueness

Lz transform technique is a new mathematical technique which improvises the application of UGF technique for MSS where its components are explicitly described by discrete state continuous time Markov process. Consider a discrete state continuous time (DSCT) Markov process $X(t) \in \{x_1, x_2, ..., x_k\}$, $t \geq 0$ with k possible states $i, (i = 1, 2, ..., k)$ where performance level associated with any state i is x_i. This Markov process is entirely defined by set of possible states $x = \{x_1, x_2, ..., x_k\}$, transition intensity matrix $A = (a_{ij}), i, j = 1, 2, ..., k$. Initial state probability distribution is $p_0 = [p_{10} = Pr\{X(0) = x_1\}, ..., p_{k0} = Pr\{X(0) = x_k\}]$. Markov process can be represented as $X(t) = \{x, A, p_0\}$.

Note: If functions $a_{ij}(t) = a_{ij}$ are constants then the DSCT Markov process is said to be time-homogeneous. When $a_{ij}(t)$ are time dependent, then the resulting Markov process is non-homogeneous.

Definition 1.7.2. *Lz transform of a discrete state continuous time Markov process $X(t) = \{x, A, p_0\}$ is a function*

$$Lz\{X(t)\} = u(z, t, p_0) = \sum_{i=1}^{k} p_i(t) z^{x_i}$$

where $p_i(t)$ *is a probability that the process is in state* i *at time instant* $t \geq 0$ *for any given initial states probability distribution and* z *in a general case is a complex variable.*

Note: $u(z, t, p_0)$ can write simply $u(z, t)$ by omitting the symbol p_0 keeping in mind that Lz transform will depend initial probability distribution p_0.

Each discrete state continuous time Markov process under certain initial condition has only one Lz transform $u(z, t)$ and each Lz transform will have only one corresponding DSCT Markov process $X(t)$ developing from these initial conditions. We can state this as an existence and uniqueness property of Lz transform.

Proposition 1.7.2. Each Discrete state continuous time Markov process $X(t)$ under certain initial conditions p_0 has unique Lz transform.

Proof. The proof of this is based on Picard theorem [Coddington and Levinson(1955)]. Among theories of differential equations, Picard theorem is an important theorem in existence and stands apart for its uniqueness of solutions to system of differential equations with a given initial value problems. State probabilities $p_i(t) = Pr\{X(t) = x_i\}$, $i = 1, 2, ..., k$ for DSCT are defined by the solution of the following system of linear differential equations

$$\begin{aligned}
\frac{d}{dt}p_1(t) &= a_{11}(t)p_1(t) + a_{12}(t)p_2(t) + \ldots + a_{1k}(t)p_k(t), \\
\frac{d}{dt}p_2(t) &= a_{21}(t)p_1(t) + a_{22}(t)p_2(t) + \ldots + a_{2k}(t)p_k(t), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
\frac{d}{dt}p_k(t) &= a_{k1}(t)p_1(t) + a_{k2}(t)p_2(t) + \ldots + a_{kk}(t)p_k(t)
\end{aligned} \tag{1.1}$$

under initial conditions

$$p_0 = \{p_{10} = p_1(t_0), p_{20} = p_2(t_0), ..., p_{k0} = p_k(t_0)\} \tag{1.2}$$

According to Picard's theorem, if coefficients $a_{ij}(t) \ \ (i, j = 1, 2, ..., k)$ are continuous functions of time t, then the system of equations (1.1) has a unique solution $p_1(t), p_2(t), ..., p_k(t)$ satisfied initial conditions (1.2).

According to the definition of Lz transform for DSCT Markov process $X(t)$ we have

$$Lz\{X(t)\} = u(z, t) = \sum_{i=1}^{K} p_i(t) z^{x_i},$$

where $p_1(t), p_2(t), ..., p_k(t)$ are evaluated as a unique solution of the system of equations (1.1) under initial conditions (1.2).

Therefore there exists unique Lz transform for DSCT markov process $X(t) = \{x, A, p_0\}$ where transition intensities $a_{ij}(t)$ are continuous functions of time. $\square$

Remark 1.7.3. The inverse statement is also correct. If $u(z, t) = \sum_{i=1}^{k} p_i(t) z^{x_i}$

where $p_i(t)$ are described as a solution of the system under initial conditions , then there exist one and only one DSCT Markov process $X(t)$ for which $Lz\{X(t)\} = u(z,t) = \sum_{i=1}^{k} p_i(t)z^{x_i}$

Remark 1.7.4. In reliability explanation Lz transform may be executed to an aging system and to a system at burn in period as well as to a system with constant failure and repair rates. The unique condition that should be satisfied is the continuity of transition intensities $a_{ij}(t)$.

Following are main properties of Lz transform:

Property 1.7.5. If a constant value a is multiplied by a DSCT Markov process then it is equal to multiplying that constant value to corresponding performance level x_i at each state i on this value

$$Lz\{aX(t)\} = \sum_{i=1}^{k} p_i(t)z^{ax_i}$$

Property 1.7.6. L_z transform from a single valued function $f[G(t), H(t)]$ of two independent DSCT Markov processes $G(t)$ and $H(t)$ can be evaluated by applying Ushakov's universal generating operator Ω_f to Lz transform from $G(t)$ and $H(t)$ processes over all time points $t \geq 0$

$$Lz\{f[G(t), H(t)]\} = \Omega_f\{Lz[G(t)], Lz[H(t)]\}.$$

The property allows Lz transform implementation to reliability analysis of multi state system.

Lz transform method is very effective for short term evaluation of power systems. The applicability of this technique to MSS reliability analysis enlarges the possibility of solving class of problems in reliability analysis.

In this work the techniques UGF and Lz transform are applied for repairable multi state system

1.8 Repairable System

Reliability is a performance measure for a non repairable system. A component or system is said to have failed if it does not operate adequately. Reliability of a system can be improved with the help of repair. According to Asher and Feingold (1984), repairable system is a system which can be restored to fully satisfactory performance by a method other than replacement of the entire system after failing to perform one or more of its function satisfactorily. In recent years analysts of repairable multi state system has been the main interest of industrial engineers, statisticians etc. If the system or its components are repairable, the reliability is an incomplete performance measure for the system as maintenance is not considered in this measure. Availability, one of the most appropriate measure, considers the failure behaviors and effects of maintenance action equally in a repairable system. Availability is defined as the probability that system or components is performing its required function at a given point of time or over a stated period of time when operated and maintained in a prescribed manner [Ebeling (1997)]. As a performance measure, availability plays a significant role for the improvement of effectiveness of the repairable system. The availability function is explained as the probability that the system is operating at time t where the reliability function is the probability that the system has operated over the interval. The reliability is an interval function while availability is a point function, explaining the behavior of the system at a specified period. The reliability function prevents the failure of the system during the interval under examination where availability function does not force any such limitations on the behavior of the system. At the steady state situation the reliability function tends to zero whereas availability function tends to some steady state value. The aim of this research work is to project the reliability modeling and it's application and evaluation of performance measures, especially discussion based on availability, in a multi state system.

Reliability modeling of a repairable multi state system handles mainly with two types of repair: perfect repair and minimal repair. Perfect repair helps a system to 'as good as' new state and minimal repair helps system to 'as bad as' old state. In perfect repair each repair is perfect and the system restores a new. In the minimal repair the system restores to like just before failure. But the reality in practical situation is in between these two extreme cases. This leads to the significance of imperfect repair. In imperfect system repair the system returns to an intermediate state between as bad as old and as good as new.

Phase Type distribution has an important role in reliability analysis due to its mathematical simplicity. In the following section the basic ideas of this distribution are discussed.

1.9 Phase Type Distribution

The concept of method of phases was first proposed by Erlang (1917-18). His idea led to a generalization of the exponential distribution which is known as Erlangian distribution. Erlang's idea was extended by Cox (1955). He gave phase representation for all probability distribution on the positive real line which has Laplace-Stieltjes transform. The numerical analysis of this is very difficult because of large use of complex analysis. Neuts (1975) overcame this drawback. Neuts proposed the theory of phase-type (PH) distributions which have nice closure properties.

Continuous Phase type Distribution

Continuous Phase type distribution (CPH) is obtained as the distribution of the time until absorption in a continuous time Markov chain. Consider that $\{X(t), t \geq 0\}$ denote an absorbing continuous time Markov chain with finite state space $\Omega = \{1, 2, ..., m, m+1\}$ and infinitesimal generator $Q = \begin{pmatrix} T & T^0 \\ 0 & 0 \end{pmatrix}$. Here T is a square matrix of dimension m, that is $T = \underset{m \times m}{(T_{ij})}$ satisfies $T_{ii} < 0$ for $1 \leq i \leq m$ and $T_{ij} > 0$ for $i \neq j, 1 \leq i, j \leq m$ and $T^0 = (T_1^0, ..., T_m^0)'_{m \times 1} = -Te$, e denotes column vector with all components equal to one for which the dimension is determined by the context. The initial probability vector is given by $\overline{\alpha} = (\alpha, \alpha_{m+1})$ such that

$\alpha e + \alpha_{m+1} = 1$. The first m states $1, ..., m$ shall be transient.

A probability distribution $F(.)$ on $[0, \infty)$ is a phase type distribution (PH distribution) if it is the distribution of the time until absorption in a finite state Markov chain $X(t)$ with generator Q. It is represented as $PH(\alpha, T)$. The states $\{1, 2, ..., m\}$ are called phases. The distribution function of the time until absorption in to state $m+1$ is given by, $F(x) = 1 - \alpha \exp(Tx)e$ for $x \geq 0$. The density function of $F(.)$ is given by $f(x) = \alpha \exp(Tx)T^0$.

We can apply the following property of continuous phase type distribution in reliability analysis of a multi state system.

Property 1.9.1. Let X and Y be independent random variables with continuous PH distributions G(.) and H(.) having representations $(\alpha, T)_m$ and $(\beta, S)_n$ respectively. Let $F_1(.)$ and $F_2(.)$ be the distributions to $max(X, Y)$ and $min(X, Y)$ respectively where $F_1(.) = G(.)H(.)$ and $F_2(.) = [1 - G(.)][1 - H(.)]$. Then $F_1(.)$ and $F_2(.)$ are also phase type. That is $F_1(.) \sim Ph(\underline{\gamma}, K)_{mn+m+n}$ where $\gamma = [\alpha \otimes \beta, \alpha_{m+1}\beta, \beta_{n+1}\alpha]$ and $\begin{pmatrix} T \otimes I + I \otimes S & I \otimes S^0 & T^0 \otimes I \\ 0 & T & 0 \\ 0 & 0 & S \end{pmatrix}$ and $F_2(.) \sim (\alpha \otimes \beta, T \otimes I + I \otimes S)$.

Property 1.9.2. The i th moment of a PH distribution with representation (α, T) is $\mu_i^{'} = (-1)^i i! \alpha T^{-i} e$

For further properties and results of PH distribution one may refer to Neuts (1975).

The following section gives a brief idea of the work presented in this thesis.

1.10 An Outline of the Present Work

The thesis is arranged into eight chapters, including this introductory part, as outlined below.

In *chapter 1*, which is an introductory chapter, basic concepts and definitions used in this thesis are given. Relevant literature review and elementary ideas of system reliability theory are also presented. Detailed description of multi state system (MSS) are presented in *chapter 2*.

In *chapter 3* we propose straight forward stochastic process approach for analyzing models and evaluating system performance measures of multi state system. The utility of this approach is illustrated in respect of three different structures of system with constat failure and repair rates.

Universal generating function (UGF) is established by using well known ordinary generating function and it is confirmed to be very helpful for analysis of numerical illustration. Combination of stochastic (Markov) process approach and UGF technique by decomposing MSS in to several subsystem is discussed in *chapter 4*. It provides comparatively small computational effort for calculating reliability indices of a multi state system. A real data obtained from a power station modeled as a multi state system which has been divided into two subsystems with many states of degradation.

Lz transform technique is proposed in *chapter 5* for avoiding the curse of dimensionality of stochastic process approach which is often used for the reliability analysis of multi state system. This technique can drastically minimize the computational burden for dynamic reliability assessment of repairable multi state system assuming variable failure rates and repair rates of components of the system. We illustrate this method for the reliability evaluation of a power station based on real data set which is used in the preceding chapter.

Reliability analysis of a multi state system with independent components having many levels of degradation has been considered in *chapter 6*. Periodic inspection and maintenance have been performed for each component of the system. For evaluating steady state probability for each component, the components have been modeled as discrete state continuous time semi Markov process. Steady state reliability indices of availability and performance deficiency are obtained using UGF technique to avoid computational complexity of random process method. Analytical procedures have been illustrated based on the same data, that was used in previous chapters, from a power station with independent generators. The availability of the whole system has been enhanced through monthly periodic inspection and maintenance.

In *chapter 7* the dynamic reliability behavior in terms of common cause failures is identified and a state space model has been formed for the evaluation of performance measure of a multi state system. The concept of renewal is employed and Markov regenerative process has been used for assessment of availability of a multi state parallel system. A system in which this technique is effectively used is illustrated in this chapter.

Phase type distribution is very useful for analytical modeling in the study of multi state reliability systems. It can be used to describe extensive random phenomena because of its versatility. In *chapter 8* we describe a parallel repairable system with single repair facility in which life time and repair time of components have phase type distribution. Steady state probability vector, steady state availability and mean time between failures (MTBF) are evaluated through simple algebraic formalism. Application of this model is illustrated with a numerical example.

Matlab and Mathematica softwares have been employed for the numerical evaluations in the thesis. The thesis is concluded with an epilogue that analyse the main contribution of the current work and provides a perspective approach to future research. A reasonably exhaustive bibliography incorporating all papers and books stated in the thesis are given at the end.

CHAPTER 2

MULTI STATE SYSTEM: BASIC CONCEPTS AND METHODOLOGIES

2.1 Definition and properties

A complete demonstration of multi state reliability theory with reliability analysis and optimization techniques furnished by Lisnianski and Levitin (2003). All technical systems are proposed to fulfilled their intended tasks in a specific environment. The system which performs their tasks with distinct levels of efficiency is called multi state system (MSS). The levels of efficiency of a system is usually known as performance values of multi state system. A binary state system which has two distinct states (perfect functioning and complete failure) is the simplest form of a multi state system. A system can be considered as a multi state system which satisfies the following conditions.

1. Any system with multi state units have a collective effect on the entire system performance and the performance rate of such system is based on the availability of its units. Different numbers of available units can provide different levels of the task performance.

2. The performance rate of element in a system can vary as a result of the deteriorating (fatigue, partial failure) or variant current conditions. Failures of components lead to a poor performance of the entire MSS.

The performance rate of components of MSS can vary from perfect function to complete failure. The failure that leads to decrease in the performance of components is called partial failure. After the partial failure components continue to work at degrading performance rate. After complete failure, the components are totally incapable to perform their task.

Consider a multi state system composed of n components. Any component j of multi state system have k_j different states to the performance level constitute by the set $g_j = \{g_{j1}, ..., g_{jk_j}\}$ where g_{j1} is performance level of component j in the state $i, i \in \{1, 2, ..., k_j\}$. The performance level of component j at any time $t \geq 0$, $G_j(t)$ is a random variable that hold its value from g_j. That is $G_j(t) \in g_j$. The probabilities corresponding to different states (performance levels) of the system component j at any time t can be represented b the set $p_j(t) = \{p_{j1}(t), p_{j2}(t), ..., p_{jk_j}(t)\}$ where $p_{ji}(t) = Pr\{G_j(t) = g_{ji}\}$. The probability distribution of performance of component

j at time t can be described by $g_{ji}, p_j(t); i = 1, 2, ..., k_j$. We can define a generic model of the multi state system that should involved the performance stochastic process $G_j(t), j = 1, 2, ...n$ for each system component j and the system structure function $G(t) = \phi(G_1(t), ..., G_n(t))$ which gives stochastic process associated to the output performance of the entire multi state system. Probability distribution of performance of the multi state system can be determined by $g_j, p_j(t); j = 1, 2, ...n$.

Main properties of multi state system are relevancy of the components of the system, Coherency and Homogeneity.

1. **Relevancy of the components of the system**

   In a multi state system particular component is relevant if the entire system state is changed because of any change in that component's state without any change in the states of other remaining components. With regard to the system structure function, the relevancy of component j means that there exist $G_1(t), ..., G_n(t)$ that for some $g_{jk} \neq g_{jm}$.

$$\begin{aligned} &\phi(G_1(t), ..., G_{j-1}(t), g_{jk}, G_{j+1}(t), ..., G_n(t)) \\ \neq\ &\phi(G_1(t), ..., G_{j-1}(t), g_{jm}, G_{j+1}(t), ..., G_n(t)). \end{aligned}$$

2. **Coherency**

   In a multi state system model, the system is coherent if and only if its structure function is increasing (non decreasing) in each argument and whole system components are relevant. This structure function property follows that the performance of system is high when the performance of components of the system is high and performance of system is low when the components of performance of system is low.

3. **Homogeneity**

   The multi state system is homogeneous when all the components of the system and system itself must have same number of significant states. Binary state system always obey this property. In real life problems most of the multi state system does not obey this property.

2.2 MSS Reliability and its measurs

Multi state System(MSS) acceptability function and reliability

The behaviour of MSS is portrayed by its development in expansion of states. The whole set of feasible system states can be divided in to two mutually exclusive subsets which corresponds to acceptable and un acceptable system functioning. Entry of the system in to the subset of unacceptable states constitutes a failure. The reliability of a multi state system can be defined as its capacity to continue to exist in the acceptable states during the functioning period. The state acceptability depends on the value of output performance $G(t)$ of the system. The acceptability of the state of the system is based on the association between the performance of the multi state system and desired level of this performance (demand) which is determined by outside of the system. Usually demand (W(t)) is a random process with values $w = \{w_1..., w_M\}$

using acceptability function $F(G(t), W(t))$, we can express relationship between the system performance and the demand. It takes non negative values if and only if the functioning of system is acceptable. The acceptable system states correspond to the $F(G(t), W(t)) \geq 0$ and the unacceptable states correspond to $F(G(t), W(t)) < 0$. Acceptability function have the form $F(G(t), W(t)) = G(t) - W(t)$ if the situation, which is more practical in real life, that performance of MSS should exceed the demand. The transitions between subsets of acceptable and unacceptable states can take place an arbitrary number of times for repairable system or for the system with variable demands.

Relevancy and Coherence in the Reliability Context of MSS

From the perspective of reliability the acceptability of the performance rate of a system point out system's ability to perform its intended task. In that circumstance, a component of the system is relevant of changes of its state without changes of the states of remaining componens of the system may lead to change in the acceptability of the behaviour of system. Relevancy of component$j$ implies that there exist $G_1(t), G_1(t), ..., G_n(t)$ for some $g_{jk} \neq g_{jm}$

$$F(\phi(G_1(t), ..., G_{j-1}(t), g_{jk}, G_{j+1}(t), ..., G_n(t)), W) < 0$$

while

$$F(\phi(G_1(t), ..., G_{j-1}(t), g_{jm}, G_{j+1}(t), ..., G_n(t)), W) \geq 0.$$

The coherency of MSS can be defined using acceptability function as like the definition of coherency of binary system. Coherency of multi state system requires the monotonic behaviour of structure function. Multi state system coherency in the context of reliability means that the improvement in the performance of the components of the system may not change the entire system transition from an acceptable state to unacceptable state.

Reliability Indices of MSS

The reliability indices of multi state system has to be determined for evaluating behaviour of a multi state system numerically. We know that the reliability of a multi state system can be explained as its capability to stay in the acceptable states during the operational period. When the system is in given time instant or in steady state, behaviour of the multi state system is determined by its performance represented as a random variable. Different types of reliability indices are explained by Lisnianski and Levitin (2003)

Instantaneous (point) availability of a multi state system is the probability that the system is one of the acceptable states at t $(t > 0)$.

$$A(t) = Pr\{F(G(t), W(t)) \geq 0\}.$$

Availability of MSS in the time interval $(0, T]$ is

$$A_T = \frac{1}{T}\int_0^T I(F(G(t), W(t)) \geq 0)dt.$$

A_T describes the portion of time if the performance rate of MSS is in an acceptable area.

Demand availability is the expected value of A_T which is defined by Aven and Jensen (1999).

$$A_D = E(A_T)$$

As time $t \to \infty$ initial state of the system has no influence on availability of the system. Stationary availability of multi state system for the constant demand level $W(t) = w$ can be determined on the basis of probability distribution of the system.

$$A(w) = \sum_{i=1}^{k} p_i I(F(g_i, w) \geq 0)$$

where

$$p_i = \lim_{t\to\infty} p_i(t)$$

is the steady state probability of the system at state i with the output performance level g_i.

If $F(G(t), W(t)) = G(t) - W(t)$, we get $F(g_i, w) = g_i - w$. Then

$$A(w) = \sum_{i=1}^{k} p_i I(g_i \geq w) = \sum_{g_i \geq w} p_i.$$

The expectation of performance can be used to evaluate measure that describes average performance of multi state system. Mean of multi state instantaneous out put performance at time t is given by

$$E_t = E(G(t)).$$

If the steady state probabilities exist expected steady state MSS performance

$$E_\infty = \sum_{i=1}^{k} p_i g_i.$$

Mean output performance for a fixed time interval $(0, T]$ is determined by

$$E_T = \frac{1}{T}\int_0^T E_t dt.$$

Generally it is significant to be aware about the measure of system performance deviation from a demand when the demand is not met. Instantaneous performance deviation which is known as instantaneous performance deficiency at instant t of a system can be defined in the case $F(G(t), W(t)) = G(t) - W(t)$ as

$$D(t) = max\{W(t) - G(t), 0\}.$$

Expected value of multi state performance deficiency at instant t

$$D_t = E(D(t)).$$

If the system is in steady state with constant demand $W(t) = w$, expected performance deficiency is not a function of time. Expected steady state performance deficiency with constant demand $W(t) = w$ can be obtained from probability distribution of system at steady state.

$$D_\infty = \sum_{i=1}^{k} p_i max(w - g_i, 0).$$

2.3 Types of Multi State System

According to generic model different types of MSS can be described by explaining the stochastic behaviour of its component and the structure function of system.

Series Structure

The series structure of components of multi state system represent that an overall system failure is resulted by the failure of a single component. That is in the series connection of system components of a multi state system, a failure of an individual component becomes the cause of the failure of entire system. The fundamental property of the series system that its operation depends on the complete availability of its components. That is the series connection should maintain its main property that the complete failure of any component of the system (corresponding to its performance rate equal to zero) will be resulted in the complete failure of the system (system performance rate equal to Zero). Different types of series MSS are distinguished by the type of performance and the nature of the interconnection of the components. First consider a system which uses its component capacity as the performance measure. Examples of these types of systems are power systems, energy or material continuous transmission systems, continuous productive systems etc. In this case for the series structure the capacity of the system is equal to the capacity of its weakest component. Another type of series is a task processing system in which the performance means it is defined by an operation time (processing speed). Examples of these types of systems are control systems, information or data processing systems, manufacturing systems with constrained operation time etc. In this case operation time of the entire system is equal to sum of the operation time of all its components. The performance of the component or system is measured in terms of processing speed (reciprocal of the operation time) the total failure corresponds to a performance rate zero. The MSS fails totally if it at least one component of the system is in a state of complete failure.

Parallel Structure

In parallel connection of component of the multi state system, failure of the system occurs only when all its components fail. That is system fails if and only if all of its components fail. The assumption is that if the components of multi state system are connected in parallel structure some of the tasks can be performed by any one of the components. In multi state system with work sharing, the performance rate of the whole system is equal to the sum of performance rate of components which are connected in parallel. Examples of parallel system with work sharing are flow transmission and task processing systems in which performance rate of the system is taken as sum of the performance rate of the components. In multi state system without work sharing performance rate of the system depends on the discipline of the activation of components. In such situation the performance rate of the system is the performance rate of the available component having greatest value. Examples of parallel system without work sharing are flow transmission system and task processing system in which performance rate of the system is taken as maximum performance rate of the available components in operation.

A multi state system composed of series and parallel structures results in series-parallel multi state systems. That is mixed combination of series and parallel structures makes the series-parallel system. The performance rate of these structures can be derived by the successive evaluation of the performance rate of a pure series or parallel sub systems. The reliability of the multi state system for the series, parallel and series-parallel structured can be described as the probabilities that the overall performance rate of the system meets a specified demand. k-out of-n system reliability is explained as the probability that at least k components out of n components are workable conditions. For details of different types of multi state system one may refer to Lisnianski and Levitin (2003).

CHAPTER 3

RELIABILITY EVALUATION FOR A MULTI STATE SYSTEM USING STOCHASTIC PROCESS METHOD

3.1 Introduction

This chapter is intended for describing the application of stochastic process approach to reliability analysis of a multi state system (MSS). Markov process with continuous time and discrete state space (continuous time Markov chain) are extensively applied in multi state system reliability analysis. A detailed study of continuous time Markov chain was given by Ross (1996). A remarkable illustration of continuous time Markov chain and its implementation in reliability theory was presented by Cocozza and

Thivet (1997). Rausand and Hoyland (2003) discussed application of continuous time Markov chain in reliability and availability of MSS. Lisnianski and Levitin (2003) demonstrated basic concepts and ideas of use of random process method in reliability theory of MSS. In this chapter multi state system reliability models will be analysed based on the Markov process. The number of failures in arbitrary time interval in many real life situations can be explained as Poisson process and also time up to the failures and repair times are generally exponentially distributed. In such cases the Markov process is extensively applied for reliability analysis. Multi state system reliability measures as availability, expected performance, performance deficiency etc can be evaluated using Markov process. But in all real life situation basic assumptions about exponential distributions of times between failures and repair times do not satisfied. In such cases complex stochastic process technique known as semi-Markov processes can be adapted.

This chapter is organized as follows. Section 3.2 briefly describes the traditional Markov process and application of continuous time Markov chain in reliability analysis. In section 3.3 a power generating system with two independent generators is illustrated. The steady state distribution and system performance characteristics are determined for the system in the case of three specific structures viz, parallel structure with work sharing of independent components, parallel structure without work sharing and series structure.

3.2 Markov Model

The implementation of conventional Markov technique to the reliability analysis of multi state system consists of two phases. One is evolution of state transition diagram. Organized description of state of the system is developed if pictorial form of the state transition diagram is more complicated. Second phase is evaluation of reliability of a system by solving a system of differential equations based on the state transition diagram or organized description of state of system.

A multi state system with several components is considered and each combination of the state of components of system represent a unique state of the system. For j^{th} component of system k_j different states are to be assigned according to the corresponding performance out put defined by the set $g_j = \{g_{j1}, ..., g_{jk_j}\}$. The performance rate is $G_j(t)$ of component j. The performance level of a multi state system can be determined for any combination of performance levels of components of that system by utilizing the structure function of the whole system. The output performance of the entire MSS $G(t)$ at any instant t is a continuous time Markov chain which takes values from $g = \{g_1, ..., g_k\}$. For applying Markov technique to the reliability analysis of MSS, at first state transition diagram for whole MSS have to be developed. It is a tedious work for the MSS which have large number of states. If the pictorial representation of state transition diagram is impossible, a detailed description of it is to be presented. The transition intensity matrix A has to be evaluated for the corresponding Markov model. Failure rates and repair rates are calculated for every component of the multi state system. the set of failure rates and repair rates in a specific order are as follows $\{\lambda_{k_j,k_{j-1}}, ..., \lambda_{k_j,1}, ..., \lambda_{3,2}, \lambda_{3,1}, \lambda_{2,1}\}$ $\{\mu_{1,2}, ..., \mu_{1,k_j}, ..., \mu_{k_j-2,k_j}, \mu_{k_j-1,k_j}\}$. The failure rate is equal to zero in failure rate set if there is no failure for the component and the repair rate is equal to zero in repair rate set if there is no repair for that component. All probable states of MSS are produced as distinct combinations of all the states of possible output performance of components of system. A set of corresponding state of the component of the system should be assigned for every state of the system. All the pairs of state of the system have to be evaluated for finding transition rates of the system. The transition rates a_{ij} for $i \neq j \quad i, j \in \Omega, \Omega = \{1, 2, ..., k\}$ for each transition are specified. Each transition will usually involves a failure or a repair. The transition rates will therefore be failure rates and repair rates and combinations of these. Transition rates a_{ij} as a matrix is called transition intensity matrix. It is also called infinitesimal generator of the process.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{10k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix},$$

where the diagonal elements of this matrix should be described as

$$a_{ii} = -\sum_{j=1, j\neq i}^{k} a_{ij}.$$

The system of differential equations characterizing the behavior of the system can be derived based on the state transition intensity matrix. The distribution of Markov process at time t is described as the row vector $p(t) = [p_1(t), ..., p_k(t)]$. We recognized that the process started in state i at time 0. The distribution $p(t)$ will be calculated from the Kolmogrove forward equations [refer Ross (1996 p.242)] given in matrix as

$$\frac{d}{dt}\ p(t) = p(t).A \tag{3.1}$$

The system of differential equations with initial conditions $p_i(0) = 1$ and $p_j(0) = 0, j \neq i$ can be solved and the probabilities $p_i(t)$ can be evaluated for all the states of system $i = 1, 2, ..., k$. Equation (3.1) is termed as state equation for the Markov process. Reliability applications are likely to be interested in long run (steady state) probabilities. The state probabilities $p_i(t)$ approached a steady state p_i as $t \to \infty$. For an irreducible Markov process [refer Ross (1996)], the limit

$$\lim_{t \to \infty} p_i(t) = p_i$$

for $i = 1, 2, ..., k$ always exist and are independent of the initial state of the process. If $p_i(t)$ tends to a constant value (steady state value) as $t \to \infty$, then

$$\lim_{t \to \infty} \frac{d}{dt} p(t) = 0$$

for $i = 1, 2, ..., k$. The steady state probability vector $p = [p_1, p_2, ..., p_k]$ must satisfy the matrix equation

$$[p_1, p_2, ..., p_k].\begin{pmatrix} a_{11} & a_{12} & \dots & a_{10k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix} = [0, 0, ..., 0] \tag{3.2}$$

and

$$\sum_{i=1}^{k} p_i = 1 \tag{3.3}$$

The steady state probabilities $p_1, p_2, ..., p_k$ are evaluated using k of the k+1 linear algebraic equations from the matrix equation (3.2) and in addition to this the sum of the steady state probabilities is always equal to one (3.3).

System Performance Measures

According to section 2.2 of the chapter 2 system performance measures at steady situation for a constant demand w can be obtained as

Steady state availability

$$A_\infty(w) = \sum_{g_i \geq w} p_i$$

Mean steady state MSS performance

$$E_{\infty} = \sum_{i=1}^{k} g_i p_i$$

Expected steady state MSS performance deficiency

$$D_{\infty} = \sum_{i=1}^{k} p_i max(w - g_i, 0)$$

3.3 Numerical Illustration

Consider a power generating system with two independent components (generators) that can have only minor failures and minor repairs for components. The minor failures cause state transition only from the state k to adjacent state $k-1$ and the minor repairs cause state transition only from the state $k-1$ to adjacent state k. Each generator has 3 possible performance levels. For generator 1 states are 1, 2 and 3 with performance outputs $g_{11} = 0MW$, $g_{12} = 80MW$ and $g_{13} = 150MW$. For generator 2 states are 1, 2 and 3 with performance outputs States of $g_{21} = 0MW$, $g_{22} = 50MW$ and $g_{23} = 100MW$.That is $G_1(t) \in \{g_{11}, g_{12}, g_{13}\} = \{0, 80, 150\}$ and $G_2(t) \in \{g_{21}, g_{22}, g_{23}\} = \{0, 50, 100\}$. The initial state is the best state 3 for each generator. The failure rates and repair rates corresponding to two generators are

$$\lambda_{21}^{(1)} = 10^{-3}, \quad \lambda_{32}^{(1)} = 2 \times 10^{-4}, \quad \lambda_{21}^{(2)} = 1.2 \times 10^{-3}, \quad \lambda_{32}^{(2)} = 2.3 \times 10^{-4}$$

$$\mu_{12}^{(1)} = \mu_{23}^{(1)} = 2 \times 10^{-2} \quad \mu_{12}^{(2)} = \mu_{23}^{(2)} = 2.5 \times 10^{-2}.$$

We evaluate the steady state reliability analysis of parallel MSS with work sharing, parallel MSS without work sharing and series MSS of independent components (generators).

Parallel MSS with work sharing

Assuming that the generators are connected in parallel and the total generating capacity of the system is determined as sum of the generating capacities of the two generators. For every state, the system output performance rate is computed based on MSS additive structure function $G(t) = \phi_p(G_1(t), G_2(t)) = G_1(t) + G_2(t)$

System State	State of Gen 1	State of Gen 2	System Out put
1	1	1	0MW
2	1	2	50MW
3	2	1	80MW
4	1	3	100MW
5	2	2	130MW
6	3	1	150MW
7	2	3	180MW
8	3	2	200MW
9	3	3	250MW

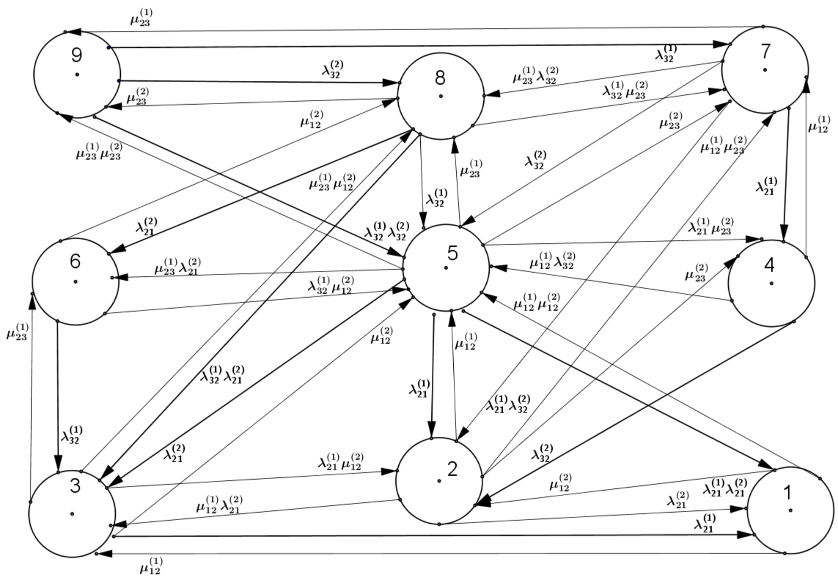

Figure 3.1: State transition diagram of the system

Transition intensity matrix

$$A = \begin{bmatrix}
\alpha 1 & \mu_{12}^{(2)} & \mu_{12}^{(1)} & 0 & \mu_{12}^{(1)}\mu_{12}^{(2)} & 0 & 0 & 0 & 0 \\
\lambda_{21}^{(2)} & \alpha 2 & \mu_{12}^{(1)}\lambda_{21}^{(2)} & \mu_{23}^{(2)} & \mu_{12}^{(1)} & 0 & \mu_{12}^{(1)}\mu_{23}^{(2)} & 0 & 0 \\
\lambda_{21}^{(1)} & \lambda_{21}^{(1)}\mu_{12}^{(2)} & \alpha 3 & 0 & \mu_{12}^{(2)} & \mu_{23}^{(1)} & 0 & \mu_{23}^{(1)}\mu_{12}^{(2)} & 0 \\
0 & \lambda_{32}^{(2)} & 0 & \alpha 4 & \mu_{12}^{(1)}\lambda_{32}^{(2)} & 0 & \mu_{12}^{(1)} & 0 & 0 \\
\lambda_{21}^{(1)}\lambda_{21}^{(2)} & \lambda_{21}^{(1)} & \lambda_{21}^{(2)} & \lambda_{21}^{(1)}\mu_{23}^{(2)} & \alpha 5 & \mu_{23}^{(1)}\lambda_{21}^{(2)} & \mu_{23}^{(2)} & \mu_{23}^{(1)} & \mu_{23}^{(1)}\mu_{23}^{(2)} \\
0 & 0 & \lambda_{32}^{(2)} & 0 & \lambda_{32}^{(1)}\mu_{12}^{(2)} & \alpha 6 & 0 & \mu_{12}^{(1)} & 0 \\
0 & \lambda_{21}^{(1)}\lambda_{32}^{(2)} & 0 & \lambda_{21}^{(1)} & \lambda_{32}^{(2)} & 0 & \alpha 7 & \mu_{23}^{(1)}\lambda_{32}^{(2)} & \mu_{23}^{(1)} \\
0 & 0 & \lambda_{32}^{(1)}\lambda_{21}^{(2)} & 0 & \lambda_{32}^{(1)} & \lambda_{21}^{(2)} & \lambda_{32}^{(1)}\mu_{23}^{(2)} & \alpha 8 & \mu_{23}^{(2)} \\
0 & 0 & 0 & 0 & \lambda_{32}^{(1)}\lambda_{32}^{(2)} & 0 & \lambda_{32}^{(1)} & \lambda_{32}^{(2)} & \alpha 9
\end{bmatrix}$$

where

$$\begin{aligned}
\alpha 1 &= -(\mu_{12}^{(2)} + \mu_{12}^{(1)} + \mu_{12}^{(1)}\mu_{12}^{(2)}) \\
\alpha 2 &= -(\lambda_{21}^{(2)} + \mu_{12}^{(1)}\lambda_{21}^{(2)} + \mu_{23}^{(2)} + \mu_{12}^{(1)} + \mu_{12}^{(1)}\mu_{23}^{(2)}) \\
\alpha 3 &= -(\lambda_{21}^{(1)} + \lambda_{21}^{(1)}\mu_{12}^{(2)} + \mu_{12}^{(2)} + \mu_{23}^{(1)} + \mu_{23}^{(1)}\mu_{12}^{(2)}) \\
\alpha 4 &= -(\lambda_{32}^{(2)} + \mu_{12}^{(1)}\lambda_{32}^{(2)} + \mu_{12}^{(1)}), \\
\alpha 5 &= -(\lambda_{21}^{(1)}\lambda_{21}^{(2)} + \lambda_{21}^{(1)} + \lambda_{21}^{(2)} + \lambda_{21}^{(1)}\mu_{23}^{(2)} + \mu_{23}^{(1)}\lambda_{21}^{(2)} + \mu_{23}^{(2)} + \mu_{23}^{(1)} + \mu_{23}^{(1)}\mu_{23}^{(2)}) \\
\alpha 6 &= -(\lambda_{32}^{(2)} + \lambda_{32}^{(1)}\mu_{12}^{(2)} + \mu_{12}^{(1)}) \\
\alpha 7 &= -(\lambda_{21}^{(1)}\lambda_{32}^{(2)} + \lambda_{21}^{(1)} + \lambda_{32}^{(2)} + \mu_{23}^{(1)}\lambda_{32}^{(2)} + \mu_{23}^{(1)}) \\
\alpha 8 &= -(\lambda_{32}^{(1)}\lambda_{21}^{(2)} + \lambda_{32}^{(1)} + \lambda_{21}^{(2)} + \lambda_{32}^{(1)}\mu_{23}^{(2)} + \mu_{23}^{(2)}) \\
\alpha 9 &= -(\lambda_{32}^{(1)}\lambda_{32}^{(2)} + \lambda_{32}^{(1)} + \lambda_{32}^{(2)}).
\end{aligned}$$

The steady state probability vector

$$p = [p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7 \quad p_8 \quad p_9]$$

satisfy the matrix equation (3.2) and normalizing condition (3.3). We get the following system of equations,

$$\alpha 1 p_1 + \lambda_{21}^{(2)} p_2 + \lambda_{21}^{(1)} p_3 + \lambda_{21}^{(1)}\lambda_{21}^{(2)} p_5 = 0$$

$$\mu_{12}^{(2)} p_1 + \alpha 2 p_2 + \lambda_{21}^{(1)}\mu_{12}^{(2)} p_3 + \lambda_{32}^{(2)} p_4 + \lambda_{21}^{(1)} p_5 + \lambda_{21}^{(1)}\lambda_{32}^{(2)} p_7 = 0$$

$$\mu_{12}^{(1)}p_1 + \mu_{12}^{(1)}\lambda_{21}^{(2)}p_2 + \alpha 3p_3 + \lambda_{21}^{(2)}p_5 + \lambda_{32}^{(2)}p_6 + \lambda_{32}^{(1)}\lambda_{21}^{(2)}p_8 = 0$$

$$\mu_{23}^{(2)}p_2 + \alpha 4p_4 + \lambda_{21}^{(1)}\mu_{23}^{(2)}p_5 + \lambda_{21}^{(1)}p_7 = 0$$

$$\mu_{12}^{(1)}\mu_{12}^{(2)}p_1 + \mu_{12}^{(1)}p_2 + \mu_{12}^{(2)}p_3 + \mu_{12}^{(1)}\lambda_{32}^{(2)}p_4 + \alpha 5p_5 + \lambda_{32}^{(1)}\mu_{12}^{(2)}p_6$$

$$+\lambda_{32}^{(2)}p_7 + \lambda_{32}^{(1)}p_8 + \lambda_{32}^{(1)}\lambda_{32}^{(2)}p_9 = 0$$

$$\mu_{23}^{(1)}p_3 + \mu_{23}^{(1)}\lambda_{21}^{(2)}p_5 + \alpha 6p_6 + \lambda_{21}^{(2)}p_8 = 0$$

$$\mu_{12}^{(1)}\mu_{23}^{(2)}p_2 + \mu_{12}^{(1)}p_4 + \mu_{23}^{(2)}p_5 + \alpha 7p_7 + \lambda_{32}^{(1)}\mu_{23}^{(2)}p_8 + \lambda_{32}^{(1)}p_9 = 0$$

$$\mu_{23}^{(1)}\mu_{12}^{(2)}p_3 + \mu_{23}^{(1)}p_5 + \mu_{12}^{(2)}p_6 + \mu_{23}^{(1)}\lambda_{32}^{(2)}p_7 + \alpha 8p_8 + \lambda_{32}^{(2)}p_9 = 0$$

$$\mu_{23}^{(1)}\mu_{23}^{(2)}p_5 + \mu_{23}^{(1)}p_7 + \mu_{23}^{(2)}p_8 + \alpha 9p_9 = 0$$

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 = 1.$$

In order to obtain steady state probabilities nine out of ten above equations should be solved.

$p_1 = 4.044 \times 10^{-7}$ $p_2 = 1.079 \times 10^{-5}$ $p_3 = 2.962 \times 10^{-6}$ $p_4 = 0.00001279$

$p_5 = 0.0004866$ $p_6 = 0.0002587$ $p_7 = 0.009821$ $p_8 = 0.008998$ $p_9 = 0.9804.$

System performance measures are evaluated for a constant demand $w = 80 \;\; MW$

$$A_\infty(w = 80MW) = 0.9999994$$

$$E_\infty = 248.77MW$$

$$D_\infty(w = 80MW) = 0.00036MW$$

Parallel MSS without work sharing

Assuming that the generators are connected in parallel and the total generating capacity of the system is determined as maximum of the generating capacities of the two generators. For every state ,the system output performance rate is computed based on MSS parallel (Maximum) structure function for every state.

$$G(t) = \phi_p(G_1(t), G_2(t)) = Max(G_1(t), G_2(t))$$

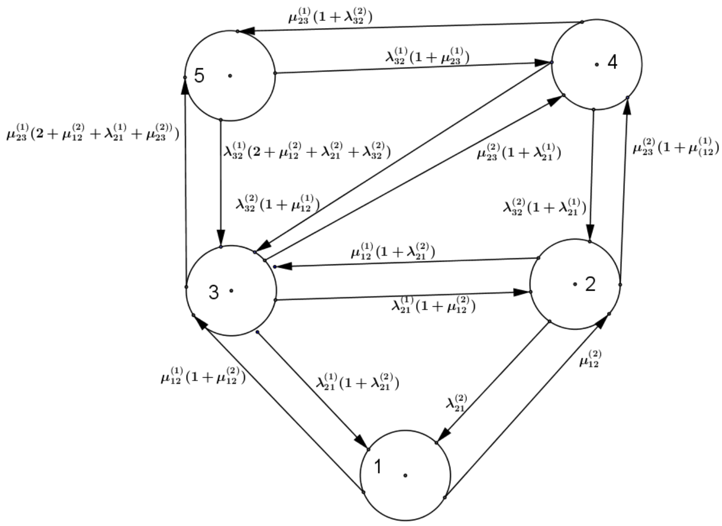

Figure 3.2: State transition diagram of the system

System State	State of Gen 1	State of Gen 2	System Output
1	1	1	0MW
2	1	2	50MW
	2	1	
3	2	2	80MW
	1	3	
4	2	3	100MW
	3	1	
	3	2	
5	3	3	150MW

Transition intensity matrix

$$A = \begin{bmatrix} \beta 1 & \mu_{12}^{(2)} & \mu_{12}^{(1)}(1+\mu_{12}^{(2)}) & 0 & 0 \\ \lambda_{21}^{(2)} & \beta 2 & \mu_{12}^{(1)}(1+\lambda_{21}^{(2)}) & \mu_{23}^{(2)}(1+\mu_{12}^{(1)}) & 0 \\ \lambda_{21}^{(1)}(1+\lambda_{21}^{(2)}) & \lambda_{21}^{(1)}(1+\mu_{12}^{(2)}) & \beta 3 & \mu_{23}^{(2)}(1+\lambda_{21}^{(1)}) & \mu_{23}^{(1)}(2+\mu_{12}^{(2)}+\lambda_{21}^{(1)}+\mu_{23}^{(2)}) \\ 0 & \lambda_{32}^{(2)}(1+\lambda_{21}^{(1)}) & \lambda_{32}^{(2)}(1+\mu_{12}^{(1)}) & \beta 4 & \mu_{23}^{(1)}(1+\lambda_{32}^{(2)}) \\ 0 & 0 & \lambda_{32}^{(1)}(2+\mu_{12}^{(2)}+\lambda_{21}^{(2)}+\lambda_{32}^{(2)}) & \lambda_{32}^{(1)}(1+\mu_{23}^{(2)}) & \beta 5 \end{bmatrix}$$

where

$$\begin{aligned} \beta 1 &= -(\mu_{12}^{(2)} + \mu_{12}^{(1)}(1+\mu_{12}^{((2))})), \\ \beta 2 &= -(\lambda_{21}^{(2)} + \mu_{12}^{(1)}(1+\lambda_{21}^{(2)}) + \mu_{23}^{(2)}(1+\mu_{12}^{(1)})), \\ \beta 3 &= -(\lambda_{21}^{(1)}(1+\lambda_{21}^{(2)}) + \lambda_{21}^{1}(1+\mu_{12}^{((2)}) + \mu_{23}^{(2)}(1+\lambda_{21}^{(1)}) + \mu_{23}^{(1)}(2+\mu_{12}^{(2)}+\lambda_{21}^{(1)}+\mu_{23}^{(2)})) \end{aligned}$$

$$
\begin{aligned}
\beta 4 &= -(\lambda_{32}^{(2)}(1+\lambda_{21}^{(1)}) + \lambda_{32}^{(2)}(1+\mu_{12}^{(1)}) + \mu_{23}^{(1)}(1+\lambda_{32}^{(2)})) \\
\beta 5 &= -(\lambda_{32}^{(1)}(2+\mu_{12}^{(2)}+\lambda_{21}^{(2)}+\lambda_{32}^{(2)}) + \lambda_{32}^{(1)}(1+\mu_{23}^{(2)})),
\end{aligned}
$$

The steady state probability vector

$$p = [p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5]$$

satisfy the equations (3.2) and (3.3).

$$
\begin{aligned}
-(\mu_{12}^{(2)} + \mu_{12}^{(1)}(1+\mu_{12}^{((2))}))p_1 + \lambda_{21}^{(2)}p_2 + \lambda_{21}^{(1)}(1+\lambda_{21}^{(2)})p_3 = 0 \\
\mu_{12}^{(2)}p_1 - (\lambda_{21}^{(2)} + \mu_{12}^{(1)}(1+\lambda_{21}^{(2)}) + \mu_{23}^{(2)}(1+\mu_{12}^{(1)}))p_2 + \lambda_{21}^{(1)}(1+\mu_{12}^{(2)})p_3 + \lambda_{32}^{(2)}(1+\lambda_{21}^{(1)})p_4 = 0 \\
\mu_{12}^{(1)}(1+\mu_{12}^{(2)})p_1 + \mu_{12}^{(1)}(1+\lambda_{21}^{(2)})p_2 \\
-(\lambda_{21}^{(1)}(1+\lambda_{21}^{(2)}) + \lambda_{21}^{(1)}(1+\mu_{12}^{((2)}) + \mu_{23}^{(2)}(1+\lambda_{21}^{(1)}) + \mu_{23}^{(1)}(2+\mu_{12}^{(2)}+\lambda_{21}^{(1)}+\mu_{23}^{(2)}))p_3 \\
+\lambda_{32}^{(2)}(1+\mu_{12}^{(1)})p_4 + \lambda_{32}^{(1)}(2+\mu_{12}^{(2)}+\lambda_{21}^{(2)}+\lambda_{32}^{(2)})p_5 = 0 \\
\mu_{23}^{(2)}(1+\mu_{12}^{(1)}))p_2 + \mu_{23}^{(2)}(1+\lambda_{21}^{(1)})p_3 - (\lambda_{32}^{(2)}(1+\lambda_{21}^{(1)}) + \lambda_{32}^{(2)}(1+\mu_{12}^{(1)}) + \mu_{23}^{(1)}(1+\lambda_{32}^{(2)}))p_4 \\
+\lambda_{32}^{(1)}(1+\mu_{23}^{(2)})p_5 = 0 \\
\mu_{23}^{(1)}(2+\mu_{12}^{(2)}+\lambda_{21}^{(1)}+\mu_{23}^{(2)})p_3 + \mu_{23}^{(1)}(1+\lambda_{32}^{(2)})p_4 \\
-(\lambda_{32}^{(1)}(2+\mu_{12}^{(2)}+\lambda_{21}^{(2)}+\lambda_{32}^{(2)}) + \lambda_{32}^{(1)}(1+\mu_{23}^{(2)}))p_5 = 0 \\
p_1+p_2+p_3+p_4+p_5 = 1.
\end{aligned}
$$

Steady state probabilities are evaluated by solving five equations in the above six system of equations.

$p_1 = 0.00015 \quad p_2 = 0.00029 \quad p_3 = 0.00554 \quad p_4 = 0.01837 \quad p_5 = 0.97566$

$$A_\infty(w = 80MW) = 0.99985$$

$$E_\infty = 148.6437MW$$

$$D_\infty(w = 80MW) = 0.0207MW$$

Series MSS

Assuming that the generators are connected in series and the total generating capacity of the system is determined as minimum of the generating capacities of the two generators. Based on the series structure function, the system output performance rate is computed as $\phi_S(G_1, G_2) = Min(G_1, G_2)$.

System State	State of Gen 1	State of Gen 2	System Output
	1	1	
	1	2	
1	1	3	0MW
	2	1	
	3	1	
	2	2	
2	3	2	50MW
3	2	3	80MW
4	3	3	100MW

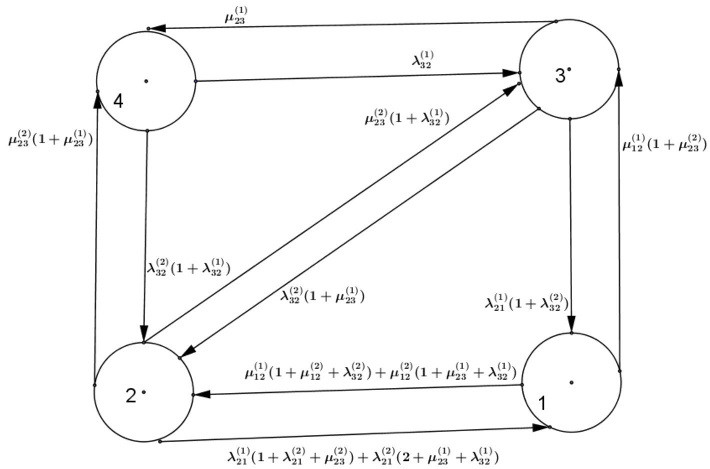

Figure 3.3: State transition diagram of the system

Transition intensity matrix

$$A = \begin{bmatrix} \gamma 1 & \mu_{12}^{(1)}(1+\mu_{12}^{(2)}+\lambda_{32}^{(2)})+\mu_{12}^{(2)}(2+\mu_{23}^{(1)}+\lambda_{32}^{(1)}) & \mu_{12}^{(1)}(1+\mu_{23}^{(2)}) & 0 \\ \lambda_{21}^{(1)}(1+\lambda_{21}^{(2)}+\mu_{23}^{(2)})+\lambda_{21}^{(2)}(2+\mu_{23}^{(1)}+\lambda_{32}^{(1)}) & \gamma 2 & \mu_{23}^{(2)}(1+\lambda_{32}^{(1)}) & \mu_{23}^{(2)}(1+\mu_{23}^{(1)}) \\ \lambda_{21}^{(1)}(1+\lambda_{32}^{(2)}) & \lambda_{32}^{(2)}(1+\mu_{23}^{(1)}) & \gamma 3 & \mu_{23}^{(1)} \\ 0 & \lambda_{32}^{(2)}(1+\lambda_{32}^{(1)}) & \lambda_{32}^{(1)} & \gamma_4 \end{bmatrix}$$

where

$$\gamma 1 = -(\mu_{12}^{(1)}(1+\mu_{12}^{(2)}+\lambda_{32}^{(2)})+\mu_{12}^{(2)}(2+\mu_{23}^{(1)}+\lambda_{32}^{(1)})+\mu_{12}^{(1)}(1+\mu_{23}^{(2)})),$$

$$\gamma 2 = -(\lambda_{21}^{(1)}(1+\lambda_{21}^{(2)}+\mu_{23}^{(2)})+\lambda_{21}^{(2)}(2+\mu_{23}^{(1)}+\lambda_{32}^{(1)})+\mu_{23}^{(2)}(1+\lambda_{32}^{(1)})+\mu_{23}^{(2)}(1+\mu_{23}^{(1)})),$$

$$\gamma 3 = -(\lambda_{21}^{(1)}(1+\lambda_{32}^{(2)})+\lambda_{32}^{(2)}(1+\mu_{23}^{(1)})+\mu_{23}^{(1)})$$

$$\gamma 4 = -(\lambda_{32}^{(2)}(1+\lambda_{32}^{(1)})+\lambda_{32}^{(1)})$$

The steady state probabilities

$$p = [p_1 \quad p_2 \quad p_3 \quad p_4]$$

satisfy equations (3.2) and (3.3).

$$-(\mu_{12}^{(1)}(1+\mu_{12}^{(2)}+\lambda_{32}^{(2)})+\mu_{12}^{(2)}(2+\mu_{23}^{(1)}+\lambda_{32}^{(1)})+\mu_{12}^{(1)}(1+\mu_{23}^{(2)}))p_1$$

$$+\lambda_{21}^{(1)}(1+\lambda_{21}^{(2)}+\mu_{23}^{(2)})+\lambda_{21}^{(2)}(2+\mu_{23}^{(1)}+\lambda_{32}^{(1)})p_2+\lambda_{21}^{(1)}(1+\lambda_{32}^{(2)})p_3=0$$

$$\mu_{12}^{(1)}(1+\mu_{12}^{(2)}+\lambda_{32}^{(2)})+\mu_{12}^{(2)}(2+\mu_{23}^{(1)}+\lambda_{32}^{(2)})p_1$$

$$-(\lambda_{21}^{(1)}(1+\lambda_{21}^{(2)}+\mu_{23}^{(2)})+\lambda_{21}^{(2)}(2+\mu_{23}^{(1)}+\lambda_{32}^{(1)}+\mu_{23}^{(2)}(1+\lambda_{32}^{(1)})+\mu_{23}^{(2)}(1+\mu_{23}^{(1)}))p_2$$

$$+\lambda_{32}^{(2)}(1+\mu_{23}^{(1)})p_3+\lambda_{32}^{(2)}(1+\lambda_{32}^{(1)})p_4=0$$

$$\mu_{12}^{(1)}(1+\mu_{23}^{(2)})p_1+\mu_{23}^{(2)}(1+\lambda_{32}^{(1)})p_2$$

$$-(\lambda_{21}^{(1)}(1+\lambda_{32}^{(2)})+\lambda_{32}^{(2)}(1+\mu_{23}^{(1)})+\mu_{23}^{(1)})p_3+\lambda_{32}^{(1)}p_4=0$$

$$\mu_{23}^{(2)}(1+\mu_{23}^{(1)})p_2+\mu_{23}^{(1)}p_3-(\lambda_{32}^{(2)}(1+\lambda_{32}^{(1)})+\lambda_{32}^{(1)})p_4=0$$

$$p_1+p_2+p_3+p_4=1.$$

Steady state probabilities are evaluated by solving four equations in the above five system of equations.

$p_1 = 0.00037 \quad p_2 = 0.00598 \quad p_3 = 0.01346 \quad p_4 = 0.98019$

$$A_{\infty}(w = 80MW) = 0.99963$$

$$E_{\infty} = 99.3948MW$$

$$D_{\infty}(w = 80MW) = 0.209MW$$

3.4 Conclusion

Comparison between system performance measures of different structure of independent generators of the power generating system helps in effective decision making. The dimension of the system of equations is the critical factor that decides the difficulty of computational complexity in random process method. The method used in this chapter is extensively applied for the multi state system reliability analysis and is more universal than other methods.

CHAPTER 4

EVALUATION OF SYSTEM PERFORMANCE MEASURES OF MULTI STATE DEGRADED SYSTEM WITH MINIMAL REPAIR

4.1 Introduction

[1] A combined method in reliability evaluation of a multi state system based on stochastic process and universal generating function (UGF) is discussed in this chapter. The traditional random process method are often suggested for evaluation of performance measures of a multi state system which is discussed in the previous chapter. But this method is not effective enough for the multi state system in which the system modeling is complicated when the number of states of the sys-

[1]Some contents of this chapter are based on Manoharan and Vidhya (2017).

tem is extremely large. Universal generating function technique has been developed by Ushakov (1986) and it is successfully implemented by Lisnianski et al. (1996), Levitin et al. (1997) and Levitin and Lisnianski (1998) in reliability analysis and optimization of power system. A detailed discussion on many technical application of universal generating function models was given by Lisnianski and Levitin (2003) and Levitin (2005). They explained application of the UGF based reliability block diagram technique which minimizes the dimension of system of equations in a stochastic process method. This technique is being implemented for multi state systems with statistically independent repairable components. At first system of equations for each component of the system is solved one by one in this combined approach and the results are consolidated using UGF technique for reliability evaluation of a multi state system. Levitin (2007) proposed a modification of the generalized reliability block diagram method for reliability assessment of multi-state systems with uncovered failures. Combined UGF and random process method described by Lisnianski and Ding (2009) for reliability evaluation of inter connected repairable multi state system . A multi state system that consists of multi state components with minor failures and minor repairs was considered by Qin et al. (2016) and combined method is considered for obtaining reliability indices of the system.

In this chapter we focuss on problem of evaluation of steady state probabilities and system performance measures of a MSS consists of independent repairable components with minor repairs. Description of model with assumptions has been presented in section 4.2. In section 4.3 the combined stochastic process and UGF technique approach is applied for multi state degraded system for avoiding dimension damnation problem of the stochastic process approach. A different approach of decomposing a system in to two or more sub systems (each sub system consists of the same type of components) has been proposed. Steady state probabilities and system performance characteristics are calculated for subsystems using the random process method and reliability indices of the entire system in steady state situation are finally evaluated using UGF technique. A more realistic system has been taken to validate the applicability of this approach. A power station with two sub systems (each sub system with three generators) has been illustrated in section 4.4 of this chapter.

4.2 Multi state degraded system

A multi state system with n subsystems is considered and each subsystem consists of m components. Any subsystem j of MSS have k_j different states with performance rates are represented by the set $g^{(j)} = \{g_1^{(j)}, g_2^{(j)}, g_3^{(j)}, \ldots, g_{k_j}^{(j)}\}$ where $g_i^{(j)}$ is the output of subsystem j in state i, $i \in \{1, 2, \ldots k_j\}$. The output $G_j(t)$ of subsystem j at any instant $t \geq 0$ is a random variable and it takes values from $g^j : G_j(t) \in g^{(j)}$.

Assumptions

- The system or subsystem may have many levels of degradation which vary from perfect functioning to a complete failure.

- The system or subsystem might fail any 'up' state to its 'down' states and it is minimally repaired.
- The components of the system might fail independently and they are operated on continuous basis.
- The components of the system are repaired independently.

4.3 Analysis of Model

State space of j^{th} subsystem is $S = \{0, 1, 2, ...k_j\}$. Components of the system have variable failure rates and variable repair rates . When a component fails a repair action is initiated to bring the component back to its initial up state. The Markov model for each subsystem, which is described in chapter 1, will be developed. The system of differential equations describing the nature of the system can be derived based on the state transition intensity matrix. The distribution of Markov process at time t is described as the row vector $p(t) = [p_1(t), ..., p_{k_j}(t)]$. We recognized that the process started in state i at time 0. The distribution $p(t)$ will be calculated from the Kolmogrove forward equations

Steady state probability vector $p = [p_1, p_2, ..., p_{k_j}]$ must satisfy the matrix equation

$$pA = 0 \tag{4.1}$$

and

$$\sum_{i=1}^{k_j} p_i = 1 \tag{4.2}$$

where A is the state transition matrix of each subsystem. The steady state probabilities p_1, p_2, ...,p_{k_j} are evaluated using k of the k+1 linear algebraic equations from the matrix equation (4.1) and in addition to this the sum of the steady state probabilities is always equal to one (4.2). This can be computed easily using computation algorithms based on Mathematica.

In general, a system consists of n subsystems with each subsystem possessing k_j states. Here $g^{(j)} = \{g_1^{(j)}, g_2{}^{(j)}, g_3^{(j)}, \ldots, g_{k_j}^{(j)}\}$ is the performance level of subsystem j. The steady state probability of j^{th} subsystem $p^{(j)} = \{p_1^{(j)}, p_2^{(j)}, \ldots p_{k_j}^{(j)}\}$ is determined by previously described stochastic process approach.

The UGF of the j^{th} subsystem is determined as

$$u^j(z) = \sum_{i=1}^{k_j} p^{(i)} z^{g^{(i)}}$$

The structure function of a MSS consisting of series and parallel subsystem may be determined by reliability block diagram method ie, iteratively composing the structure functions of the independent subsystems. In order to find u-function for the entire MSS the corresponding operators Ω_Φ operators should be applied. $\Omega_{\Phi s}$ and $\Omega_{\Phi p}$ are used for the subsystems connected in series and parallel respectively.

For MSS with n subsystem connected in parallel the system structure function is in the form

$$U(z) = \Omega_{\Phi p}\{u^1(z), u^2(z), \ldots u^n(z)\}$$

which corresponds to output probability distribution $g = \{g_1, g_2, ..., g_k\}$, $p = \{p_1, p_2, ..., p_k\}$

Reliability indices of the system in steady state situation

We have multi state system probability distribution (PD) in the form of universal generating function as

$$U(z) = \sum_{i=1}^{k} p_i z^{g_i}$$

1. **Steady state MSS availability**

   Steady state MSS availability can be obtained for any arbitrary constant demand w using the following operator δ_A.

$$A_\infty(w) = \delta_A(U(z), w) = \delta_A(\sum_{i=1}^{k} p_i z^{g_i}, w) = \sum_{i=1}^{k} p_i I(F(g_i, w) \geq 0)$$

   where $F(g_i, w)$ is an acceptability function.

2. **Expected Steady state MSS performance**

   Expected steady state output performance (Mean Steady state performance) can be obtained for the given $U(z)$ using the following operator δ_E.

$$E_\infty = \delta_E(U(z)) = \delta_E(\sum_{i=1}^{k} p_i z^{g_i}) = \sum_{i=1}^{k} p_i g_i$$

3. **Expected steady state MSS performance deficiency**

   Expected steady state MSS performance deficiency (mean performance deficiency for the given $U(z)$ can be obtained for an arbitrary demand w using the following operator δ_D.

$$D_\infty(w) = \delta_D(U(z), w) = \delta_D(\sum_{i=1}^{k} p_i z^{g_i}, w) = \sum_{i=1}^{k} p_i.max(w - g_i, 0)$$

4.4 Numerical Example

In this section the aforesaid method is applied to carry out the reliability analysis based on the data collected from Kuttiady Hydro Electric Project, governed by Kerala State Electricity Board (KSEB) under Govt. of Kerala, located at Kakkayam, Kozhikode district. With an installed capacity 75 MW (3 generators each with 25 MW), the Kuttiady power station was commissioned on 30-09-1972. The next generator with installed capacity 50MW was commissioned on 27-01-2001. Last two generators with installed capacity each 50 MW were commissioned on 30-10-2010 and 10-11-2010. Total capacity of the power station is 225 MW. The generation of the power station is controlled by state Load Despatch Centre, a functional unit of KSEB, which is the apex body to ensure integrated operation of the power system in Kerala. According to the centre the production of generators are categorized into three - either in full generation mode or half generation mode or zero generation mode. Three generators with installed capacity 75MW (each with 25MW) have same features. States and outputs of Generator 1, 2 and 3 (G1, G2 and G3) are respectively 1 (0 MW), 2 (12.5 MW) and 3 (25 MW) and these constitute Subsystem 1. Other three generators with installed capacity 150 MW (each with 50 MW) have same features. States and outputs of Generators 4, 5 and 6 (G4, G5 and G6) are respectively 1 (0MW), 2 (25 MW)and 3 (50 MW) and these generators constitute subsystem 2. The generators in subsystems are connected in parallel and two subsystems are also connected in parallel. That is total output of the subsystems is equal to the sum of the outputs of the generators and total output of the system is the sum of the outputs of the subsystems.

Here the system is a power station and is made up of two sub systems. Each sub systems contains same type of generators. The steady state probabilities of each sub system can be evaluated as following.

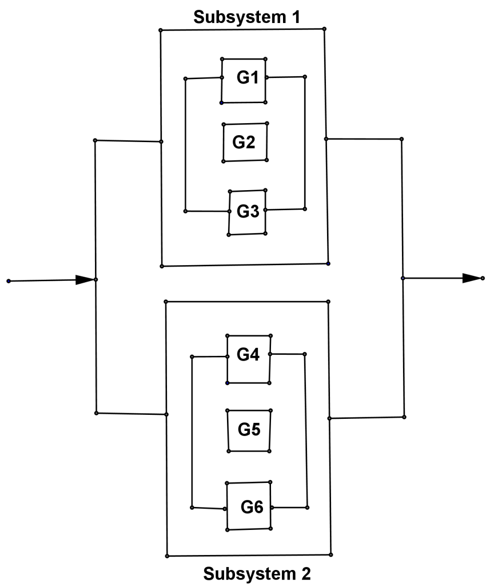

Figure 4.1: Reliability block diagram of the system

Subsystem 1

Sub system 1 consists of three generators (components) G1, G2 and G3. Each component of the sub system has three states with corresponding outputs 0 MW, 12.5 MW and 25 MW and whole sub system has seven states with corresponding out puts 0 MW, 12.5 MW, 25 MW, 37.5 MW, 50 MW, 62.5 MW and 75 MW. The steady state probability vector $p = [p_1^{(1)}\ p_2^{(1)}\ p_3^{(1)}\ p_4^{(1)}\ p_5^{(1)}\ p_6^{(1)}\ p_7^{(1)}]$ is obtained using equations (4.1) and (4.2).

$$a_{11}p_1^{(1)} + a_{21}p_2^{(1)} + a_{31}p_3^{(1)} + a_{41}p_4^{(1)} + a_{51}p_5^{(1)} + a_{61}p_6^{(1)} + a_{71}p_7^{(1)} = 0$$

$$a_{12}p_1^{(1)} + a_{22}p_2^{(1)} + a_{32}p_3^{(1)} + a_{42}p_4^{(1)} + a_{52}p_5^{(1)} + a_{62}p_6^{(1)} + a_{72}p_7^{(1)} = 0$$

$$a_{13}p_1^{(1)} + a_{23}p_2^{(1)} + a_{33}p_3^{(1)} + a_{43}p_4^{(1)} + a_{53}p_5^{(1)} + a_{63}p_6^{(1)} + a_{73}p_7^{(1)} = 0$$

$$a_{14}p_1^1 + a_{24}p_2^1 + a_{34}p_3^1 + a_{44}p_4^1 + a_{54}p_5^1 + a_{64}p_6^1 + a_{74}p_7^1 = 0$$

$$a_{25}p_2^{(1)} + a_{35}p_3^{(1)} + a_{45}p_4^{(1)} + a_{55}p_5^{(1)} + a_{65}p_6^{(1)} + a_{75}p_7^{(1)} = 0$$

$$a_{36}p_3^{(1)} + a_{46}p_4^{(1)} + a_{56}p_5^{(1)} + a_{66}p_6^{(1)} + a_{76}p_7^{(1)} = 0$$

$$a_{47}p_4^{(1)} + a_{57}p_5^{(1)} + a_{67}p_6^{(1)} + a_{77}p_7^{(1)} = 0$$

$$p_1^{(1)} + p_2^{(1)} + p_3^{(1)} + p_4^{(1)} + p_5^{(1)} + p_6^{(1)} + p_7^{(1)} = 1$$

where

$$a_{21} = \lambda_{21}^{(1)} + \lambda_{21}^{(2)} + \lambda_{21}^{(3)}$$

$$a_{31} = \lambda_{21}^{(1)}\lambda_{21}^{(2)} + \lambda_{21}^{(1)}\lambda_{21}^{(3)} + \lambda_{21}^{(2)}\lambda_{21}^{(3)} + \lambda_{31}^{(2)} + \lambda_{31}^{(1)} + \lambda_{31}^{(3)}$$

$$a_{41} = \lambda_{21}^{(1)}\lambda_{21}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{31}^{(2)} + \lambda_{21}^{(1)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{21}^{(2)} + \lambda_{31}^{(1)}\lambda_{21}^{(3)} + \lambda_{31}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(2)}\lambda_{31}^{(3)}$$

$$a_{51} = \lambda_{31}^{(1)}\lambda_{21}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{31}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{21}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{31}^{(2)}$$

$$a_{61} = \lambda_{31}^{(1)}\lambda_{31}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{31}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{21}^{(2)}\lambda_{31}^{(2)}$$

$$a_{71} = \lambda_{31}^{(1)}\lambda_{31}^{(2)}\lambda_{31}^{(3)}$$

$$a_{12} = \mu_{12}^{(1)} + \mu_{12}^{(2)} + \mu_{12}^{(3)}$$

$$a_{32} = 2\lambda_{21}^{(1)} + 2\lambda_{21}^{(2)} + 2\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{21}^{(2)}\mu_{12}^{(3)} + \lambda_{21}^{(1)}\mu_{12}^{(2)}\lambda_{21}^{(3)} + \mu_{12}^{(1)}\lambda_{21}^{(2)}\lambda_{21}^{(3)} + \mu_{12}^{(1)}\lambda_{31}^{(2)}$$

$$+\lambda_{32}^{(2)}\mu_{12}^{(3)} + \lambda_{31}^{(1)}\mu_{12}^{(2)} + \lambda_{31}^{(1)}\mu_{12}^{(3)} + \lambda_{32}^{(2)} + \lambda_{32}^{(1)} + \lambda_{32}^{(3)} + \mu_{12}^{(1)}\lambda_{31}^{(3)} + \mu_{12}^{(2)}\lambda_{31}^{(3)}$$

$$a_{42} = 3\lambda_{31}^{(1)} + 3\lambda_{31}^{(2)} + 2\lambda_{31}^{(3)} + \lambda_{21}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{21}^{(2)} + \lambda_{21}^{(1)}\lambda_{32}^{(2)} + \lambda_{21}^{(1)}\lambda_{31}^{(2)}\mu_{12}^{(3)}$$

$$+\lambda_{21}^{(1)}\mu_{12}^{(2)}\lambda_{31}^{(3)} + \lambda_{21}^{(1)}\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\lambda_{21}^{(2)} + \lambda_{31}^{(1)}\lambda_{21}^{(2)}\mu_{12}^{(3)} + \lambda_{32}^{(1)}\lambda_{21}^{(3)} + \lambda_{31}^{(1)}\mu_{12}^{(2)}\lambda_{21}^{(3)}$$

$$+\mu_{12}^{(1)}\lambda_{31}^{(2)}\lambda_{21}^{(3)} + \lambda_{32}^{(2)}\lambda_{21}^{(3)} + \mu_{12}^{(1)}\lambda_{21}^{(2)}\lambda_{31}^{(3)} + \lambda_{21}^{(2)}\lambda_{32}^{(3)}$$

$$a_{52} = \lambda_{32}^{(1)}\lambda_{21}^{(2)}\lambda_{21}^{(3)} + \lambda_{31}^{(1)}\lambda_{21}^{(3)} + \lambda_{31}^{(1)}\lambda_{21}^{(2)} + \lambda_{31}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{32}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{31}^{(2)}$$

$$+\lambda_{21}^{(2)}\lambda_{31}^{(3)} + \lambda_{21}^{(1)}\lambda_{31}^{(3)} + \lambda_{21}^{(1)}\lambda_{21}^{(2)}\lambda_{32}^{(3)} + \mu_{12}^{(1)}\lambda_{31}^{(2)}\lambda_{32}^{(3)} + \lambda_{32}^{(2)}\lambda_{31}^{(1)} + \lambda_{31}^{(2)}\lambda_{32}^{(3)}$$

$$+\lambda_{32}^{(1)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\mu_{12}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\lambda_{31}^{(2)} + \lambda_{31}^{(1)}\lambda_{32}^{(2)} + \lambda_{31}^{(1)}\lambda_{31}^{(2)}\mu_{12}^{(3)}$$

$$a_{62} = \lambda_{32}^{(1)}\lambda_{31}^{(2)}\lambda_{21}^{(3)} + \lambda_{31}^{(1)}\lambda_{32}^{(2)}\lambda_{21}^{(3)} + \lambda_{31}^{(1)}\lambda_{31}^{(2)} + \lambda_{31}^{(2)}\lambda_{31}^{(3)} + \lambda_{21}^{(1)}\lambda_{32}^{(2)}\lambda_{31}^{(3)}$$

$$+\lambda_{21}^{(1)}\lambda_{31}^{(2)}\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\lambda_{21}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{21}^{(2)}\lambda_{32}^{(3)}$$

$$a_{72} = \lambda_{32}^{(1)}\lambda_{31}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{32}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{31}^{(2)}\lambda_{32}^{(3)}$$

$$a_{13} = \mu_{12}^{(1)}\mu_{12}^{(2)} + \mu_{12}^{(1)}\mu_{12}^{(3)} + \mu_{12}^{(2)}\mu_{12}^{(3)}$$

$$a_{23} = 2\mu_{12}^{(1)} + 2\mu_{12}^{(2)} + 2\mu_{12}^{(3)} + \mu_{23}^{(1)} + \mu_{23}^{(2)} + \mu_{23}^{(3)} + \mu_{12}^{(1)}\mu_{12}^{(2)} + \mu_{12}^{(2)}\mu_{12}^{(3)} + \mu_{12}^{(1)}\mu_{12}^{(3)}$$

$$a_{43} = 3\lambda_{21}^{(1)} + 3\lambda_{21}^{(2)} + 2\lambda_{21}^{(3)} + 2\lambda_{32}^{(1)} + 2\lambda_{32}^{(2)} + \lambda_{32}^{(3)} + \mu_{12}^{(3)} + \lambda_{31}^{(3)} + \lambda_{21}^{(1)}\mu_{23}^{(2)}\lambda_{21}^{(3)}$$

$$+\mu_{23}^{(1)}\lambda_{21}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{21}^{(2)}\mu_{23}^{(3)} + \lambda_{31}^{(2)}\mu_{12}^{(3)} + \mu_{23}^{(1)}\lambda_{31}^{(3)} + \mu_{12}^{(3)}\lambda_{31}^{(3)} + \lambda_{21}^{(1)}\lambda_{32}^{(2)}\mu_{12}^{(3)}$$

$$+\mu_{23}^{(1)}\lambda_{31}^{(3)} + \lambda_{32}^{(1)}\mu_{12}^{(2)}\lambda_{21}^{(3)} + \lambda_{31}^{(1)}\mu_{12}^{(2)} + \lambda_{31}^{(1)}\mu_{23}^{(2)} + \lambda_{32}^{(1)}\mu_{12}^{(2)}\lambda_{21}^{(3)} + \lambda_{31}^{(2)}\mu_{12}^{(3)} + \lambda_{31}^{(1)}\mu_{23}^{(3)}$$

$$+\mu_{12}^{(1)}\lambda_{32}^{(2)}\lambda_{21}^{(3)} + \mu_{12}^{(1)}\lambda_{31}^{(2)} + \lambda_{31}^{(2)}\mu_{23}^{(3)} + \mu_{12}^{(1)}\lambda_{31}^{(3)} + \mu_{12}^{(1)}\lambda_{21}^{(2)}\lambda_{32}^{(3)} + \mu_{23}^{(2)}\lambda_{31}^{(3)}$$

$$a_{53} = 2\lambda_{31}^{(1)} + 2\lambda_{31}^{(2)} + 2\lambda_{31}^{(3)} + \lambda_{32}^{(1)} + \lambda_{32}^{(1)}\lambda_{21}^{(2)} + \lambda_{31}^{(1)}\mu_{12}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(2)}\lambda_{21}^{(3)} + \lambda_{31}^{(1)}\lambda_{21}^{(2)}\mu_{23}^{(3)}$$

$$+\lambda_{32}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\lambda_{32}^{(2)} + \mu_{23}^{(1)}\lambda_{31}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\mu_{23}^{(2)}\lambda_{31}^{(3)} + \mu_{23}^{(1)}\lambda_{21}^{(2)}\lambda_{31}^{(3)}$$

$$+\lambda_{21}^{(1)}\lambda_{21}^{(2)} + \mu_{12}^{(1)}\lambda_{32}^{(2)}\lambda_{31}^{(3)} + \mu_{12}^{(1)}\lambda_{31}^{(2)}\lambda_{32}^{(3)} + \lambda_{32}^{(2)}\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\mu_{12}^{(2)}\lambda_{31}^{(3)}$$

$$+\lambda_{32}^{(1)}\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\lambda_{31}^{(2)}\mu_{12}^{(3)} + \lambda_{32}^{(1)}\lambda_{31}^{(2)}\mu_{12}^{(2)} + \lambda_{31}^{(1)}\lambda_{32}^{(2)}\mu_{12}^{(3)}$$

$$a_{63} = \lambda_{32}^{(1)}\lambda_{32}^{(2)}\lambda_{21}^{(3)} + \lambda_{32}^{(1)}\lambda_{31}^{(2)} + \lambda_{31}^{(1)}\lambda_{32}^{(2)} + \lambda_{31}^{(1)}\lambda_{21}^{(3)} + \lambda_{31}^{(2)}\lambda_{21}^{(3)} + \lambda_{31}^{(1)}\lambda_{31}^{(2)}\mu_{23}^{(3)}$$

$$+\lambda_{32}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(2)}\lambda_{32}^{(3)} + \lambda_{21}^{(1)}\lambda_{32}^{(2)}\lambda_{32}^{(3)} + \lambda_{21}^{(1)}\lambda_{31}^{(3)} + \mu_{23}^{(1)}\lambda_{31}^{(2)}\lambda_{31}^{(3)} + \lambda_{21}^{(1)}\lambda_{31}^{(2)}$$

$$+\lambda_{32}^{(1)}\lambda_{31}^{(3)} + \lambda_{32}^{(1)}\lambda_{21}^{(2)}\lambda_{32}^{(3)} + \lambda_{31}^{(1)}\lambda_{32}^{(3)} + \lambda_{31}^{(1)}\mu_{23}^{(2)}\lambda_{31}^{(3)} + \lambda_{21}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{21}^{(2)}$$

$$a_{73} = \lambda_{32}^{(1)}\lambda_{32}^{(2)} + \lambda_{32}^{(1)}\lambda_{32}^{(3)} + \lambda_{32}^{(2)}\lambda_{32}^{(3)} + \lambda_{31}^{(1)}\lambda_{31}^{(3)} + \lambda_{31}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(1)}\lambda_{21}^{(2)}$$

$$a_{14} = \mu_{12}^{(1)}\mu_{12}^{(2)}\mu_{12}^{(3)}$$

$$a_{24} = \mu_{12}^{(1)}\mu_{12}^{(3)} + \mu_{23}^{(1)}\mu_{12}^{(2)} + \mu_{23}^{(1)}\mu_{12}^{(3)} + \mu_{12}^{(1)}\mu_{12}^{(3)} + \mu_{12}^{(1)}\mu_{23}^{(2)} + \mu_{23}^{(2)}\mu_{12}^{(3)} + \mu_{12}^{(1)}\mu_{12}^{(2)}$$

$$+\mu_{12}^{(1)}\mu_{23}^{(3)} + \mu_{12}^{(2)}\mu_{23}^{(3)}$$

$$a_{34} = 3\mu_{12}^{(1)} + 3\mu_{12}^{(2)} + 3\mu_{12}^{(3)} + 2\mu_{23}^{(1)} + 2\mu_{23}^{(2)} + 2\mu_{23}^{(3)} + \mu_{23}^{(1)}\mu_{12}^{(3)}\lambda_{21}^{(2)} + \lambda_{21}^{(1)}\mu_{23}^{(2)}\mu_{12}^{(3)}$$

$$+\mu_{23}^{(1)}\mu_{12}^{(2)}\lambda_{21}^{(3)} + \lambda_{21}^{(1)}\mu_{12}^{(2)}\mu_{23}^{(3)} + \mu_{12}^{(1)}\mu_{23}^{(2)}\lambda_{21}^{(3)} + \mu_{12}^{(1)}\lambda_{21}^{(2)}\mu_{23}^{(3)} + \mu_{12}^{(1)}\lambda_{32}^{(2)}\mu_{12}^{(3)}$$

$$+\lambda_{32}^{(1)}\mu_{12}^{(2)}\mu_{12}^{(3)} + \mu_{12}^{(1)}\mu_{12}^{(2)}\lambda_{32}^{(3)}$$

$$a_{54} = 3\lambda_{21}^{(1)} + 2\lambda_{21}^{(2)} + \lambda_{21}^{(3)} + 3\lambda_{32}^{(1)} + 3\lambda_{32}^{(2)} + 3\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\mu_{23}^{(2)}\lambda_{21}^{(3)} + \lambda_{32}^{(1)}\lambda_{21}^{(2)}\mu_{23}^{(3)} + \lambda_{31}^{(1)}\mu_{23}^{(2)}$$

$$+\lambda_{31}^{(1)}\mu_{23}^{(3)} + \lambda_{31}^{(2)}\mu_{23}^{(3)} + \mu_{23}^{(1)}\lambda_{32}^{(2)}\lambda_{21}^{(3)} + \mu_{23}^{(1)}\lambda_{31}^{(2)} + \lambda_{21}^{(1)}\lambda_{32}^{(2)}\mu_{23}^{(3)} + \mu_{23}^{(2)}\lambda_{31}^{(3)} + \mu_{23}^{(1)}\lambda_{21}^{(2)}\lambda_{32}^{(3)}$$

$$+\lambda_{21}^{(1)}\mu_{23}^{(2)}\lambda_{32}^{(3)} + \mu_{12}^{(1)}\lambda_{32}^{(2)}\lambda_{32}^{(3)} + \mu_{12}^{(1)}\lambda_{31}^{(3)} + \mu_{12}^{(1)}\lambda_{31}^{(2)} + \lambda_{32}^{(1)}\mu_{12}^{(2)}\lambda_{32}^{(3)} + \mu_{12}^{(1)}\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\mu_{12}^{(2)}$$

$$+\lambda_{31}^{(1)}\lambda_{31}^{(2)}\mu_{12}^{(3)} + \lambda_{32}^{(2)}\mu_{12}^{(3)} + \lambda_{31}^{(1)}\mu_{12}^{(3)}$$

$$a_{64} = 2\lambda_{31}^{(1)} + 2\lambda_{31}^{(2)} + 2\lambda_{31}^{(3)} + \lambda_{32}^{(1)}\lambda_{32}^{(2)} + \lambda_{32}^{(1)}\lambda_{21}^{(3)} + \lambda_{32}^{(1)}\lambda_{31}^{(2)}\mu_{23}^{(3)} + \lambda_{32}^{(2)}\lambda_{21}^{(3)} + \lambda_{31}^{(1)}\lambda_{32}^{(2)}\mu_{23}^{(3)}$$

$$+\lambda_{32}^{(2)}\lambda_{32}^{(3)} + \mu_{23}^{(1)}\lambda_{32}^{(2)}\lambda_{31}^{(3)} + \mu_{23}^{(1)}\lambda_{31}^{(2)}\lambda_{32}^{(3)} + \lambda_{21}^{(1)}\lambda_{32}^{(3)} + \lambda_{21}^{(1)}\lambda_{32}^{(2)} + \lambda_{32}^{(1)}\lambda_{32}^{(3)}$$

$$+\lambda_{31}^{(1)}\mu_{23}^{(2)}\lambda_{31}^{(3)} + \lambda_{32}^{(1)}\lambda_{21}^{(2)} + \lambda_{21}^{(2)}\lambda_{32}^{(3)} + \mu_{23}^{(2)}\lambda_{32}^{(3)}$$

$$a_{74} = \lambda_{32}^{(1)}\lambda_{32}^{(2)}\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\lambda_{31}^{(3)} + \lambda_{32}^{(1)}\lambda_{31}^{(2)} + \lambda_{32}^{(2)}\lambda_{31}^{(3)} + \lambda_{31}^{(2)}\lambda_{32}^{(3)} + \lambda_{31}^{(1)}\lambda_{32}^{(3)} + \lambda_{31}^{(1)}\lambda_{32}^{(2)}$$

$$a_{25} = \mu_{23}^{(1)}\mu_{12}^{(2)}\mu_{12}^{(3)} + \mu_{12}^{(1)}\mu_{23}^{(2)}\mu_{12}^{(2)} + \mu_{12}^{(1)}\mu_{12}^{(2)}\mu_{23}^{(3)}$$

$$a_{35} = \mu_{23}^{(1)}\mu_{12}^{(3)} + \mu_{23}^{(2)}\mu_{12}^{(3)} + \mu_{23}^{(1)}\mu_{23}^{(2)} + \mu_{23}^{(1)}\mu_{12}^{(2)} + \mu_{12}^{(2)}\mu_{23}^{(3)} + \mu_{23}^{(1)}\mu_{23}^{(3)} + \mu_{12}^{(1)}\mu_{23}^{(2)} + \mu_{12}^{(1)}\mu_{23}^{(3)}$$

$$+\mu_{23}^{(2)}\mu_{23}^{(3)} + \mu_{12}^{(1)}\mu_{12}^{(3)} + \mu_{12}^{(2)}\mu_{12}^{(3)} + \mu_{12}^{(1)}\mu_{12}^{(2)}$$

$$a_{45} = 3\mu_{23}^{(1)} + 3\mu_{23}^{(2)} + 3\mu_{23}^{(3)} + 2\mu_{12}^{(1)} + 2\mu_{12}^{(2)} + 2\mu_{12}^{(3)} + \lambda_{21}^{(1)}\mu_{23}^{(2)}\mu_{23}^{(3)} + \mu_{23}^{(1)}\lambda_{21}^{(2)}\mu_{23}^{(3)}$$

$$+\mu_{23}^{(1)}\mu_{23}^{(2)}\lambda_{21}^{(3)} + \mu_{23}^{(1)}\lambda_{32}^{(2)}\mu_{12}^{(3)} + \mu_{23}^{(1)}\mu_{12}^{(2)}\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\mu_{23}^{(2)}\mu_{12}^{(3)}$$

$$+\lambda_{32}^{(1)}\mu_{12}^{(2)}\mu_{23}^{(3)} + \mu_{12}^{(1)}\lambda_{32}^{(2)}\mu_{23}^{(3)} + \mu_{12}^{(1)}\mu_{23}^{(2)}\lambda_{32}^{(3)}$$

$$a_{65} = \lambda_{21}^{(1)} + \lambda_{21}^{(2)} + \lambda_{21}^{(3)} + 2\lambda_{32}^{(1)} + 2\lambda_{32}^{(2)} + 2\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\lambda_{32}^{(2)}\mu_{23}^{(3)} + \lambda_{31}^{(1)}\mu_{23}^{(3)} + \lambda_{31}^{(2)}\mu_{23}^{(3)}$$

$$+\mu_{23}^{(1)}\lambda_{32}^{(2)}\lambda_{32}^{(3)} + \mu_{23}^{(1)}\lambda_{31}^{(2)} + \mu_{23}^{(1)}\lambda_{31}^{(3)} + \lambda_{32}^{(1)}\mu_{23}^{(2)}\lambda_{32}^{(3)} + \lambda_{31}^{(1)}\mu_{23}^{(2)} + \mu_{23}^{(2)}\lambda_{31}^{(3)}$$

$$a_{75} = \lambda_{32}^{(2)}\lambda_{32}^{(3)} + \lambda_{32}^{(1)}\lambda_{32}^{(2)} + \lambda_{32}^{(1)}\lambda_{32}^{(3)} + \lambda_{31}^{(1)} + \lambda_{31}^{(2)} + \lambda_{31}^{(3)}$$

$$a_{36} = \mu_{23}^{(1)}\mu_{23}^{(2)}\mu_{12}^{(3)} + \mu_{23}^{(1)}\mu_{12}^{(2)}\mu_{23}^{(3)} + \mu_{12}^{(1)}\mu_{23}^{(2)}\mu_{23}^{(3)}$$

$$a_{46} = \mu_{23}^{(1)}\mu_{23}^{(2)} + \mu_{23}^{(2)}\mu_{23}^{(3)} + \mu_{23}^{(1)}\mu_{23}^{(3)} + \mu_{23}^{(1)}\mu_{12}^{(3)} + \mu_{23}^{(1)}\mu_{12}^{(2)} + \mu_{23}^{(2)}\mu_{12}^{(3)} + \mu_{12}^{(2)}\mu_{23}^{(3)}$$

$$+\mu_{12}^{(1)}\mu_{23}^{(3)} + \mu_{12}^{(1)}\mu_{23}^{(2)}$$

$$a_{56} = 2\mu_{23}^{(1)} + 2\mu_{23}^{(2)} + 2\mu_{23}^{(3)} + \mu_{12}^{(1)} + \mu_{12}^{(2)} + \mu_{12}^{(3)} + \lambda_{32}^{(1)}\mu_{23}^{(2)}\mu_{23}^{(3)} + \mu_{23}^{(1)}\mu_{23}^{(2)}\lambda_{32}^{(3)}$$

$$a_{76} = \lambda_{32}^{(1)} + \lambda_{32}^{(2)} + \lambda_{32}^{(3)} \quad a_{47} = \mu_{23}^{(1)}\mu_{23}^{(2)}\mu_{23}^{(3)}$$

$$a_{57} = \mu_{23}^{(2)}\mu_{23}^{(3)} + \mu_{23}^{(1)}\mu_{23}^{(3)} + \mu_{23}^{(1)}\mu_{23}^{(2)} \quad a_{67} = \mu_{23}^{(1)} + \mu_{23}^{(2)} + \mu_{23}^{(3)}$$

$$a_{11} = -(a_{12} + a_{13} + a_{14}) \quad a_{22} = -(a_{21} + a_{23} + a_{24} + a_{25})$$

$$a_{33} = -(a_{31} + a_{32} + a_{34} + a_{35} + a_{36})$$

$$a_{44} = -(a_{41} + a_{42} + a_{43} + a_{45} + a_{46} + a_{47})$$

$$a_{55} = -(a_{51} + a_{52} + a_{53} + a_{54} + a_{56} + a_{57})$$

$$a_{66} = -(a_{61} + a_{62} + a_{63} + a_{64} + a_{65} + a_{67})$$

$$a_{77} = -(a_{71} + a_{72} + a_{73} + a_{74} + a_{75} + a_{76})$$

Transition rates of the generators G1,G2 and G3 per hour(h^{-1}) are calculated from the collected data and are given in the table below.

Table 4.1: Transition Rates

Generator	μ_{12}	μ_{23}	λ_{21}	λ_{32}	λ_{31}
G1	7.1×10^{-2}	6.4×10^{-2}	3×10^{-3}	6.7×10^{-2}	3.3×10^{-3}
G2	7.3×10^{-2}	6.5×10^{-2}	3×10^{-3}	6.8×10^{-2}	3.3×10^{-3}
G3	7.4×10^{-2}	6.1×10^{-2}	3×10^{-3}	6.2×10^{-2}	3.3×10^{-3}

Using Mathematica, the steady state probabilities are obtained as $p_1^{(1)} = 0.0056$, $p_2^{(1)} = 0.0421$, $p_3^{(1)} = 0.093$, $p_4^{(1)} = 0.156$,$p_5^{(1)} = 0.2021$,$p_6^{(1)} = 0.271$ and $p_7^{(1)} = 0.2302$.

Subsystem 2

Sub system 2 consists of three generators (components) G4, G5 and G6. Each component of the sub system has three states with corresponding outputs 0 MW, 25 MW and 50 MW and whole sub system has seven states with corresponding out puts 0 MW, 25 MW, 50 MW, 75 MW, 100 MW, 125 MW and 150 MW. The steady state probability vector $p = [p_1^{(2)}\ p_2^{(2)}\ p_3^{(2)}\ p_4^{(2)}\ p_5^{(2)}\ p_6^{(2)}\ p_7^{(2)}]$ is obtained using equations

(4.1) and (4.2).

$$
\begin{aligned}
b_{11}p^{1}_{(2)} + b_{21}p^{(2)}_{2} + b_{31}p^{(2)}_{3} + b_{41}p^{(2)}_{4} + b_{51}p^{(2)}_{5} + b_{61}p^{(2)}_{6} + b_{71}p^{(2)}_{7} &= 0 \\
b_{12}p^{(2)}_{1} + b_{22}p^{(2)}_{2} + b_{32}p^{(2)}_{3} + b_{42}p^{(2)}_{4} + b_{52}p^{(2)}_{5} + b_{62}p^{(2)}_{6} + b_{72}p^{(2)}_{7} &= 0 \\
b_{13}p^{(2)}_{1} + b_{23}p^{(2)}_{2} + b_{33}p^{(2)}_{3} + b_{43}p^{(2)}_{4} + b_{53}p^{(2)}_{5} + b_{63}p^{(2)}_{6} + b_{73}p^{(2)}_{7} &= 0 \\
b_{14}p^{(2)}_{1} + b_{24}p^{(2)}_{2} + b_{34}p^{(2)}_{3} + b_{44}p^{(2)}_{4} + b_{54}p^{(2)}_{5} + b_{64}p^{(2)}_{6} + b_{74}p^{(2)}_{7} &= 0 \\
b_{25}p^{(2)}_{2} + b_{35}p^{(2)}_{3} + b_{45}p^{(2)}_{4} + b_{55}p^{(2)}_{5} + b_{65}p^{(2)}_{6} + b_{75}p^{(2)}_{7} &= 0 \\
b_{36}p^{1}_{3} + b_{46}p^{1}_{4} + b_{56}p^{1}_{5} + b_{66}p^{1}_{6} + b_{76}p^{1}_{7} &= 0 \\
b_{47}p^{1}_{4} + b_{57}p^{1}_{5} + b_{67}p^{1}_{6} + b_{77}p^{1}_{7} &= 0 \\
p^{2}_{1} + p^{2}_{2} + p^{2}_{3} + p^{2}_{4} + p^{2}_{5} + p^{2}_{6} + p^{2}_{7} &= 1
\end{aligned}
$$

where

$$
\begin{aligned}
b_{21} &= \lambda^{(4)}_{21} + \lambda^{(5)}_{21} + \lambda^{(6)}_{21} \\
b_{31} &= \lambda^{(4)}_{21}\lambda^{(5)}_{21} + \lambda^{(4)}_{21}\lambda^{(6)}_{21} + \lambda^{(5)}_{21}\lambda^{(6)}_{21} + \lambda^{(5)}_{31} + \lambda^{(4)}_{31} + \lambda^{(6)}_{31} \\
b_{41} &= \lambda^{(4)}_{21}\lambda^{(5)}_{21}\lambda^{(6)}_{21} + \lambda^{(4)}_{21}\lambda^{(5)}_{31} + \lambda^{(4)}_{21}\lambda^{(6)}_{31} + \lambda^{(4)}_{31}\lambda^{(5}_{21} + \lambda^{(4)}_{31}\lambda^{(6)}_{21} + \lambda^{(5)}_{31}\lambda^{(6)}_{21} + \lambda^{(5)}_{21}\lambda^{(6)}_{31} \\
b_{51} &= \lambda^{(4)}_{31}\lambda^{(5)}_{21}\lambda^{(6)}_{21} + \lambda^{(4)}_{21}\lambda^{(5)}_{31}\lambda^{(6)}_{21} + \lambda^{(4)}_{21}\lambda^{(5)}_{21}\lambda^{(6)}_{31} + \lambda^{(5)}_{31}\lambda^{(6)}_{31} + \lambda^{(4)}_{31}\lambda^{(6)}_{31} + \lambda^{(4)}_{31}\lambda^{(5)}_{31} \\
b_{61} &= \lambda^{(4)}_{31}\lambda^{(5)}_{31}\lambda^{(6)}_{21} + \lambda^{(4)}_{21}\lambda^{(5)}_{31}\lambda^{(6)}_{31} + \lambda^{(4)}_{31}\lambda^{(5)}_{21}\lambda^{(5)}_{31}
\end{aligned}
$$

$$b_{71} = \lambda_{31}^{(4)}\lambda_{31}^{(5)}\lambda_{31}^{(6)}$$

$$b_{12} = \mu_{12}^{(4)} + \mu_{12}^{(5)} + \mu_{12}^{(6)}$$

$$b_{32} = 2\lambda_{21}^{(4)} + 2\lambda_{21}^{(5)} + 2\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\lambda_{21}^{(5)}\mu_{12}^{(6)} + \lambda_{21}^{(4)}\mu_{12}^{(5)}\lambda_{21}^{(6)} + \mu_{12}^{(4)}\lambda_{21}^{(5)}\lambda_{21}^{(6)} + \mu_{12}^{(4)}\lambda_{31}^{(5)}$$

$$+\lambda_{32}^{(5}\mu_{12}^{(6)} + \lambda_{31}^{(4)}\mu_{12}^{(5)} + \lambda_{31}^{(4)}\mu_{12}^{(6)} + \lambda_{32}^{(5)} + \lambda_{32}^{(4)} + \lambda_{32}^{(6)} + \mu_{12}^{(4)}\lambda_{31}^{(6)} + \mu_{12}^{(5)}\lambda_{31}^{(6)}$$

$$b_{42} = 3\lambda_{31}^{(4)} + 3\lambda_{31}^{(5)} + 2\lambda_{31}^{(6)} + \lambda_{21}^{(5)}\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\lambda_{21}^{(5)} + \lambda_{21}^{(4)}\lambda_{32}^{(5)} + \lambda_{21}^{(4)}\lambda_{31}^{(5)}\mu_{12}^{(6)}$$

$$+\lambda_{21}^{(4)}\mu_{12}^{(5)}\lambda_{31}^{(6)} + \lambda_{21}^{(4)}\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\lambda_{21}^{(5)} + \lambda_{31}^{(4)}\lambda_{21}^{(5)}\mu_{12}^{(6)} + \lambda_{32}^{(4)}\lambda_{21}^{(6)} + \lambda_{31}^{(4)}\mu_{12}^{(5)}\lambda_{21}^{(6)}$$

$$+\mu_{12}^{(4)}\lambda_{31}^{(5)}\lambda_{21}^{(6)} + \lambda_{32}^{(5)}\lambda_{21}^{(6)} + \mu_{12}^{(4)}\lambda_{21}^{(5)}\lambda_{31}^{(6)} + \lambda_{21}^{(5)}\lambda_{32}^{(6)}$$

$$b_{52} = \lambda_{32}^{(4)}\lambda_{21}^{(5)}\lambda_{21}^{(6)} + \lambda_{31}^{(4)}\lambda_{21}^{(6)} + \lambda_{31}^{(4)}\lambda_{21}^{(5)} + \lambda_{31}^{(5)}\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\lambda_{32}^{(5)}\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\lambda_{31}^{(5)}$$

$$+\lambda_{21}^{(5)}\lambda_{31}^{(6)} + \lambda_{21}^{(4)}\lambda_{31}^{(6)} + \lambda_{21}^{(4)}\lambda_{21}^{(5)}\lambda_{32}^{(6)} + \mu_{12}^{(4)}\lambda_{31}^{(5)}\lambda_{32}^{(6)} + \lambda_{32}^{(5)}\lambda_{31}^{(4)} + \lambda_{31}^{(5)}\lambda_{32}^{(6)}$$

$$+\lambda_{32}^{(4)}\lambda_{31}^{(6)} + \lambda_{31}^{(4)}\mu_{12}^{(5)}\lambda_{31}^{(6)} + \lambda_{31}^{(4)}\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\lambda_{31}^{(5)}$$

$$+\lambda_{31}^{(4)}\lambda_{32}^{(5)} + \lambda_{31}^{(4)}\lambda_{31}^{(5)}\mu_{12}^{(6)}$$

$$b_{62} = \lambda_{32}^{(4)}\lambda_{31}^{(5)}\lambda_{21}^{(6)} + \lambda_{31}^{(4)}\lambda_{32}^{(5)}\lambda_{21}^{(6)} + \lambda_{31}^{(4)}\lambda_{31}^{(5)} + \lambda_{31}^{(5)}\lambda_{31}^{(6)} + \lambda_{21}^{(4)}\lambda_{32}^{(5)}\lambda_{31}^{(6)}$$

$$+\lambda_{21}^{(4)}\lambda_{31}^{(5)}\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\lambda_{21}^{(5)}\lambda_{31}^{(6)} + \lambda_{31}^{(4)}\lambda_{31}^{(6)} + \lambda_{31}^{(4)}\lambda_{21}^{(5)}\lambda_{32}^{(6)}$$

$$b_{72} = \lambda_{32}^{(4)}\lambda_{31}^{(5)}\lambda_{31}^{(6)} + \lambda_{31}^{(4)}\lambda_{32}^{(5)}\lambda_{31}^{(6)} + \lambda_{31}^{(4)}\lambda_{31}^{(5)}\lambda_{32}^{(6)}$$

$$b_{13} = \mu_{12}^{(4)}\mu_{12}^{(5)} + \mu_{12}^{(4)}\mu_{12}^{(6)} + \mu_{12}^{(5)}\mu_{12}^{(6)}$$

$$b_{23} = 2\mu_{12}^{(4)} + 2\mu_{12}^{(5)} + 2\mu_{12}^{(6)} + \mu_{23}^{(4)} + \mu_{23}^{(5)} + \mu_{23}^{(6)} + \mu_{12}^{(4)}\mu_{12}^{(5)} + \mu_{12}^{(5)}\mu_{12}^{(6)} + \mu_{12}^{(4)}\mu_{12}^{(6)}$$

$$b_{43} = 3\lambda_{21}^{(4)} + 3\lambda_{21}^{(5)} + 2\lambda_{21}^{(6)} + 2\lambda_{32}^{(4)} + 2\lambda_{32}^{(5)} + \lambda_{32}^{(6)} + \mu_{12}^{(6)} + \lambda_{31}^{(6)} + \lambda_{21}^{(4)}\mu_{23}^{(5)}\lambda_{21}^{(6)}$$

$$+\mu_{23}^{(4)}\lambda_{21}^{(5)}\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\lambda_{21}^{(5)}\mu_{23}^{(6)} + \lambda_{31}^{(5)}\mu_{12}^{(6)} + \mu_{23}^{(4)}\lambda_{31}^{(6)} + \mu_{12}^{(3)}\lambda_{31}^{(3)} + \lambda_{21}^{(4)}\lambda_{32}^{(5)}\mu_{12}^{(6)}$$

$$+\mu_{23}^{(4)}\lambda_{31}^{(6)} + \lambda_{32}^{(4)}\mu_{12}^{(5)}\lambda_{21}^{(6)} + \lambda_{31}^{(4)}\mu_{12}^{(5)} + \lambda_{31}^{(4)}\mu_{23}^{(5)} + \lambda_{32}^{(4)}\mu_{12}^{(5)}\lambda_{21}^{(6)} + \lambda_{31}^{(5)}\mu_{12}^{(6)} + \lambda_{31}^{(4)}\mu_{23}^{(6)}$$

$$+\mu_{12}^{(4)}\lambda_{32}^{(5)}\lambda_{21}^{(6)} + \mu_{12}^{(4)}\lambda_{31}^{(5)} + \lambda_{31}^{(5)}\mu_{23}^{(6)} + \mu_{12}^{(4)}\lambda_{31}^{(6)} + \mu_{12}^{(4)}\lambda_{21}^{(5)}\lambda_{32}^{(6)} + \mu_{23}^{(5)}\lambda_{31}^{(6)}$$

$$b_{53} = 2\lambda_{31}^{(4)} + 2\lambda_{31}^{(5)} + 2\lambda_{31}^{(6)} + \lambda_{32}^{(4)} + \lambda_{32}^{(4)}\lambda_{21}^{(5)} + \lambda_{31}^{(4)}\mu_{12}^{(5)}\lambda_{21}^{(6)} + \lambda_{21}^{(5)}\lambda_{21}^{(6)} + \lambda_{31}^{(4)}\lambda_{21}^{(5)}\mu_{23}^{(6)}$$

$$+\lambda_{32}^{(5)}\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\lambda_{32}^{(5)} + \mu_{23}^{(4)}\lambda_{31}^{(5)}\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\mu_{23}^{(5)}\lambda_{31}^{(6)} + \mu_{23}^{(4)}\lambda_{21}^{(5)}\lambda_{31}^{(6)}$$

$$+\lambda_{21}^{(4)}\lambda_{21}^{(5)} + \mu_{12}^{(4)}\lambda_{32}^{(5)}\lambda_{31}^{(6)} + \mu_{12}^{(4)}\lambda_{31}^{(5)}\lambda_{32}^{(6)} + \lambda_{32}^{(5)}\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\mu_{12}^{(5)}\lambda_{31}^{(6)}$$

$$+\lambda_{32}^{(4)}\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\lambda_{31}^{(5)}\mu_{12}^{(6)} + \lambda_{32}^{(4)}\lambda_{31}^{(5)}\mu_{12}^{(6)} + \lambda_{31}^{(4)}\lambda_{32}^{(5)}\mu_{12}^{(6)}$$

$$b_{63} = \lambda_{32}^{(4)}\lambda_{32}^{(5)}\lambda_{21}^{(6)} + \lambda_{32}^{(4)}\lambda_{31}^{(5)} + \lambda_{31}^{(4)}\lambda_{32}^{(5)} + \lambda_{31}^{(4)}\lambda_{21}^{(6)} + \lambda_{31}^{(5)}\lambda_{21}^{(6)} + \lambda_{31}^{(4)}\lambda_{31}^{(5)}\mu_{23}^{(6)}$$

$$+\lambda_{32}^{(5)}\lambda_{31}^{(6)} + \lambda_{31}^{(5)}\lambda_{32}^{(6)} + \lambda_{21}^{(4)}\lambda_{32}^{(5)}\lambda_{32}^{(6)} + \lambda_{21}^{(4)}\lambda_{31}^{(6)} + \mu_{23}^{(4)}\lambda_{31}^{(5)}\lambda_{31}^{(6)} + \lambda_{21}^{(4)}\lambda_{31}^{(5)}$$

$$+\lambda_{32}^{(4)}\lambda_{31}^{(6)} + \lambda_{32}^{(4)}\lambda_{21}^{(5)}\lambda_{32}^{(6)} + \lambda_{31}^{(4)}\lambda_{32}^{(6)} + \lambda_{31}^{(4)}\mu_{23}^{(5)}\lambda_{31}^{(6)} + \lambda_{21}^{(5)}\lambda_{31}^{(6)} + \lambda_{31}^{(4)}\lambda_{21}^{(5)}$$

$$b_{73} = \lambda_{32}^{(4)}\lambda_{32}^{(5)} + \lambda_{32}^{(4)}\lambda_{32}^{(6)} + \lambda_{32}^{(5)}\lambda_{32}^{(6)} + \lambda_{31}^{(4)}\lambda_{31}^{(6)} + \lambda_{31}^{(5)}\lambda_{31}^{(6)} + \lambda_{31}^{(4)}\lambda_{21}^{(5)}$$

$$b_{14} = \mu_{12}^{(4)}\mu_{12}^{(5)}\mu_{12}^{(6)}$$

$$b_{24} = \mu_{12}^{(4)}\mu_{12}^{(6)} + \mu_{23}^{(4)}\mu_{12}^{(5)} + \mu_{23}^{(4)}\mu_{12}^{(6)} + \mu_{12}^{(4)}\mu_{12}^{(6)} + \mu_{12}^{(4)}\mu_{23}^{(5)} + \mu_{23}^{(5)}\mu_{12}^{(6)} + \mu_{12}^{(4)}\mu_{12}^{(5)}$$

$$+\mu_{12}^{(4)}\mu_{23}^{(6)} + \mu_{12}^{(5)}\mu_{23}^{(6)}$$

$$b_{34} = 3\mu_{12}^{(4)} + 3\mu_{12}^{(5)} + 3\mu_{12}^{(6)} + 2\mu_{23}^{(4)} + 2\mu_{23}^{(5)} + 2\mu_{23}^{(6)} + \mu_{23}^{(4)}\mu_{12}^{(6)}\lambda_{21}^{(5)} + \lambda_{21}^{(4)}\mu_{23}^{(5)}\mu_{12}^{(6)}$$

$$+\mu_{23}^{(4)}\mu_{12}^{(5)}\lambda_{21}^{(6)} + \lambda_{21}^{(4)}\mu_{12}^{(5)}\mu_{23}^{(6)} + \mu_{12}^{(4)}\mu_{23}^{(5)}\lambda_{21}^{(6)} + \mu_{12}^{(4)}\lambda_{21}^{(5)}\mu_{23}^{(6)} + \mu_{12}^{(4)}\lambda_{32}^{(5)}\mu_{12}^{(6)}$$

$$+\lambda_{32}^{(4)}\mu_{12}^{(5)}\mu_{12}^{(6)} + \mu_{12}^{(4)}\mu_{12}^{(5)}\lambda_{32}^{(6)}$$

$$b_{54} = 3\lambda_{21}^{(4)} + 2\lambda_{21}^{(5)} + \lambda_{21}^{(6)} + 3\lambda_{32}^{(4)} + 3\lambda_{32}^{(5)} + 3\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\mu_{23}^{(5)}\lambda_{21}^{(6)} + \lambda_{32}^{(4)}\lambda_{21}^{(5)}\mu_{23}^{(6)} + \lambda_{31}^{(4)}\mu_{23}^{(5)}$$

$$+\lambda_{31}^{(4)}\mu_{23}^{(6)} + \lambda_{31}^{(5)}\mu_{23}^{(6)} + \mu_{23}^{(4)}\lambda_{32}^{(5)}\lambda_{21}^{(6)} + \mu_{23}^{(4)}\lambda_{31}^{(5)} + \lambda_{21}^{(4)}\lambda_{32}^{(5)}\mu_{23}^{(6)} + \mu_{23}^{(5)}\lambda_{31}^{(6)} + \mu_{23}^{(4)}\lambda_{21}^{(5)}\lambda_{32}^{(6)}$$

$$+\lambda_{21}^{(4)}\mu_{23}^{(5)}\lambda_{32}^{(6)} + \mu_{12}^{(4)}\lambda_{32}^{(5)}\lambda_{32}^{(6)} + \mu_{12}^{(4)}\lambda_{31}^{(6)} + \mu_{12}^{(4)}\lambda_{31}^{(5)} + \lambda_{32}^{(4)}\mu_{12}^{(5)}\lambda_{32}^{(6)} + \mu_{12}^{(4)}\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\mu_{12}^{(5)}$$

$$+\lambda_{31}^{(4)}\lambda_{31}^{(5)}\mu_{12}^{(6)} + \lambda_{32}^{(5)}\mu_{12}^{(6)} + \lambda_{31}^{(4)}\mu_{12}^{(6)}$$

$$b_{64} = 2\lambda_{31}^{(4)} + 2\lambda_{31}^{(5)} + 2\lambda_{31}^{(6)} + \lambda_{32}^{(4)}\lambda_{32}^{(5)} + \lambda_{32}^{(4)}\lambda_{21}^{(6)} + \lambda_{32}^{(4)}\lambda_{31}^{(5)}\mu_{23}^{(6)} + \lambda_{32}^{(5)}\lambda_{21}^{(6)} + \lambda_{31}^{(4)}\lambda_{32}^{(5)}\mu_{23}^{(6)}$$
$$+\lambda_{32}^{(5)}\lambda_{32}^{(6)} + \mu_{23}^{(4)}\lambda_{32}^{(5)}\lambda_{31}^{(6)} + \mu_{23}^{(4)}\lambda_{31}^{(5)}\lambda_{32}^{(6)} + \lambda_{21}^{(4)}\lambda_{32}^{(6)} + \lambda_{21}^{(4)}\lambda_{32}^{(5)} + \lambda_{32}^{(4)}\lambda_{32}^{(6)}$$
$$+\lambda_{31}^{(4)}\mu_{23}^{(5)}\lambda_{31}^{(6)} + \lambda_{32}^{(4)}\lambda_{21}^{(5)} + \lambda_{21}^{(5)}\lambda_{32}^{(6)} + \mu_{23}^{(5)}\lambda_{32}^{(6)}$$

$$b_{74} = \lambda_{32}^{(4)}\lambda_{32}^{(5)}\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\lambda_{31}^{(6)} + \lambda_{32}^{(4)}\lambda_{31}^{(5)} + \lambda_{32}^{(5)}\lambda_{31}^{(6)} + \lambda_{31}^{(5)}\lambda_{32}^{(6)} + \lambda_{31}^{(4)}\lambda_{32}^{(6)} + \lambda_{31}^{(4)}\lambda_{32}^{(5)}$$

$$b_{25} = \mu_{23}^{(4)}\mu_{12}^{(5)}\mu_{12}^{(6)} + \mu_{12}^{(4)}\mu_{23}^{(5)}\mu_{12}^{(6)} + \mu_{12}^{(4)}\mu_{12}^{(5)}\mu_{23}^{(6)}$$

$$b_{35} = \mu_{23}^{(4)}\mu_{12}^{(6)} + \mu_{23}^{(5)}\mu_{12}^{(6)} + \mu_{23}^{(4)}\mu_{23}^{(5)} + \mu_{23}^{(4)}\mu_{12}^{(5)} + \mu_{12}^{(5)}\mu_{23}^{(6)} + \mu_{23}^{(4)}\mu_{23}^{(6)} + \mu_{12}^{(4)}\mu_{23}^{(5)} + \mu_{12}^{(4)}\mu_{23}^{(6)}$$
$$+\mu_{23}^{(5)}\mu_{23}^{(6)} + \mu_{12}^{(4)}\mu_{12}^{(6)} + \mu_{12}^{(5)}\mu_{12}^{(6)} + \mu_{12}^{(4)}\mu_{12}^{(5)}$$

$$b_{45} = 3\mu_{23}^{(4)} + 3\mu_{23}^{(5)} + 3\mu_{23}^{(6)} + 2\mu_{12}^{(4)} + 2\mu_{12}^{(5)} + 2\mu_{12}^{(6)} + \lambda_{21}^{(4)}\mu_{23}^{(5)}\mu_{23}^{(6)} + \mu_{23}^{(4)}\lambda_{21}^{(5)}\mu_{23}^{(6)}$$
$$+\mu_{23}^{(4)}\mu_{23}^{(5)}\lambda_{21}^{(6)} + \mu_{23}^{(4)}\lambda_{32}^{(5)}\mu_{12}^{(6)} + \mu_{23}^{(4)}\mu_{12}^{(5)}\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\mu_{23}^{(5)}\mu_{12}^{(6)}$$
$$+\lambda_{32}^{(4)}\mu_{12}^{(5)}\mu_{23}^{(6)} + \mu_{12}^{(4)}\lambda_{32}^{(5)}\mu_{23}^{(6)} + \mu_{12}^{(4)}\mu_{23}^{(5)}\lambda_{32}^{(6)}$$

$$b_{65} = \lambda_{21}^{(4)} + \lambda_{21}^{(5)} + \lambda_{21}^{(6)} + 2\lambda_{32}^{(4)} + 2\lambda_{32}^{(5)} + 2\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\lambda_{32}^{(5)}\mu_{23}^{(6)} + \lambda_{31}^{(4)}\mu_{23}^{(6)} + \lambda_{31}^{(5)}\mu_{23}^{(6)}$$
$$+\mu_{23}^{(4)}\lambda_{32}^{(5)}\lambda_{32}^{(6)} + \mu_{23}^{(4)}\lambda_{31}^{(5)} + \mu_{23}^{(4)}\lambda_{31}^{(6)} + \lambda_{32}^{(4)}\mu_{23}^{(5)}\lambda_{32}^{(6)} + \lambda_{31}^{(4)}\mu_{23}^{(5)} + \mu_{23}^{(5)}\lambda_{31}^{(6)}$$

$$b_{75} = \lambda_{32}^{(5)}\lambda_{32}^{(6)} + \lambda_{32}^{(4)}\lambda_{32}^{(5)} + \lambda_{32}^{(4)}\lambda_{32}^{(6)} + \lambda_{31}^{(4)} + \lambda_{31}^{(5)} + \lambda_{31}^{(6)}$$

$$b_{36} = \mu_{23}^{(4)}\mu_{23}^{(5)}\mu_{12}^{(6)} + \mu_{23}^{(4)}\mu_{12}^{(5)}\mu_{23}^{(6)} + \mu_{12}^{(4)}\mu_{23}^{(5)}\mu_{23}^{(6)}$$

$$b_{46} = \mu_{23}^{(4)}\mu_{23}^{(5)} + \mu_{23}^{(5)}\mu_{23}^{(6)} + \mu_{23}^{(4)}\mu_{23}^{(6)} + \mu_{23}^{(4)}\mu_{12}^{(6)} + \mu_{23}^{(4)}\mu_{12}^{(5)} + \mu_{23}^{(5)}\mu_{12}^{(6)} + \mu_{12}^{(5)}\mu_{23}^{(6)}$$
$$+\mu_{12}^{(4)}\mu_{23}^{(6)} + \mu_{12}^{(4)}\mu_{23}^{(5)}$$

$$b_{56} = 2\mu_{23}^{(4)} + 2\mu_{23}^{(5)} + 2\mu_{23}^{(6)} + \mu_{12}^{(4)} + \mu_{12}^{(5)} + \mu_{12}^{(6)} + \lambda_{32}^{(4)}\mu_{23}^{(5)}\mu_{23}^{(6)} + \mu_{23}^{(4)}\mu_{23}^{(5)}\lambda_{32}^{(6)}$$

$$b_{76} = \lambda_{32}^{(4)} + \lambda_{32}^{(5)} + \lambda_{32}^{(6)} \quad a_{47} = \mu_{23}^{(4)}\mu_{23}^{(5)}\mu_{23}^{(6)}$$

$$b_{57} = \mu_{23}^{(5)}\mu_{23}^{(6)} + \mu_{23}^{(4)}\mu_{23}^{(6)} + \mu_{23}^{(4)}\mu_{23}^{(5)} \quad a_{67} = \mu_{23}^{(4)} + \mu_{23}^{(5)} + \mu_{23}^{(6)}$$

$$b_{11} = -(b_{12} + b_{13} + b_{14}) \quad b_{22} = -(b_{21} + b_{23} + b_{24} + b_{25})$$

$$b_{33} = -(b_{31} + b_{32} + b_{34} + b_{35} + b_{36})$$

$$b_{44} = -(b_{41} + b_{42} + b_{43} + b_{45} + b_{46} + b_{47})$$

$$b_{55} = -(b_{51} + b_{52} + b_{53} + b_{54} + b_{56} + b_{57})$$

$$b_{66} = -(b_{61} + b_{62} + b_{63} + b_{64} + b_{65} + b_{67})$$

$$b_{77} = -(b_{71} + b_{72} + b_{73} + b_{74} + b_{75} + b_{76})$$

Transition rates of the generators G4 , G5 and G6 per hour (h^{-1}) are calculated from the collected data and are given in the table below.

Table 4.2: Transition Rates

Generator	μ_{12}	μ_{23}	λ_{21}	λ_{32}	λ_{31}
G4	7.8×10^{-2}	6.6×10^{-2}	3.3×10^{-3}	6.9×10^{-2}	3×10^{-3}
G5	7.8×10^{-2}	6.4×10^{-2}	3.4×10^{-3}	6.7×10^{-2}	3×10^{-3}
G6	7.9×10^{-2}	6.4×10^{-2}	3.3×10^{-3}	6.7×10^{-2}	3×10^{-3}

Using Mathematica,the steady state probabilities are obtained as $p_1^{(2)} = 0.0042$,$p_2^{(2)} = 0.0323$, $p_3^{(2)} = 0.0814$, $p_4^{(2)} = 0.15$, $p_5^{(2)} = 0.2081$, $p_6^{(2)} = 0.2842$ and $p_7^{(2)} = 0.2398$.

For **Subsystem 1** we have

$$g^{(1)} = \{0, 12.5, 25, 37.5, 50, 62.5, 75\},$$

$$p^{(1)} = \{p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, p_4^{(1)}, p_5^{(1)}, p_6^{(1)}, p_7^{(1)}\},$$

$$u_1(z) = p_1^{(1)} z^0 + p_2^{(1)} z^{12.5} + p_3^{(1)} z^{25} + p_4^{(1)} z^{37.5} + p_5^{(1)} z^{50} + p_6^{(1)} z^{62.5} + p_7^{(1)} z^{75}.$$

For **Subsystem 2** we have

$$g^{(2)} = \{0, 25, 50, 75, 100, 125, 150\},$$

$$p^{(2)} = \{p_1^{(2)}, p_2^{(2)}, p_3^{(2)}, p_4^{(2)}, p_5^{(2)}, p_6^{(2)}, p_7^{(2)}\},$$

$$u_2(z) = p_1^{(2)} z^0 + p_2^{(2)} z^{25} + p_3^{(2)} z^{50} + p_4^{(2)} z^{75} + p_5^{(2)} z^{100} + p_6^{(2)} z^{125} + p_7^{(2)} z^{150}.$$

For **System** we have

$$g = \{0, 12.5, 25, 37.5, 50, 62.5, 75, 87.5, 100, 112.5,$$

$$125, 137.5, 150, 162.5, 175, 187.5, 200, 212.5, 225\},$$

$$p = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_{17}, p_{18}, p_{19}\}.$$

The u-function of the structure of entire system in which two subsystems are connected in parallel (total output of the power station is determined as the outputs of the two sub systems) is

$$U(z) = \Omega_{\phi p}[u_1(z), u_2(z)]$$

$$= \Omega_{\phi p}(p_1^{(1)} z^0 + p_2^{(1)} z^{12.5} + p_3^{(1)} z^{25} + p_4^{(1)} z^{37.5} + p_5^{(1)} z^{50} + p_6^{(1)} z^{62.5} + p_7^{(1)} z^{75}$$

$$, p_1^{(2)} z^0 + p_2^{(2)} z^{25} + p_3^{(2)} z^{50} + p_4^{(2)} z^{75} + p_5^{(2)} z^{100} + p_6^{(2)} z^{125} + p_7^{(2)} z^{150})$$

$$= p_1 z^0 + p_2 z^{12.5} + p_3 z^{25} + p_4 z^{37.5} + p_5 z^{50} + p_6 z^{62.5} + p_7 z^{75} + p_8 z^{87.5}$$
$$+ p_9 z^{100} + p_{10} z^{112.5} + p_{11} z^{125} + p_{12} z^{137.5} + p_{13} z^{150} + p_{14} z^{162.5}$$
$$+ p_{15} z^{175} + p_{16} z^{187.5} + p_{17} z^{200} + p_{18} z^{212.5} + p_{19} z^{225}$$

where

$$p_1 = p_1^1 p_1^2 = 0.00002, \quad p_2 = p_2^1 p_1^2 = 0.00018$$
$$p_3 = p_1^1 p_2^2 + p_3^1 p_1^2 = 0.0006, \quad p_4 = p_2^1 p_2^2 + p_4^1 p_1^2 = 0.002$$
$$p_5 = p_1^1 p_3^2 + p_3^1 p_2^2 + p_5^1 p_1^2 = 0.0043, \quad p_6 = p_2^1 p_3^2 + p_4^1 p_2^2 + p_6^1 p_1^2 = 0.0096$$
$$p_7 = p_1^1 p_4^2 + p_3^1 p_3^2 + p_5^1 p_2^2 + p_7^1 p_1^2 = 0.016, \quad p_8 = p_2^1 p_4^2 + p_4^1 p_3^2 + p_6^1 p_2^2 = 0.0278$$
$$p_9 = p_1^1 p_5^2 + p_3^1 p_4^2 + p_5^1 p_3^2 + p_7^1 p_2^2 = 0.039$$
$$p_{10} = p_2^1 p_5^2 + p_4^1 p_4^2 + p_6^1 p_3^2 = 0.0542$$
$$p_{11} = p_1^1 p_6^2 + p_3^1 p_5^2 + p_5^1 p_4^2 + p_7^1 p_3^2 = 0.07$$
$$p_{12} = p_2^1 p_6^2 + p_4^1 p_5^2 + p_6^1 p_4^2 = 0.0851, \quad p_{13} = p_1^1 p_7^2 + p_3^1 p_6^2 + p_5^1 p_5^2 = 0.1044$$
$$p_{14} = p_2^1 p_7^2 + p_4^1 p_6^2 + p_6^1 p_5^2 = 0.1108, \quad p_{15} = p_3^1 p_7^2 + p_5^1 p_6^2 + p_7^1 p_5^2 = 0.1276$$
$$p_{16} = p_4^1 p_7^2 + p_6^1 p_6^2 = 0.1144, \quad p_{17} = p_5^1 p_7^2 + p_7^1 p_6^2 = 0.1139$$
$$p_{18} = p_6^1 p_7^2 = 0.065, \quad p_{19} = p_7^1 p_7^2 = 0.0552$$

According to Load Despatch Centre average demand for a particular month of this power station is $w = 108.4$ MW.

Steady state availability of the power station for this demand level is given by

$$A_\infty(w = 108.4) = \delta_A(U(z), w = 108.4) = \delta_A(\sum_{i=1}^{19} p_i z^{g_i}, 108.4)$$
$$= p_{10} + p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{16} + p_{17} + p_{18} + p_{19} = 0.9006$$

Mean Steady state performance

$$E_\infty = \sum_{i=1}^{k} p_i g_i = 161.45MW$$

Expected steady state MSS performance deficiency For w = 108.4 MW

$$D_\infty(w = 108.4) = \sum_{i=1}^{19} p_i max(108.4 - g_i, 0) = 2.35MW$$

4.5 Conclusion

Mathematical model based on straight forward stochastic process is not effective enough for system with several components with huge number of states. A different approach has been presented in this chapter by decomposing the entire MSS in to several subsystems. By using the method of combination of markov process and UGF technique, multi state system modeling and solving of system of differential equations will need only a little effort. So analysis of system has been greatly simplified and reliability indices of MSS with minimal repair can be predicted easily.

CHAPTER 5

DYNAMIC RELIABILITY ANALYSIS OF POWER GENERATING SYSTEM USING Lz TRANSFORM

5.1 Introduction

[1] Generally stochastic process method is used for evaluating the multi state system reliability measures. The disadvantage of this method is that the stochastic process models are very difficult for application to real world MSS consisting of many elements with many states. We discussed the disadvantage of this method earlier and during the recent years a specific approach named universal generating function (UGF) procedure has been used for MSS reliability analysis which was explained in chapter 4. UGF technique is essentially based on moment generating functions

[1] Some contents of this chapter are based on Vidhya and Manoharan (2018a).

and it is a mathematical concept for random variables. UGF plays a crucial role in the steady state analysis of MSS. In previous chapter we successfully applied this technique to steady state reliability evaluation of a power generating system with six generators. The disadvantage of this powerful technique is that it can only be theoretically applied to random variable and so only steady state reliability is evaluated through this technique. Lisnianski et al. (2010) proposed a special transform called z transform connected with stochastic process. Lz transform is similar to this transform for discrete state continuous time stochastic process and Usakov's operator can be enforced in this transform. For a Discrete State Continuous Time (DSCT) Markov process Lz transform which is an extension of UGF technique is introduced by Lisnianski (2011). The Lz transform method was applied by him for dynamic reliability analysis of some MSS. After that Lz method was used by Frenkel et al. (2012) for computation of availability for complex aging refrigeration system. Lisnianski and Ben-Haim (2013) employed this method for short term evaluation of a power generating system and also several important indices of the system have been evaluated. An instantaneous availability model for multi state repairable system with common bus performance sharing has been proposed by Yu et al. (2014). Here the idea used by the above mentioned authors have been adapted and employed in the case of a power generating system of n components with multiple states connected in parallel. If we want to learn about the short term behavior of a multi state system, it can not be evaluated through steady state behaviours. Important short term reliability performance measures such as availability, loss of load probability and expected generating capacity deficiency are basically distinct from important steady state reliability performance measures. State probabilities $p_i(t)$ are out of universal generating function technique because it is time dependent. Lz transform technique can be described below by the following steps.

1. In order to get state probabilities of each component of the multi state system, traditional differential equations for Markov model of each component are taken into account. State probabilities are obtained as functions of time. Lz transform of each component has been determined using these state probabilities.

2. The Lz transform has been obtained for the entire multi state system by using corresponding Ushakov's universal generating operators. Multi state system short term reliability performance measures can be evaluated using the Lz transform of the entire system.

We have to evaluate short term behavior of the same power generating system in chapter 4 with six components each having three states. In order to find state probabilities for the system we can use classical Markov model. If we use Markov model $3^6 = 729$ states are contained in that model and corresponding differential equations can be solved for evaluation of system reliability. For the steady state reliability analysis of this power system we applied UGF technique in previous chapter. Evidently total output of the system in the above power generating system model is equal to the sum of the outputs of the individual components. We determine Lz transform for each individual element after solving differential equations for Markov model of each system element in order to obtain state probabilities as functions of time. By using Ushakov's universal generating operator (UGO) we get Lz transform for the entire system output and also determine the reliability measures of corresponding power system. Application of this approach on power system reliability analysis is illustrated by a numerical example. Based on these lines six sections in this paper are developed and presented.

5.2 Model Description and Methodology

Assumptions

- The power generating system may have many levels of degradation which vary from perfect functioning to complete failure.
- The system might fail from any 'up' state to its 'down' states and it is minimally repaired.
- The components of the system might fail independently and they are operated on continuous basis.
- The components of the system can be repaired independently.
- The failure rates and repair rates from one state to other state are varying for each component of the system.

Consider a system with n components each having $1, 2, ...k_j$ states where k_j is the best functioning state and 1 is the worst state. The state space of the component of the system is $S = \{1, 2, ...k_j\}$.

Consider a power station with n generators, having states $k_1, k_2, ..., k_n$ respectively. Markov model of whole power station will have $k_1 \times k_2 \times ... \times k_n$ states. This model can be analysed for finding reliability indices of the power station. If conventional stochastic process approach is applied, it will require huge effort even for relatively small n and $k_j, \quad j = 1, 2, ..., n$. UGF technique can be applied for avoiding this dimension- damnation problem. Since UGF defined for random variables, one needs to consider only the steady state behavior of the power system. Here Lz transform method is applied for finding reliability indices for short term reliability evaluation of a power station consisting of numerous different generating units. Consider a discrete state continuous time (DSCT) Markov process $X(t) \in \{x_1, x_2 \ldots x_k\}$ which has k possible states $i, (i = 1, 2 \ldots k)$ where performance level associated with state i is x_i. This Markov process is completely defined by set of possible states $X = \{x_1, x_2 \ldots x_k\}$, transition intensities matrix $A = (a_{ij}(t)), \quad i, j = 1, 2 \ldots k$ and probability distribution of initial states. Probability distribution of initial states is represented by corresponding set

$$p_0 = [p_{10} = Pr\{X(0) = x_1\}, p_{20} = Pr\{X(0) = x_2\}, \ldots\ldots\ldots p_{k0} = Pr\{X(0) = x_k\}].$$

In general case j^{th} component of a power generating system ($j \in \{1, 2 \ldots n\}$) have k_j different states corresponding to different performances. It is represented by the set $g_j = \{g_{j1}, g_{j2}, \ldots g_{jk_j}\}$ where g_{ji} is the performance level of component j in the state i, ($i \in \{1, 2 \ldots k_j\}$ and $j \in \{1, 2 \ldots n\}$).

According to Lisnianski (2011) Lz transform of a DSCT Markov process X(t) is a function defined as follows

$$Lz\{X(t)\} = \sum_{i=1}^{k} p_i(t) z^{g_i}$$

where $p_i(t)$ is a probability that the process is in state i at time instant $t \geq 0$ for a given initial states probability distribution p_0 and z in general case is a complex variable.

At first stage a Markov model should be constructed for each multi state element in MSS. Solving the following system of linear differential equation of j^{th} component [refer Trivedi (2002)].

$$\begin{aligned}
\frac{d}{dt} p_{j1}(t) &= a_{11}(t) p_{j1}(t) + a_{12}(t) p_{j2}(t) + \ldots + a_{1k_j}(t) p_{jk_j}(t) \\
\frac{d}{dt} p_{j2}(t) &= a_{21}(t) p_{j1}(t) + a_{22}(t) p_{j2}(t) + \ldots + a_{2k_j}(t) p_{jk_j}(t) \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{d}{dt} p_{jk_j}(t) &= a_{k_j1}(t) p_{j1}(t) + a_{k_j2}(t) p_{j2}(t) + \ldots + a_{k_jk_j}(t) p_{jk_j}(t)
\end{aligned}$$

under the given initial conditions $p_0 = \{p_{10}, p_{20} \ldots p_{k_j 0}\}$ we get the probabilities $p_{ji}(t), i = 1.2...k_j, j = 1, 2, ...n$ The individual Lz transform for each component j can be obtained by the formula

$$Lz\{G_j(t)\} = \sum_{i=1}^{k_j} p_{ji}(t) z^{g_{ji}}, j = 1, 2...n \tag{5.1}$$

Lz transform of whole MSS can be obtained based on Lz transform for each component and system structure function f. By applying Ushakov's operator Ω_f over all Lz transform of individual elements we get the resulting Lz transform, $Lz\{G(t)\}$ linked with output performance stochastic process $G(t)$ of the whole MSS. Employing Ushakov's Universal Generating Operator to all individual Lz transforms $Lz\{G_j(t)\}$ which is obtained by equation (5.1) over all time point $t \geq 0$ we can obtain

$$Lz\{G(t)\} = \Omega_f\{Lz\{G_1(t)\}, Lz\{G_2(t)\}, \ldots Lz\{G_n(t)\}\} \tag{5.2}$$

Ushakov's operator is well defined by Lisnianski et al. (2010) for many different structure functions. By using technique of Lz transform we can drastically minimize computational burden and $Lz\{G(t)\}$ is associated with the output performance of the entire MSS. Multi state system reliability measure can be obtained from the resulting Lz transform $Lz\{G(t)\}$, as summarized in the next section.

5.3 System Reliability Measures

If Lz-transform

$$Lz\{G(t)\} = \sum_{k=1}^{K} p_k(t) z^{g_k} \tag{5.3}$$

of the entire MSS's output stochastic process

$G(t) \in \{g_1, g_2, \dots g_k\}$ is known, then important system reliability measures can be easily obtained [refer Yu et al. (2014)].

The power station availability for demand level w is defined as system ability to provide power supply to consumers with summarized load w. That is, the power station should be in states with generating capacity more or equal w.

Therefore the system availability for the constant demand w at instant $t \geq 0$ is given by

$$A_w(t) = \sum_{g_k \geq w} p_k(t). \tag{5.4}$$

In order to find MSS instantaneous availability we should summarize all probabilities in Lz transform from terms where powers of z are greater or equal to demand w.

Loss of load occurs when the system load exceeds the generating capacity available for use. Loss of Load Probability (LOLP) is a projected value of how much time, in the long run, the load on a power system is expected to be greater than the capacity of the available generating resources. It is defined as the probability that the load will exceed the available generation.

Loss of Load probability ($LOLP_w$) for a given level w is then obtained as

$$LOLP_w(t) = 1 - A_w(t). \tag{5.5}$$

The expected generating capacity deficiency (ECD_w) of the system is given by

$$ECD_w(t) = \sum_{k=1}^{K} p_k(t)(w - g_k)I_{(w-g_k)} \tag{5.6}$$

where

$$I_{w-g_K} = \begin{cases} 1 & \text{if } w - g_k > 0 \\ \\ 0 & \text{if } w - g_k \leq 0 \end{cases}$$

.

Reliability measures for a power system depend strongly on initial states of units.

5.4 Reliability Evaluation of a power station with six generating units connected in parallel

In this section we apply the methods presented in the foregoing section to carry out the availability analysis based on the same data in the previous chapter. Taking up the terminology used earlier, we have a Markov model at hand and we shall apply the Lz transform technique to evaluate the dynamic reliability of the power system.

Transition intensities a_{ij} drawn up as a matrix, called infinitesimal generator of the process is given by

$$A = |a_{ij}| = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k_j} \\ a_{21} & a_{22} & \dots & a_{2k_j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k_j1} & a_{k_j2} & \dots & a_{k_jk_j} \end{pmatrix}$$

$$= \begin{pmatrix} -\mu_{12}^j & \mu_{12}^j & 0 \\ \lambda_{21}^j & -(\lambda_{21}^j + \mu_{23}^j) & \mu_{23}^j \\ \lambda_{31}^j & \lambda_{32}^j & -(\lambda_{31}^j + \lambda_{32}^j) \end{pmatrix}, \ j = 1, 2, 3, 4, 5, 6$$

We have Kolmogorov forward equation in matrix term

$$\frac{d}{dt} p_j(t) = p_j(t)\ A.$$

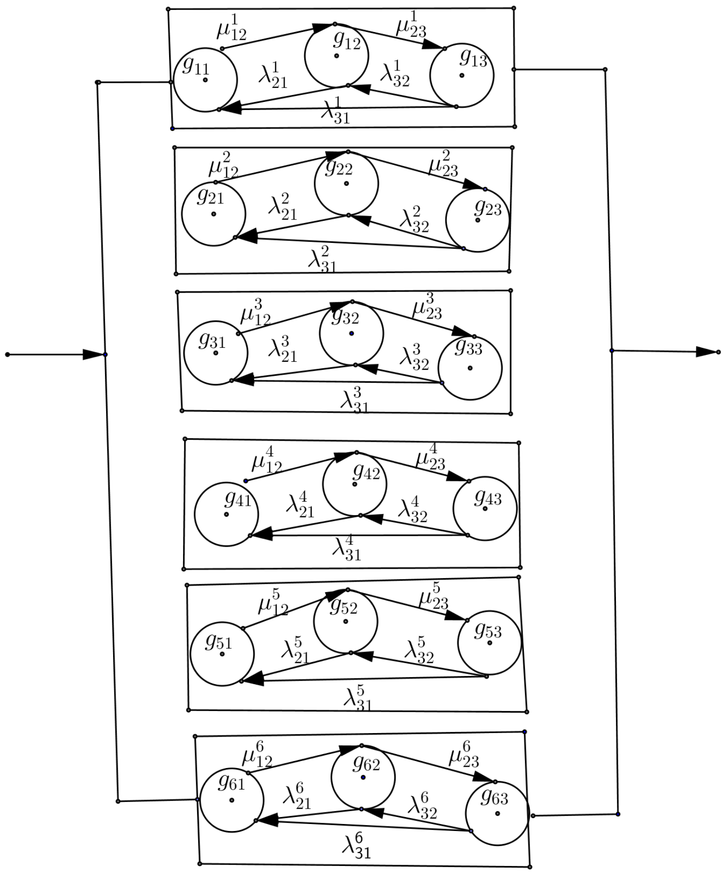

Figure 5.1: State space diagram of the power generating system with six generators

Table 5.1: Transition rates of the generators per hour(h^{-1})

Generator	μ_{12}	μ_{23}	λ_{21}	λ_{32}	λ_{31}
G1	7.1×10^{-2}	6.4×10^{-2}	3×10^{-3}	6.7×10^{-2}	3.3×10^{-3}
G2	7.3×10^{-2}	6.5×10^{-2}	3×10^{-3}	6.8×10^{-2}	3.3×10^{-3}
G3	7.4×10^{-2}	6.1×10^{-2}	3×10^{-3}	6.2×10^{-2}	3.3×10^{-3}
G4	7.8×10^{-2}	6.6×10^{-2}	3.3×10^{-3}	6.9×10^{-2}	3×10^{-3}
G5	7.8×10^{-2}	6.4×10^{-2}	3.4×10^{-3}	6.7×10^{-2}	3×10^{-3}
G6	7.9×10^{-2}	6.4×10^{-2}	3.3×10^{-3}	6.7×10^{-2}	3×10^{-3}

For elements $j = 1, 2, 3$;

$$g_j = \{g_{j1}, g_{j2}, g_{j3}\} = \{0, 12.5, 25\}$$

$$p_j(t) = \{p_{j1}(t), p_{j2}(t), p_{j3}(t)\}$$

$$\begin{aligned}
\frac{d}{dt}p_{j1}(t) &= -\mu_{12}^{j}(t)p_{j1}(t) + \lambda_{21}^{j}p_{j2}(t) + \lambda_{31}^{j}p_{j3}(t) \\
\frac{d}{dt}p_{j2}(t) &= \mu_{12}^{j}(t)p_{j1}(t) - (\lambda_{21}^{j} + \mu_{23}^{j})p_{j2}(t) + \lambda_{32}^{j}p_{j3}(t) \\
\frac{d}{dt}p_{j3}(t) &= -\mu_{23}^{j}(t)p_{j2}(t) - (\lambda_{31}^{j} + \lambda_{32}^{j})p_{j3}(t)
\end{aligned}$$

Initial conditions are $p_{j3}(t) = 1, p_{j2}(t) = p_{j1}(t) = 0$.

For elements $j = 4, 5, 6$;

$$g_j = \{g_{j1}, g_{j2}, g_{j3}\} = \{0, 25, 50\}$$

$$p_j(t) == \{p_{j1}(t), p_{j2}(t), p_{j3}(t)\}$$

$$\begin{aligned}
\frac{d}{dt}p_{j1}(t) &= -\mu_{12}^{j}(t)p_{j1}(t) + \lambda_{21}^{j}p_{j2}(t) + \lambda_{31}^{j}p_{j3}(t)\\
\frac{d}{dt}p_{j2}(t) &= \mu_{12}^{j}(t)p_{j1}(t) - (\lambda_{21}^{j} + \mu_{23}^{j})p_{j2}(t) + \lambda_{32}^{j}p_{j3}(t)\\
\frac{d}{dt}p_{j3}(t) &= -\mu_{23}^{j}(t)p_{j2}(t) - (\lambda_{31}^{j} + \lambda_{32}^{j})p_{j3}(t)
\end{aligned}$$

Initial conditions are $p_{j3}(t) = 1, p_{j2}(t) = p_{j1}(t) = 0$.

After solving six separate system of differential equations under the given initial conditions, we get the state probabilities $p_j(t)$ for $j = 1, 2, 3, 4, 5, 6$. Having the sets $g_j, p_j(t)$ we can define Lz transforms for each individual element j as follows.

$$Lz\{G_1(t)\} = p_{11}(t)z^{g_{11}} + p_{12}(t)z^{g_{12}} + p_{13}(t)z^{g_{13}}$$

$$= p_{11}(t)z^{0} + p_{12}(t)z^{12.5} + p_{13}(t)z^{25}$$

$$Lz\{G_2(t)\} = p_{21}(t)z^{g_{21}} + p_{22}(t)z^{g_{22}} + p_{23}(t)z^{g_{23}}$$

$$= p_{21}(t)z^{0} + p_{22}(t)z^{12.5} + p_{23}(t)z^{25}$$

$$Lz\{G_3(t)\} = p_{31}(t)z^{g_{31}} + p_{32}(t)z^{g_{32}} + P_{33}(t)z^{g_{33}}$$

$$= p_{31}(t)z^{0} + p_{32}(t)z^{12.5} + p_{33}(t)z^{25}$$

$$Lz\{G_4(t)\} = p_{41}(t)z^{g_{41}} + p_{42}(t)z^{g_{42}} + p_{43}(t)z^{g_{43}}$$

$$= p_{41}(t)z^{0} + p_{42}(t)z^{25} + p_{43}(t)z^{50}$$

$$Lz\{G_5(t)\} = p_{51}(t)z^{g_{51}} + p_{52}(t)z^{g_{52}} + p_{53}(t)z^{g_{53}}$$

$$= p_{51}(t)z^{0} + p_{52}(t)z^{25} + p_{53}(t)z^{50}$$

$$Lz\{G_6(t)\} = p_{61}(t)z^{g_{61}} + p_{62}(t)z^{g_{62}} + p_{63}(t)z^{g_{63}}$$
$$= p_{61}(t)z^0 + p_{62}(t)z^{25} + p_{63}(t)z^{50}$$

Using composition operator Ω_f par for MSS elements 1,2,3,4,5 and 6 connected in parallel we get the Lz transform

$$\begin{aligned}Lz\{G(t)\} = \Omega_f par\{&p_{11}(t)z^0 + p_{12}(t)z^{12.5} + p_{13}(t)z^{25},\\ &p_{21}(t)z^0 + p_{22}(t)z^{12.5} + p_{23}(t)z^{25},\\ &p_{31}(t)z^0 + p_{32}(t)z^{12.5} + p_{33}(t)z^{25},\\ &p_{41}(t)z^0 + p_{42}(t)z^{25} + p_{43}(t)z^{50},\\ &p_{61}(t)z^0 + p_{62}(t)z^{25} + p_{63}(t)z^{50}\}\end{aligned}$$

$$\begin{aligned}&= a_1b_1z^0 + a_2b_1z^{12.5} + [a_1b_2 + a_3b_1]z^{25} + [a_2b_2 + a_4b_1]z^{37.5} + [a_1b_3 + a_3b_2 + a_5b_1]z^{50}\\ &+[a_2b_3 + a_4b_2 + a_6b_1]z^{62.5} + [a_1b_4 + a_3b_3 + a_5b_2 + a_7b_1]z^{75} + [a_2b_4 + a_4b_3 + a_6b_2]z^{87.5}\\ &+[a_1b_5 + a_3b_4 + a_5b_3 + a_7b_2]z^{100} + [a_2b_5 + a_4b_4 + a_6b_3]z^{112.5}\\ &+[a_1b_6 + a_3b_5 + a_5b_4 + a_7b_3]z^{125} + [a_2b_6 + a_4b_5 + a_6b_4]z^{137.5}\\ &+[a_1b_7 + a_3b_6 + a_5b_5 + a_7b_4]z^{150} + [a_2b_7 + a_4b_6 + a_6b_5]z^{162.5}\\ &+[a_3b_7 + a_5b_6 + a_7b_5]z^{175} + [a_4b_7 + a_6b_6]z^{187.5}\\ &+[a_5b_7 + a_7b_6]z^{200} + a_6b_7z^{212.5} + a_7b_7z^{225}\end{aligned}$$

where

$$a_1 = p_{11}(t)p_{21}(t)p_{31}(t)$$

$$a_2 = p_{11}(t)p_{21}(t)p_{32}(t) + p_{11}(t)p_{22}(t)p_{31}(t) + p_{12}(t)p_{21}(t)p_{31}(t)$$

$$a_3 = p_{11}(t)p_{21}(t)p_{33}(t) + p_{11}(t)p_{22}(t)p_{32}(t) + p_{11}(t)p_{23}(t)p_{31}(t) + p_{12}(t)p_{21}(t)p_{32}(t)$$

$$+p_{12}(t)p_{22}(t)p_{31}(t) + p_{13}(t)p_{21}(t)p_{31}(t)$$

$$a_4 = p_{11}(t)p_{22}(t)p_{33}(t) + p_{11}(t)p_{23}(t)p_{32}(t) + p_{12}(t)p_{21}(t)p_{33}(t) + p_{12}(t)p_{22}(t)p_{32}(t)$$

$$+p_{12}(t)p_{23}(t)p_{31}(t) + p_{13}(t)p_{21}(t)p_{32}(t) + p_{13}(t)p_{22}(t)p_{31}(t)$$

$$a_5 = p_{11}(t)p_{23}(t)p_{33}(t) + p_{12}(t)p_{22}(t)p_{33}(t) + p_{12}(t)p_{23}(t)p_{32(t)} + p_{13}(t)p_{21}(t)p_{33}(t)$$

$$+p_{13}(t)p_{22}(t)p_{32}(t) + p_{13}(t)p_{23}(t)p_{31}(t)$$

$$a_6 = p_{12}(t)p_{23}(t)p_{33}(t) + p_{13}(t)p_{22}(t)p_{33}(t) + p_{13}(t)p_{23}(t)p_{32}(t)a_7 = p_{13}(t)p_{23}(t)p_{33}(t)$$

$$b_1 = p_{41}(t)p_{51}(t)p_{61}(t)$$

$$b_2 = p_{41}(t)p_{51}(t)p_{62}(t) + p_{41}(t)p_{52}(t)p_{61}(t) + p_{42}(t)p_{51}(t)p_{61}(t)$$

$$b_3 = p_{41}(t)p_{51}(t)p_{63}(t) + p_{41}(t)p_{52}(t)p_{62}(t) + p_{41}(t)p_{53}(t)p_{61}(t) + p_{42}(t)p_{51}(t)p_{62}(t)$$

$$+p_{42}(t)p_{52}(t)p_{61}(t) + p_{43}(t)p_{51}(t)p_{61}(t)$$

$$b_4 = p_{41}(t)p_{52}(t)p_{63}(t) + p_{41}(t)p_{53}(t)p_{62}(t) + p_{42}(t)p_{51}(t)p_{63(t)} + p_{42}(t)p_{52}(t)p_{62}(t)$$

$$+p_{42}(t)p_{53}(t)p_{61}(t) + p_{43}(t)p_{51}(t)p_{62}(t) + P_{43}(t)p_{52}(t)p_{61}(t)$$

$$b_5 = p_{41}(t)p_{53}(t)p_{63}(t) + p_{42}(t)p_{52}(t)p_{63}(t) + p_{42}(t)p_{53}(t)p_{62}(t) + p_{43}(t)p_{51}(t)p_{63}(t)$$

$$+p_{43}(t)p_{52}(t)p_{62}(t) + p_{43}(t)p_{53}(t)p_{61}(t)$$

$$b_6 = p_{42}(t)p_{53}(t)p_{63}(t) + p_{43}(t)p_{52}(t)p_{63}(t) + p_{43}(t)p_{53}(t)p_{62}(t)$$

$$b_7 = p_{43}(t)p_{53}(t)p_{63}(t)$$

Thus

$$Lz\{G(t)\} = \sum_{k=1}^{19} p_k(t) z^{g_k} \tag{5.7}$$

The state probabilities of the components of this power generating system can be calculated by solving system of differential equations of each component under given initial conditions. We can apply eigen value-eigen vector method for finding solution of differential equation [refer Braun (1993)] and also with the help of Matlab software we can evaluate state probabilities.

For Element 1:

$$p_{11}(t) = 0.0405 - 0.4445 \exp\{-0.0743t\} + 0.404 \exp\{-0.134t\}$$

$$p_{12}(t) = 0.9594 - 0.057 \exp\{-0.0743t\} - 0.9024 \exp\{-0.134t\}$$

$$p_{13}(t) = 0.0000005 + 0.5015753 \exp\{-0.0743t\} + 0.4984242 \exp\{-0.134t\}$$

For Element 2:

$$p_{21}(t) = 0.04 - 0.43 \exp\{-0.0763t\} + 0.39 \exp\{-0.136t\}$$

$$p_{22}(t) = 0.96 - 0.069 \exp\{-0.0763t\} - 0.891 \exp\{-0.136t\}$$

$$p_{23}(t) = 0.000002 + 0.501940 \exp\{-0.0763t\} + 0.498058 \exp\{-0.136t\}$$

For Element 3:

$$p_{31}(t) = 0.0418 - 0.3495\exp\{-0.0771t\} + 0.3077\exp\{-0.1261t\}$$

$$p_{32}(t) = 0.956 - 0.163\exp\{-0.0771t\} + 0.793\exp\{-0.1261t\}$$

$$p_{33}(t) = 0.0018 + 0.5131\exp\{-0.0771t\} + 0.4851\exp\{-0.1261t\}$$

For Element 4:

$$p_{41}(t) = 0.04 - 0.39\exp\{-0.081t\} + 0.35\exp\{-0.1383t\}$$

$$p_{42}(t) = 0.957 - 0.116\exp\{-0.081t\} - 0.841\exp\{-0.1383t\}$$

$$p_{43}(t) = 0.003 + 0.504\exp\{-0.081t\} + 0.493\exp\{-0.1383t\}$$

For Element 5:

$$p_{51}(t) = 0.039 - 0.364\exp\{-0.0809t\} + 0.325\exp\{-0.1345t\}$$

$$p_{52}(t) = 0.955 - 0.138\exp\{-0.0809t\} - 0.817\exp\{-0.1345t\}$$

$$p_{53}(t) = 0.0052 + 0.5034\exp\{-0.0809t\} + 0.4914\exp\{-0.1345t\}$$

For Element 6:

$$P_{61}(t) = 0.038 - 0.352\exp\{-0.0819t\} + 0.314\exp\{-0.1344t\}$$

$$P_{62}(t) = 0.957 - 0.154\exp\{-0.0819t\} - 0.803\exp\{-0.1344t\}$$

$$P_{63}(t) = 0.0041 + 0.5058\exp\{-0.0819t\} + 0.4901\exp\{-0.1344t\}$$

Based on the resulting Lz transform Lz{G(t)} of the entire MSS (5.7), we can obtain instantaneous availability for the given demand w.

According to Load Despatch Centre average demand for a particular month of this power station is $w = 108.4$ MW. Reliability performance measures are obtained by equation(5.4), (5.5) and (5.6).

The power station availability for this demand level is given by

$$A_{108.4}(t) = \sum_{g_k \geq 108.4} p_k(t)$$

$$= [a_2b_5 + a_4b_4 + a_6b_3] + [a_1b_6 + a_3b_5 + a_5b_4 + a_7b_3] + [a_2b_6 + a_4b_5 + a_6b_4]$$

$$+[a_1b_7 + a_3b_6 + a_5b_5 + a_7b_4] + [a_2b_7 + a_4b_6 + a_6b_5] + [a_3b_7 + a_5b_6 + a_7b_5]$$

$$+[a_4b_7 + a_6b_6] + [a_5b_7 + a_7b_6] + a_6b_7 + a_7b_7.$$

The LOLP for the demand $w = 108.4$ is

$$LOLP_{108.4}(t) = 1 - A_{108.4}(t).$$

The Expected generating capacity deficiency for the demand $w = 108.4$ is

$$ECD_{w=108.4}(t)$$

$$= a_1b_1 \times 108.4 + a_2b_1 \times 95.9 + [a_1b_2 + a_3b_1] \times 83.4 + [a_2b_2 + a_4b_1] \times 70.9$$

$$+[a_1b_3 + a_3b_2 + a_5b_1] \times 58.4 + [a_2b_3 + a_4b_2 + a_6b_1] \times 45.9$$

$$+[a_1b_4 + a_3b_3 + a_5b_2 + a_7b_1] \times 33.4 + [a_2b_4 + a_4b_3 + a_6b_2] \times 20.9$$

$$+[a_1b_5 + a_3b_4 + a_5b_3 + a_7b_2] \times 8.4.$$

The MSS instantaneous availability A(t) of the power system is calculated for different hours and the computed values are presented in the figure 2. The figure

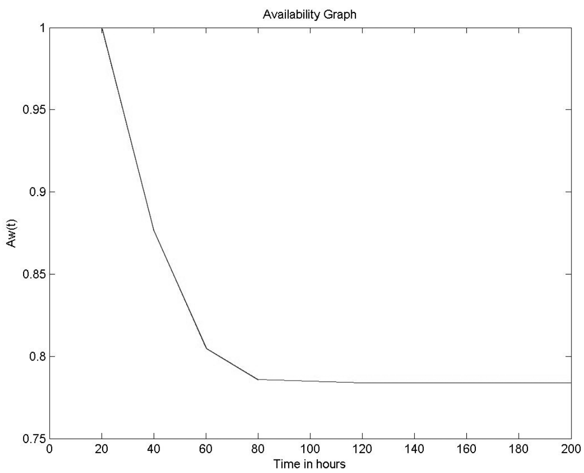

Figure 5.2: Graph of power system Availability (for the demand w=108.4 MW) as a function of time

shows that instantaneous availability of the power system is one up to few hours and decreases after few hours and later eventually attains a stable value.

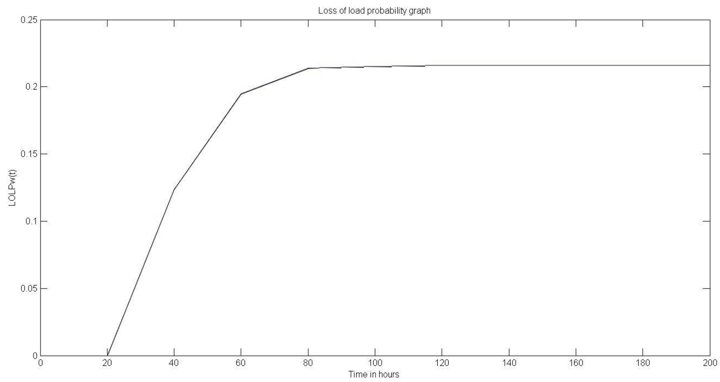

Figure 5.3: Graph of loss of load probability (for the demand w=108.4 MW) as a function of time

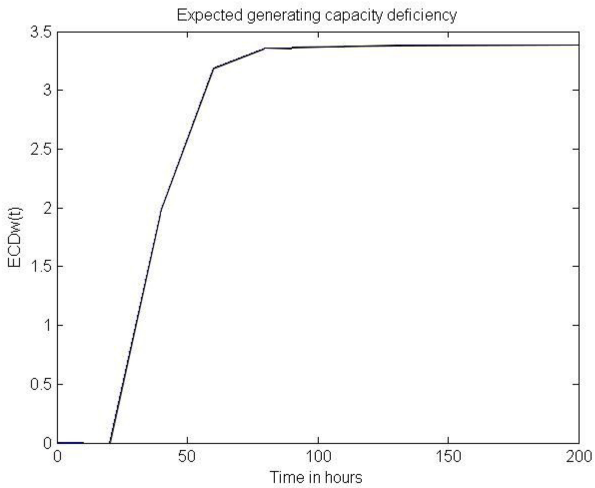

Figure 5.4: Graph of the expected generating capacity deficiency(for the demand w=108.4 MW) as a function of time

5.5 Conclusion

In this chapter Lz transform for discrete state continuous time Markov process is presented for a power generating system of multiple components with multiple states connected in parallel. The methods are employed as a case study for a power station with six generating units connected in parallel. Lz transform is obtained using simple algebra and it is proved to be a very effective method. The idea in this chapter supports the engineering decision making by providing required availability measure for such complex multi-state system with multiple components. Lz-transform remarkably simplifies the reliability evaluation of MSS when it compared with straightforward Markov method.

CHAPTER 6

RELIABILITY ANALYSIS OF PERIODICALLY MAINTAINED SYSTEMS USING SEMI-MARKOV PROCESS BASED UGF TECHNIQUE

6.1 Introduction

The degeneration of a multi state system or component is due to failure or deterioration or transition from one state to other. If we are not aware of the failure of a component, the repair of the component will not take place and this leads to break down of the component or the system. As such the scheduled or periodic inspection takes place and there by lead to periodically maintained system which will be discussed in this chapter. The periodic inspection and preventive maintenance improve the component and system performance. The main intention of a preventive maintenance technique is to prevent the failure of a component before it actually happens. A degenerated repairable MSS with an imperfect preventive maintenance (PM) policy was examined and a model for assessing the availability, the production rate and the reliability function of the multi-state degraded system subjected to minimal repairs and imperfect preventive maintenance was developed by Issac et al. (2010)

In various real life problems, the life time and repair time distribution need not be exponentially distributed. The main disadvantage of Markov process is that it does not allow any other distribution for sojourn time, other than exponential. A semi-Markov process, that allows any distribution for transition between states, is used in reliability analysis for many real life problems. Here steady state probabilities of different states of a component is evaluated using semi-Markov process. Semi Markov process are excellently applicable for reliability analysis of multi state system and it is very effective in many engineering problems. When we apply Markov process in reliability analysis, we have a closure property that is product of independent Markov process is also a Markov process. But this closure property is not valid for semi Markov process. This is the main difficulty in using semi Markov process in reliability analysis of MSS. But the semi Markov process gives a better model for reliability analysis of multi state system than the Markov process. In real life problems of multi state system inspection is generally performed at deterministic time interval and inspection and repair times are generally distributed. For the foundations of semi-

Markov process we can refer Cinlar (1975). A detailed study of semi-Markov process and it's application in reliability theory were discussed by Limnios and Oprisan (2001). Steady state reliability analysis using continuous time semi-Markov process (CTSMP) was presented by Lisnianski and Levitin (2003). Application of semi-Markov process models in reliability analysis for a repairable system and its reliability characteristics were demonstrated by Grabski (2007). The process of degradation with the corrective action of minimal repair, major repair and restart technique using a semi- Markov process was modeled by Resham and Dharmaraja (2012). The main dependability measures of periodically maintained system which is modeled as CTSMP, were studied by Sonia et al. (2014).

Several components of different levels of degradation of a system is rather involved in the application of any random process method and in particular semi-Markov technique. In this chapter we propose a combined semi-Markov process and UGF technique for the reliability analysis of a complex multi state system with periodic inspection and maintenance. We build discrete state continuous time semi-Markov process for each MSS element. We evaluate steady state probabilities of different states of the components. Universal generating function has been obtained for each component and by using composition operators in UGF technique, universal generating function of the whole MSS has been evaluated. Combined Markov process and UGF technique was discussed and combined semi-Markov process and UGF technique were also mentioned by Lisnianski and Levitin (2003). A combination of Markov process and UGF technique was applied for reliability analysis of a MSS by decomposing whole system into several sub systems in chapter 4. In this chapter assumptions, descriptions and analysis of model are presented in section 6.2. In section 6.3 analytical expressions for steady state reliability indices have been derived. We consider the same power station which is analysed in previous chapters and here a periodic inspection within a period of thirty days is introduced for that power generating system. The power station with six independent generators as components has been discussed as an illustrative example in section 6.4. A periodic inspection and thereafter maintenance has been performed for each component. Steady state reliability indices using combined semi-Markov process and UGF technique have been evaluated for this MSS and its performance is assessed.

6.2 Model Description and Analysis

Assumptions

- The system consists of several independent components and components may have many levels of degradation corresponding to discrete performance rate which vary from perfect working to complete failure.
- The system might fail from any state (ie, from the perfect functioning state or any degraded state) and it is minimally repaired.
- All transition times are arbitrarily distributed. Here in particular case, the transition times are exponentially distributed.

- The components of the system are periodically inspected. That is the components are tested every T time units.

A multi state system (MSS) consists of 'n' independent components and the components of the system are initially in its perfect functioning state. As time progress, it can either go to the first degraded state because of degradation or transition or it can go to a failed state. A minimal repair has been performed in a failed state. When a system reaches the first degraded state it can either go to the second degraded state because of degradation or can go to a failed state. The same procedure will continue in all degraded state. Periodic inspection during specific interval is conducted in any state and when a failure is detected in periodic inspection, maintenance has been performed.

Components : $j = 1, 2, ..., n$

States: $i = 1, 2, ..., k_j$

State 1 : Perfect functioning

State $2i - 1$: Degraded state; $i = 2, 3, ..., d$

State $2i$: Failed from an operational state; $i = 1, 2..., d$

α^j_{ik} : Degradation or transition rate from one state to other for j^{th} component; $i = 1, 3, ..., (2d-3), (2d-1),\ k = 1, 2, ..., (2d-1)$

λ^j_{ik}: Failure rate of j^{th} component; $i = 1, 3, ...(2d-3),\ k = 2, 4, ..., 2d$

μ^j_{ik} : Repair rate of j^{th} component; $i = 2, 4, ...2d,\ k = 1, 3, ..., (2d-3)$

β_i^j: Probability of detecting a failure when the component is in state i; $i = 1, 3, ...(2d-3)$

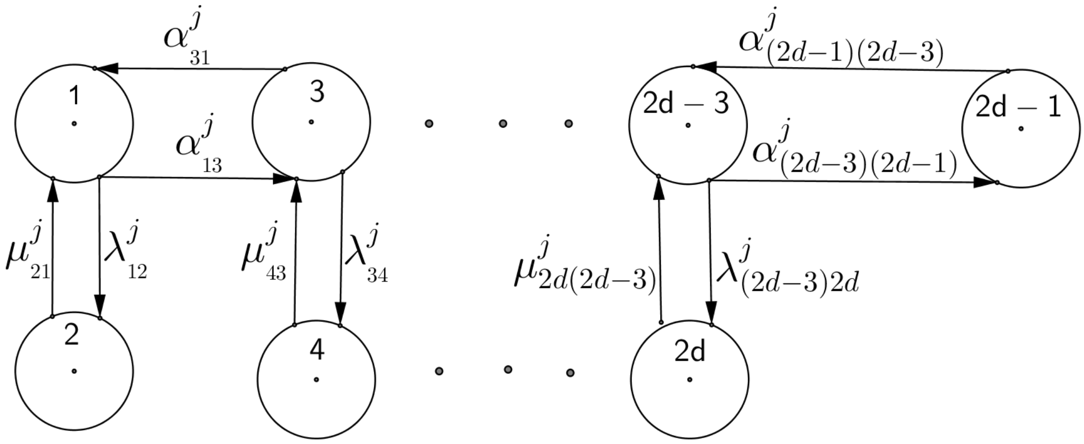

Figure 6.1: State transition diagram of a component of a system

At any time instant $t \geq 0$ the component of the system is in one of the possible states with performance rate $g_1, g_2 \dots g_{k_j}$. The component characteristic is defined by the discrete state continuous time stochastic performance process $G(t) \in \{g_1, g_2 \dots g_{k_j}\}$

Let $\{X(t)\}_{t \geq 0}$ be the semi-Markov process that gives the progress in time of the above j^{th} component with state space $S = \{1, 2 \dots k_j\}$. The kernel matrix $|Q(t)|$ and the initial state completely gives the stochastic behaviour of semi-Markov process. Each element $Q_{ik}(t)$ of the kernel matrix decides the probability that transition from state i to state k happens during time period [0,t].

One step transition probabilities for embedded Markov chain can be evaluated with the help of kernel matrix

$$\Pi_{ik} = \lim_{t\to\infty} Q_{ik}(t).$$

Cumulative distribution function (cdf) of conditional sojourn time in the state i can be calculated as

$$F^*_{ik}(t) = \frac{Q_{ik}(t)}{\Pi_{ik}}.$$

Cdf of unconditional sojourn time T_i in any state i based on the kernel matrix can be defined as

$$F_i(t) = \sum_{k=1}^{k_j} Q_{ik}(t).$$

The mean unconditional sojourn time in the state i can be evaluated as

$$\overline{T_i} = \sum_{k=1}^{k_j} \Pi_{ik}\overline{T_{ik}}^*.$$

where $\overline{T_{ik}}^*$ mean conditional sojourn time in the state i given the component or the system transits from state i to state k.

p_i, $i = 1, 2 \ldots k_j$ are steady state probabilities of embedded Markov chain. These probabilities are obtained from the solution of the following system of equations.

$$p_k = \sum_{i=1}^{k_j} p_i \Pi_{ik}$$
$$\sum_{i=1}^{k_j} p_i = 1$$

For computing steady state probabilities of each component of the system using the formula.

$$\theta_i = \frac{p_i \overline{T_i}}{\sum_{i=1}^{k_j} p_i \overline{T_i}}$$

In this model the system consists of n components with each component possessing k_j states. $g^j = \{g_1^j, g_2^j \ldots g_{k_j}^j\}$ and $\theta^j = \{\theta_1^j, \theta_2^j \ldots \theta_{k_j}^j\}$ are performance level and steady state probabilities of j^{th} component which can be obtained by semi-Markov approach. The Universal generating function of j^{th} component is defined as

$$u^j(z) = \sum_{i=1}^{k_j} \theta_i^j z^{g_i^j}, j = 1, 2 \ldots n \tag{6.1}$$

To find u-function of the entire system the corresponding composition operators $\Omega_{\Phi(s)}$ and $\Omega_{\Phi(p)}$ or their combinations can be applied over the u-function of the individual components. For MSS with n components connected in parallel the u function of entire system is in the form.

$$U(z) = \Omega_{\Phi(p)}\{u^1(z), u^2(z), ..., u^n(z)\} \tag{6.2}$$

6.3 System Reliability Indices in Steady State Situation

The resulting u function of whole system is given by (6.2). We get this function as

$$U(z) = \sum_{k=1}^{t} \theta_k z^{g_k}$$

Based on this function, reliability indices in steady state situation can be obtained.

1. **Steady state MSS availability**

   Steady state MSS availability can be obtained for any arbitrary constant demand w

$$A_\infty(w) = \delta_A(U(z), w) = \sum_{k=1}^{t}(\theta_k z^{g_k}, w) \tag{6.3}$$

2. **Mean Steady state MSS performance**

   Mean Steady state MSS performance is

$$E_\infty = \delta_E(U(z)) = \delta_E(\sum_{k=1}^{t} \theta_k z^{g_k}) = \sum_{k=1}^{t} \theta_k g_k \tag{6.4}$$

3. **Expected Steady state MSS performance deficiency**

   Expected steady state MSS performance deficiency can be obtained for any constant demand w

$$D_\infty(w) = \delta_D(U(z), w) = \delta_D(\sum_{k=1}^{t} \theta_k z^{g_k}, w) = \sum_{k=1}^{t} \theta_k.max(w - g_k, 0) \qquad (6.5)$$

6.4 Numerical Illustration

Consider the same power station with six independent generators (3 generators each with 25 MW and 3 generators each with 50 MW) which is discussed as the numerical example in previous chapters. Total capacity of the power station is 225 MW and repair time, failure time and transition time are exponentially distributed for each generator. Each generator is considered as a component of the system and each generator is periodically inspected and maintained. So the transition is affected by the specific inspection interval of 30 days. Thus the Markovian property of the system is not valid, even if the repair and the failure time distributions are exponential. State space of each component is $S = \{1, 2, 3, 4, 5\}$. β_1^j be the probability of detecting a failure when the component is in state 1 and β_2^j be the probability of detecting a failure when the component is in state 3 through periodic inspection for $j = 1, 2, 3, 4, 5, 6$.

For generators 1,2 and 3 (G1, G2, and G3)

States	1	2	3	4	5
Performance Rate	25	0	12.5	0	0

For generators 4, 5 and 6 (G4, G5 and G6)

States	1	2	3	4	5
Performance Rate	50	0	25	0	0

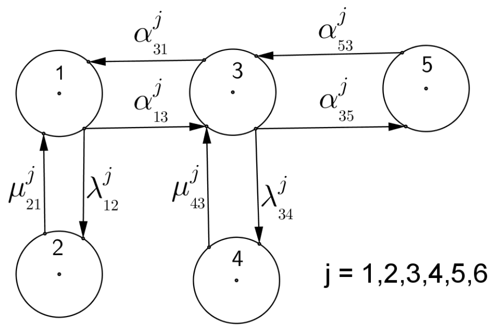

Figure 6.2: State space diagram of the component of power generating system

Semi-Markov kernel of the transition probabilities of states is

$$Q(t) = \begin{bmatrix} 0 & Q_{12}(t) & Q_{13}(t) & 0 & 0 \\ Q_{21}(t) & 0 & 0 & 0 & 0 \\ Q_{31}(t) & 0 & 0 & Q_{34}(t) & Q_{35}(t) \\ 0 & 0 & Q_{43}(t) & 0 & 0 \\ 0 & 0 & Q_{53}(t) & 0 & 0 \end{bmatrix}$$

where

$$Q_{12}(t) = \frac{\beta_1^j \lambda_{12}^j}{\alpha_{13}^j + \lambda_{12}j}[1 - \exp(-(\alpha_{13}^j + \lambda_{12}^j)t)]$$

$$Q_{13}(t) = \frac{\alpha_{13}^j}{\alpha_{13}^j + \lambda_{12}^j}[1 - \exp(-(\alpha_{13}^j + \lambda_{12}^j)t)]$$

$$Q_{21}(t) = [1 - \exp(-\mu_{21}^j t)]Q_{31}(t) = \frac{\alpha_{31}^j}{\alpha_{31}^j + \lambda_{34}^k + \alpha_{35}^j}[1 - \exp^(-(\alpha_{31}^j + \lambda_{34}^j + \alpha_{35}^j)t)]$$

$$Q_{34}(t) = \frac{\beta_2^j \lambda_{34}^j}{\alpha_{31}^j + \lambda_{34}^j + \alpha_{35}^j}[1 - \exp(-(\alpha_{31}^j + \lambda_{34}^j + \alpha_{35}^j)t)]$$

$$Q_{35}(t) = \frac{\alpha_{35}^j}{\alpha_{31}^j + \lambda_{34}^j + \alpha_{35}^j}[1 - \exp(-(\alpha_{13}^j + \lambda_{34}^j + \alpha_{35}^j)t)]$$

$$Q_{43}(t) = [1 - \exp(-\mu_{43}^j t)]$$

$$Q_{53}(t) = [1 - \exp(-\alpha_{53}^j t)].$$

Table 6.1: Transition rates of the generators per hour(h^{-1}) are calculated from the collected data and are given the table below

Generator	α_{13}	α_{31}	α_{35}	α_{53}	λ_{12}	μ_{21}	λ_{34}	μ_{43}
G1	6.7×10^{-2}	6.4×10^{-2}	1.4×10^{-3}	3.3×10^{-2}	3.3×10^{-3}	3.3×10^{-2}	1.6×10^{-3}	3.8×10^{-2}
G2	6.8×10^{-2}	6.5×10^{-2}	1.4×10^{-3}	3.5×10^{-2}	3.3×10^{-3}	3.3×10^{-2}	1.6×10^{-3}	3.8×10^{-2}
G3	6.2×10^{-2}	6.1×10^{-2}	1.4×10^{-3}	3.6×10^{-2}	3.3×10^{-3}	3.3×10^{-2}	1.6×10^{-3}	3.8×10^{-2}
G4	6.9×10^{-2}	6.6×10^{-2}	1.3×10^{-3}	3.8×10^{-2}	3×10^{-3}	3.8×10^{-2}	2×10^{-3}	4×10^{-2}
G5	6.7×10^{-2}	6.4×10^{-2}	1.3×10^{-3}	3.8×10^{-2}	3×10^{-3}	3.8×10^{-2}	2.1×10^{-3}	4×10^{-2}
G6	6.7×10^{-2}	6.4×10^{-2}	1.3×10^{-3}	3.9×10^{-2}	3×10^{-3}	3.8×10^{-2}	2×10^{-3}	4×10^{-2}

U function of the individual elements are determined by the set $\{g^j, \theta^j\}$.

For elements 1,2 and 3 (G1, G2 and G3)

$$g^j = \{25, 0, 12.5, 0, 0, 0\}$$

$$\theta^j = \{\theta_1^j, \theta_2^j, \theta_3^j, \theta_4^j, \theta_5^j\}$$

$$u^j(z) = \sum_{i=1}^{k_j} \theta_i^j z^{g_i^j} \quad j = 1, 2, 3$$

$$u^1(z) = 0.3521z^{25} + 0.0101z^0 + 0.6225z^{12.5} + 4.3 \times 10^{-3}z^0 + 0.011z^0$$

$$u^2(z) = 0.3524z^{25} + 0.0101z^0 + 0.6228z^{12.5} + 4.7 \times 10^{-3}z^0 + 0.01z^0$$

$$u^3(z) = 0.359z^{25} + 0.01z^0 + 0.6156z^{12.5} + 4.5 \times 10^{-3}z^0 + 0.0109z^0$$

For elements 4,5 and 6 (G4, G5 and G6)

$$g^j = \{50, 0, 25, 0, 0, 0\}$$

$$\theta^j = \{\theta_1^j, \theta_2^j, \theta_3^j, \theta_4^j, \theta_5^j\}$$

$$u^j(z) = \sum_{i=1}^{k_j} \theta_i^j z^{g_i^j} \quad j = 4, 5, 6$$

$$u^4(z) = 0.3543z^{50} + 0.008z^0 + 0.6236z^{25} + 0.009z^0 + 0.009z^0$$

$$u^5(z) = 0.3542z^{50} + 0.008z^0 + 0.6234z^{25} + 0.0054z^0 + 0.009z^0$$

$$u^6(z) = 0.3497z^{50} + 0.0078z^0 + 0.6307z^{25} + 0.0031z^0 + 0.0087z^0$$

U function of the entire system in which six independent components are connected in parallel is

$$\begin{aligned} u(z) &= \Omega_{\Phi(p)}\{u^1(z), u^2(z), u^3(z), u^4(z), u^5(z), u^6(z)\} \\ &= \sum_{i=1}^{19} \theta_j z^{g_i} = 2 \times 10^{-10} z^0 + 1 \times 10^{-8} z^{12.5} \\ &+3 \times 10^{-7} z^{25} + 1 \times 10^{-6} z^{37.5} + 3 \times 10^{-5} z^{50} + 0.0003z^{62.5} + 0.0011z^{75} \\ &+0.0075z^{87.5} + 0.0189z^{100} + 0.0808z^{112.5} + 0.1282z^{125} + 0.1791z^{137.5} \\ &+0.1961z^{150} + 0.1624z^{162.5} + 0.118z^{175} + 0.0666z^{187.5} + 0.0288z^{200} \\ &+0.0103z^{212.5} + 0.002z^{225} \end{aligned}$$

We mentioned in chapter 4 that according to Load Despatch Center average demand for a particular month of this power station $w = 108.4$ MW. Reliability indices are obtained by the equations (6.3), (6.4) and (6.5).

Steady State MSS Availability for the constant w = 108.4 MW

$$A_\infty(w = 108.4) = \theta_{10} + \theta_{11} + \theta_{12} + \theta_{13} + \theta_{14} + \theta_{15} + \theta_{16} + \theta_{17} + \theta_{18} + \theta_{19} = 0.9816$$

Mean steady state performance

$$E_\infty = \sum_{k=1}^{19} \theta_k g_k = 150MW$$

Steady state performance deficiency for w = 108.4 MW

$$\begin{aligned} D_\infty(w = 108.4) = \sum_{k=1}^{19} \theta_k max(108.4 - g_k, 0) = \theta_1 \times 108.4 + \theta_2 \times 95.9 \\ +\theta_3 \times 83.4 + \theta_4 \times 70.9 + \theta_5 \times 58.4 + \theta_6 \times 45.9 \\ +\theta_7 \times 33.4 + \theta_8 \times 20.9 + \theta_9 \times 8.4 = 0.3679MW \end{aligned}$$

6.5 Conclusion

This chapter explores steady state reliability analysis of MSS with several independent components. A periodic inspection during a specific interval has been done for each component and maintenance has been performed after detecting a failure.

Here a combination of semi-Markov process and UGF technique has been employed and the methodology applied for analysis of real set of data of a power station with six independent generators. The component has been modeled as a continuous time discrete state semi-Markov process. For avoiding dimension damnation problem, method of combination of semi-Markov process and UGF technique has been used for evaluating steady state reliability indices.

CHAPTER 7

AVAILABILITY ANALYSIS OF A MULTI STATE SYSTEM WITH COMMON CAUSE FAILURES USING MARKOV REGENERATIVE PROCESS

7.1 Introduction

[1] Failure of multiple components of a system due to a common cause is called Common Cause Failures (CCF). CCF is one of the most important issues in evaluation of system reliability. When compared to random failures, which affect individual components, the frequency of CCF has relatively low expectancy. According to Rausand and Hoyland (2004) common cause failures is a dependent failure in which two or more component fault states exist simultaneously or within a short interval of time

[1] Some contents of this chapter are based on Vidhya and Manoharan (2018b).

and are direct result of a shared cause. The CCFs are single faults that causes failure of multiple components. The design error deficiencies, complexity of equipment, human error in installation, maintenance and operation, extreme operating conditions such as high temperature, humidity and external shocks generated by earth quakes, floods etc. are some of the reasons involved in the happening of CCFs. Beta(β) factor model which is introduced by Fleming (1974), is the most commonly used model for common cause failures of the multi state system. The β factor model describes the correlation between the independent random component failures and common cause failures in a redundant multi state system. It is an approximation method applied for quantitative assessment of CCFs. In this method the likelihood of the CCF is estimated in relation to the random failure rate of the component of the system. A β factor is determined such that $\beta\%$ of the failure rate is assigned to the CCF and $(1-\beta)\%$ is assigned to the random failure rate of the component. In the β factor model the failure rate of component $[\lambda]$ is divided in to an independent part $[(1-\beta)\lambda]$ and a dependent part $[\beta\lambda]$ due to common cause. This factor is described as a fraction of the component failures which is due to common cause.

A set of powerful techniques that proved for the solution of non-Markovian models is based on the ideas grouped under the Markov renewal theory. The application of Markov renewal theory for finding reliability and availability of stochastic systems is discussed By Kulkarni (1995). We know that Semi-Markov process is the most widely used and adopted non Markovian model for evaluating reliability and availability of multi state system. Semi Markov modeling based UGF tecnique has been discussed for the reliability analysis of multi state system in chapter 6. The stationary character of Markov regeneratve process (MRGP) has been studied by Pyke and Schaufele (1966). Most of the theoretical foundations of Markov regeneratve process (MRGP) were discussed by Cinlar (1975) in which it is named as semi regenerative process. The transient and steady state analysis of stochastic petri nets was discussed analytically and numerically by Choi et al. (1994). MRGPs have been used to evaluating reliability and availability of the system. We discussed in chapter 1 that some examples concerning reliability and availability of power plants and fault tree systems could be found in Wereley and Walker (1988), Fricks et al. (1997), Perman et al.(1997), Fricks et al. (2002) . Many other examples and applications of MRGP in the dependability context has been solved using SHARPE software [refer Sahner et al. (1995)] by Xie et al.(2003). Semi Markov, Markov regenerative models and Phase type expansion with a number of solved examples were discussed Trivedi and Bobbio (2017).

In this chapter we present a Markov regenerative model for a multi state system with common cause failures. In forthcoming section, we describe the application of Markov regenerative process technique in reliability analysis in transient state and steady state of a multi state system. A parallel system with single repair facility with CCF is analysed using this technique in section 7.3. Steady state probability vector and steady state availability is evaluated using this technique for a numerical example in section 7.4 which is followed by a brief conclusion.

7.2 Markov Regenerative Process

The definition of regenerative process $\{Z(t), t \geq 0\}$ with state space Ω is briefly described in chapter 1. In a Markov Regenerative Process (MRGP) the stochastic evolution between two successive regeneration points depends only on the state of regeneration and not on the evolution before regeneration.

According to Choi et al. (1994) a stochastic process $\{Z(t), t \geq 0\}$ on Ω is called an MRGP if there exist a Markov renewal sequence $\{(Y_n, S_n), n \geq 0\}$ of random variable such that all conditional finite dimensional distribution of $\{Z(S_n + t), t \geq 0\}$ given $\{Z(u), 0 \leq u \leq S_n, Y_n = i\}$ $i \in \Omega$ are the same as those of $\{Z(t), t \geq 0\}$ given $Y_0 = i$.

From the above definition we obtain embedded Markov chain in $\{Z(t), t \geq 0\}$. Global kernel $K(t)$ gives a description of the evolution of process from the Markovian regenerative moment without describing the happenings between regenerative moments.

$$K(t) = K_{ij}(t) = P\{Y_1 = j, S_1 \leq t/Y_0 = i\} \quad \forall i, j \in \Omega$$

An MRGP can change states between two consecutive Markov renewal moments and we have to take these changes through the local kernel matrix $E(t)$. $E(t)$ explains the state probabilities of the process during the interval between successive Markov regenerative moments.

$$E(t) = E_{ij}(t) = Pr\{Z(t) = j, S_1 > t/Y_0 = i\} \quad \forall i, j \in \Omega$$

Conditional transition probabilities are given by

$$V_{ij}(t) = Pr\{Z(t) = j/Z(0) = Y_0 = i\} \quad \forall i, j \in \Omega$$

In many real life problems involving Markov Renewal Process our primary aim is to compute $V_{ij}(t)$ effectively and hence it leads to several performance measures of interest like Availability, Reliability based on $V_{ij}(t)$

The conditional transition probabilities $V_{ij}(t)$ at any instant t can be computed as [refer Choi et al. (1994)].

$$V_{ij}(t) = Pr\{Z(t) = j, S_1 > t/Z(0) = i\} + \sum_{k \in \Omega} \int_0^t dK(u) V_{kj}(t-u) \quad \forall i, j \in \Omega$$

A Markov renewal equation is defined by this set of integral equations. Equation can be expressed in Matrix form as

$$V(t) = E(t) + K(t).V(t)$$

Laplace-Steiltjes transform $K(s)$ and $E(s)$ of $K(t)$ and $E(t)$ respectively can obtained as

$$K(s) = \int_0^{\infty} e^{-st} dK(t)$$
$$E(s) = \int_0^{\infty} e^{-st} dE(t)$$

Then

$$V(s) = E(s) + K(s)V(s) = [I - K(s)]^{-1}E(s) \tag{7.1}$$

$V(t)$ can be obtained by taking inverse laplace transform of $V(s)$ which is given by equation (7.1)

$$P(t)_{1\times\Omega} = P(0)_{1\times\Omega} \times V(t)_{\Omega\times\Omega}$$

For the purpose of the steady state analysis of an MRGP the following two matrices $\phi = [\phi_{ij}]$ and $\alpha = [\alpha_{ij}]$should be calculated. $\phi = [\phi_{ij}]$ is the one step transition probability matrix of the Embedded Markov Chain. α_{ij} is the Mean time the process from state i spends in state j.
The two matrices are defined as

$$\phi = \lim_{t\to\infty} K(t) = \lim_{s\to 0} K(s) \tag{7.2}$$

$$\alpha = \int_{t=0}^{\infty} E(t) \quad dt = \lim_{s\to 0} \frac{1}{s} E(s) \tag{7.3}$$

To obtain the steady-state probabilities of the MRGP, at first we have to solve the steady-state probabilities of the embedded discrete time Markov chain by solving the following system of equations.

$$\begin{aligned} \nu &= \nu.\phi \\ \nu.e &= 1 \end{aligned} \tag{7.4}$$

where e is a column vector with its elements equal to 1 and ν is a Steady state probability vector.

$$\nu = [\nu_1, \nu_2, \ldots, \nu_k] \quad \text{where } k \in \Omega$$

The steady state probability of the MRGP is given by

$$\pi = \frac{\nu\alpha}{\nu\alpha e} \tag{7.5}$$

Steady state Availability of system

Let $\Omega = \{0, 1, \ldots, k\}$ be the set of all possible states of a system. Let Ω' denote the subset of states in which the system is functioning and let $F = \Omega - \Omega'$ denote the states in which the system is failed. The long term availability of the system is the mean proportion of time when the system is functioning. Steady state system availability can be obtained by

$$A_\infty = \sum_{j \in \Omega'} \pi_j \tag{7.6}$$

7.3 Parallel System with Single Repair Facility and CCF

Consider a system which consists of two components named A and B. A single repairman is assigned for the system with the First Come First Served (FCFS) scheduling policy for repair. When the components A or B fails the repairman begins to repair if he is not busy. When one component is already under repair and the other component fails then the second component has to wait for repair till the repairman is free. The lifetime of components A and B are exponentially distributed with the rates λ_A and λ_B respectively. The distribution function of the repair times of components A and B are $G_A(t)$ and $G_B(t)$ respectively. Let $\mu_A(t)$ and $\mu_B(t)$ be the respective repair rates of components A and B. Also in this case common cause failure involving both components A and B can occur with probability β. We can define the stochastic process $Z = \{Z(t); t \geq 0\}$ to represent the system state at any instant t.
$Z(t) \in \{1, 2, 3, 4, 5\}$

System is in state

1, if both components are working at time t

2, if component A is under repair while component B is working at time t

3, if component B is under repair while component A is working at time t

4, if component A is under repair while component B is waiting for repair at time t or due to common cause failure in which the repairman randomly selects component A is the first to be repaired

5, if component B is under repair while component A is waiting for repair at at time t or due to common cause failure in which the repairman randomly selects component B is the first to be repaired

We can define that all state transitions correspond to Markov renewal moments $S = \{S_n; n \in N\}$ and the embedded Markov chain $Y_n; n \in N$ such that Y_n is the state of the system at time S_{n^+} (i.e,$Y_n = Z(S_{n^+})$)

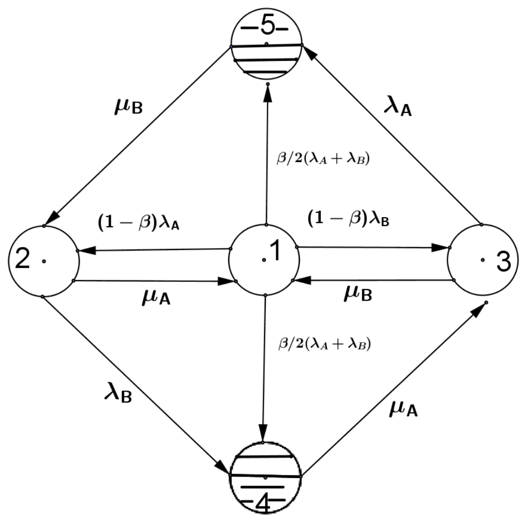

Figure 7.1: State transition diagram

Analysis of the resultant reliability transition diagram shows that Z is an MRGP with an embedded Markov Chain (EMC) defined by the states 1, 2 and 3. We can observe the transition to states 4 and 5 do not belong to the EMC since they are non-renewal moments. System is in state 1 if both A and B are up states and the repairman is free. Component A can fail at rate λ_A and reach state 2. The component A is repaired with cdf $G_A(t)$ to bring the system back to state 1. If component B falls down during repair time of component A, the system jumps to state 4. When the component B is down the system reaches the state 3 and when B is repaired with repair time cdf $G_B(t)$ to back the system state 1. But the component A fails jumping from state 3 to state 5. To find the distribution of Z for MRGP we have to construct kernel matrices [Global kernel matrix $K(t)$ and local kernel matrix $E(t)$]. Let R_A, R_B be the time to repair and L_A and L_B be the times to failure of A and B respectively.

$$K(t) = \begin{pmatrix} 0 & k_{12}(t) & k_{13}(t) \\ k_{21}(t) & 0 & k_{23}(t) \\ k_{31}(t) & k_{32}(t) & 0 \end{pmatrix},$$

$K_{12}(t) = \Pr\{$If A fail before B or common cause failures occurs and repairman chose to repair A first and complete the repair action$\}$

$$= Pr\{Z(S_1) = 2, S_1 \leq t/Z_0 = 1\} = Pr\{(L_A \leq t \cap L_B > L_A) \cup (R_A \leq t \cap (L_A = L_B) \leq R_A)\}$$
$$= (1-\beta)\lambda_A \int_0^t e^{-(\lambda_A+\lambda_B)u} du + \frac{\beta}{2}(\lambda_A + \lambda_B) \int_0^t e^{-(\lambda_A+\lambda_B)u} G_A(t-u) du$$

$$K_{13}(t) = (1-\beta)\lambda_B \int_0^t e^{-(\lambda_A+\lambda_B)u} du + \frac{\beta}{2}(\lambda_A + \lambda_B) \int_0^t e^{-(\lambda_A+\lambda_B)u} G_B(t-u) du$$

$K_{21}(t) = \Pr\{$Repair A is finished up to time t and B has not failed during repair A$\}$

$$= Pr\{Z(S_1) = 1, S_1 \leq t/Z_0 = 2\} == Pr\{R_A \leq t \cap L_B > R_A\} = \int_0^t e^{-\lambda_B u} dG_A(u)$$

$K_{23}(t) = \Pr\{$Repair A is not finished up to time t and B failed during the repair A$\}$

$$= Pr\{Z(S_1) = 3, S_1 \leq t/Z_0 = 2\} = \int_0^t (1 - e^{-\lambda_B u}) dG_A(u)$$

$$K_{31}(t) = Pr\{Z(S_1) = 3, S_1 \leq t/Z_0 = 3\} = \int_0^t e^{-\lambda_A u} dG_B(u)$$

$$K_{32}(t) = Pr\{Z(S_1) = 2, S_1 \leq t/Z_0 = 3\} = \int_0^t (1 - e^{-\lambda_A u}) dG_B(u)$$

$$E(t) = \begin{pmatrix} E_{11}(t) & 0 & 0 & E_{14}(t) & E_{15}(t) \\ 0 & E_{22}(t) & 0 & E_{24}(t) & 0 \\ 0 & 0 & E_{33}(t) & 0 & E_{35}(t) \end{pmatrix},$$

$E_{11}(t) = \Pr\{$Remaining state 1 until time t$\}$

$$= Pr\{Z(t) = 1, S_1 > t/Z_0 = 1\} = (1 - \beta)e^{-(\lambda_A + \lambda_B)t}$$

$E_{22}(t)$=Pr{repair A is not finished up to time t and B has not failed}

$$= Pr\{Z(t) = 2, S_1 > t/Z_0 = 2\} = (1 - G_A(t))e^{-\lambda_B t}$$

$$E_{33}(t) = (1 - G_B(t)e^{-\lambda_A t}$$
$$E_{14}(t) = \frac{\beta}{2}e^{-(\lambda_A+\lambda_B)t}$$
$$E_{15}(t) = \frac{\beta}{2}e^{-(\lambda_A+\lambda_B)t}$$

$E_{24}(t)$ =Pr{repair A is not finished up to time t and B has not failed}

$$= (1 - G_A(t))(1 - e^{-\lambda_B t})$$

$$E_{35}(t) = (1 - G_B(t))(1 - e^{-\lambda_A t})$$

Laplace-Steiltjes transform of Global Kernel Matrix is

$$K(s) = \begin{pmatrix} 0 & \frac{(1-\beta)\lambda_A}{s+\lambda_A+\lambda_B} + \frac{\beta(\lambda_A+\lambda_B)G_A(s)}{2(s+\lambda_A+\lambda_B)} & \frac{(1-\beta)\lambda_B}{s+\lambda_A+\lambda_B} + \frac{\beta(\lambda_A+\lambda_B)G_A(s)}{2(s+\lambda_A+\lambda_B)} \\ G_A(s+\lambda_B) & 0 & G_A(s) - G_A(s+\lambda_B) \\ G_B(s+\lambda_A) & G_B(s) - G_B(s+\lambda_A) & 0 \end{pmatrix},$$

Laplace-Steiltjes transform of Local Kernel Matrix is

$$E(s) = \begin{pmatrix} E_1(s) & E_2(s) \end{pmatrix},$$

where

$$E_1(s) = \begin{pmatrix} \frac{(1-\beta)s}{s+\lambda_A+\lambda_B} & 0 & 0 \\ 0 & \frac{s}{s+\lambda_B}(1 - G_A(s+\lambda_B)) & 0 \\ 0 & 0 & \frac{s}{s+\lambda_A}(1 - G_B(s+\lambda_A)) \end{pmatrix}$$

and

$$E_2(s) = \begin{pmatrix} \frac{\beta s}{2(s+\lambda_A+\lambda_B)} & \frac{\beta s}{2(s+\lambda_A+\lambda_B)} \\ \frac{\lambda_B}{s+\lambda_B} - G_A(s) + \frac{s}{s+\lambda_B} G_A(s+\lambda_B) & 0 \\ 0 & \frac{\lambda_A}{s+\lambda_A} - G_B(s) + \frac{s}{s+\lambda_A} G_B(s+\lambda_A) \end{pmatrix}$$

7.4 Numerical Illustration

Consider a numerical example which has deterministic repair-time distribution function.

$$G_A(t) = u(t - \mu_A), \mu_A > 0$$

$$G_B(t) = u(t - \mu_B), \mu_B > 0$$

where u(t) is the unit step function. The units are hours for repair-time (parameters μ_A and μ_B) and $hour^{-1}$ for the failure rates (parameters λ_A and λ_B). The values of parameters of the system are given below

Component	λ	μ
A	0.01	5
B	0.01	5

$$K(s) = \begin{pmatrix} 0 & \frac{(1-\beta)0.01}{s+0.02} + \frac{\beta 0.02e^{-5s}}{2(s+0.02)} & \frac{(1-\beta)0.01}{s+0.02} + \frac{\beta 0.02e^{-5s}}{2(s+0.02)} \\ e^{-5(s+0.01)} & 0 & e^{-5s} - e^{-5(s+0.01)} \\ e^{-5(s+0.01)} & e^{-5s} - e^{-5(s+0.01)} & 0 \end{pmatrix},$$

$$E_1(s) = \begin{pmatrix} \frac{(1-\beta)s}{s+0.02} & 0 & 0 \\ 0 & \frac{s}{s+0.01}(1 - e^{-5(s+0.01)}) & 0 \\ 0 & 0 & \frac{s}{s+0.01}(1 - e^{-5(s+0.01)}) \end{pmatrix},$$

$$E_2(s) = \begin{pmatrix} \frac{\beta s}{2(s+0.02)} & \frac{\beta s}{2(s+0.02)} \\ \frac{0.01}{s+0.01} - e^{-5s} + \frac{s}{s+0.01}e^{-5(s+0.01)} & 0 \\ 0 & \frac{0.01}{s+0.01} - e^{-5s} + \frac{s}{s+0.01}e^{-5(s+0.01)} \end{pmatrix}$$

The matrices ϕ and α are obtained by solving the equations (7.2) and (7.3).

$$\phi = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.951229424 & 0 & 0.048770576 \\ 0.951229424 & 0.048770576 & 0 \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 50(1-\beta) & 0 & 0 & 25\beta & 25\beta \\ 0 & 4.877058 & 0 & 0.877058 & 0 \\ 0 & 0 & 4.877058 & 0 & 0.877058 \end{pmatrix}$$

Steady state probabilities of embedded Markov chain can be evaluated by solving the system of equation (7.4)

$$[\nu_1, \nu_2, \nu_3] = [0.487503, 0.256249, 0.256249]$$

The Steady state probability vector is obtained by the equation (7.5)

$$\left[\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\right] =$$
$$\left[0.0.892074(1-\beta), 0.045738, 0.045738, 0.446037\beta + 0.008225, 0.446037\beta + 0.008225\right]$$

We get steady state Availability by the equation (7.6)

$$A_\infty = \pi_1 + \pi_2 + \pi_3 = 0.98355(1-\beta)$$

Impact of the common cause failures on the system should be evaluated for the corresponding model. The MRGP steady state availability can be calculated for varying common cause failure probability β value. By analyzing the MRGP for the above numerical values, the graph depicted in Fig. 2 is obtained.

The graph reveals the variation of the steady state availability (A_∞) of the system by changing the common cause failure probability β from 0 to 0.5. On viewing the graph we can observe a clear linear trend of the A_∞ with respect to β.

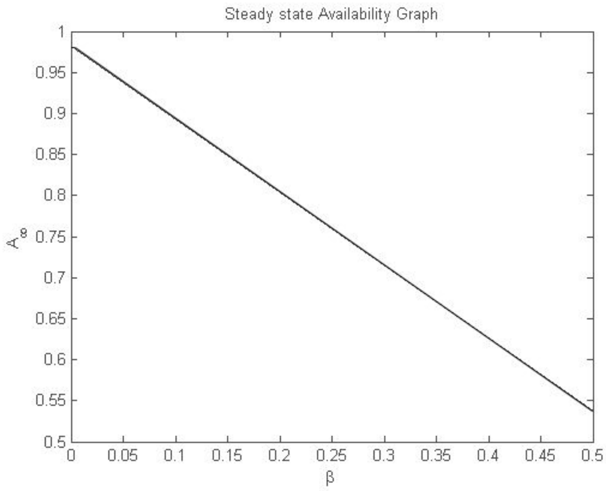

Figure 7.2: Steady state availability of the system for varying β

7.5 Conclusion

In this chapter analytical techniques based on Markov regenerative process are explored for modeling and evaluation of availability of multi state system. A parallel system of two components with common cause failures were elaborated to show the applicability of MRGP in the evaluation of performance measures with numerical example. Since Markov regenerative process can overcome limitations of semi Markov process to some extent, one can solve wide range of problems in system reliability on similar lines.

CHAPTER 8

PHASE TYPE MODELING IN MULTI STATE SYSTEM RELIABILITY

8.1 Introduction

Phase type distribution has been introduced as an instrument for integrating different types of stochastic models and initially it has been applied effectively in various queueing models. Even though it has been used in reliability analysis for quite some time, now the possibility of this important distribution is more actively discussed in the reliability analysis of multi state system. In the reliability theory, assessed distribution functions are often continuous. Life time models are determined from various distributions such as exponential, Erlang, Weibull etc. which have been verified as beneficial in practical applications. If the model is fabricated from observations, always it is not easy to adopt one specific distribution because none of the known distributions nearly fits the data set. In continuous case we can solve this problem by considering the distribution for the data set by a continuous phase type distribution. Variety of phase type distribution are involved in many stochastic models. Phase based techniques has a significant applicability in multi state system reliability theory than other models in which we use Markov process semi Markov process and Markov regenerative process.

In this chapter we aim to demonstrate usefulness of phase type distribution in the evaluation of reliability analysis of repairable parallel multi state system with single repair facility. Here we take a little bit of effort to show how effective the phase type distribution is in reliability analysis. We know that phase type distribution was introduced by Neuts (1981). The main advantage of employing phase type (PH) distribution in reliability theory is its mathematical simplicity. Phase type distribution gives excellent computational performance which makes analytic modeling easy. The complicated numerical differentiation and integration can be turned to the appropriate matrix operations in models involving phase type distribution. System with two components without external failures using phase type distribution was considered by Neuts and Meier (1981). Another important advantage of PH distribution is that it is closed under some operations which are helpful in reliability analysis of multi state system. Closure properties of phase type distributions under some operations which are applicable in multi state reliability theory were discussed by Assaf and Levikson (1982). A parallel system of 'n' identical components having a single repairman with operational times exponentially distributed and the repair times distributed as phase type was discussed by Chakravarthy (1983) from the viewpoint of queueing theory. Manoharan et al. (1992) have obtained a preservation result of phase type distribution under Poisson shock models. Repairable models with operating and repair times distributed as phase type have been discussed Neuts et al. (2000) and performance measures of the systems were evaluated for the proposed models. Two models of a repairable two unit system with phase type operational and repair times were presented by Perez-Ocon and Ruiz Castro (2004). Phase type distribution has been fruitfully employed for the study of shock models in reliability theory [refer Montoro-Cazorla (2007), Montoro-Cazorla (2009) and Segovia and Labeau (2013)]. Phase type modeling has been suggested for dynamic assessment of non repairable multi state system by Eryilmaz (2015) when the system degrades based on Markov process. Phase type distribution enables us to use more complex models in practical situations. Systems with different repair strategies can be learned through this phase type assumption. Denseness of PH distribution can effectively enhance the expressive ability of the model.

In this chapter we use continuous phase type distribution for reliability analysis of two component parallel system with single repair facility. The definition of continuous phase type distribution is explained briefly in the introductory chapter. Some of the definitions of the matrix operations which are very useful for the calculations in this chapter are given below.

Definition 8.1: If A and B are rectangular matrices of dimensions $m_1 \times m_2$ and $n_1 \times n_2$, respectively, their Kronecker product $A \otimes B$ is the matrix of dimensions $m_1 n_1 \times m_2 n_2$, written in compact form as $(a_{ij}B)$

A useful property of this product is the following equality $(A \otimes B)(C \otimes D) = AC \otimes BD$, which holds whenever the ordinary matrix product is well defined.

Definition 8.2: If A and B are matrices of dimensions $m \times m$ and $n \times n$, respectively, their Kronecker sum $A \oplus B$ is the matrix dimensions $mn \times mn$ written as

$$A \oplus B = A \otimes I_n + I_m \otimes B$$

where I_n and I_m are the identity matrices of order n and m respectively. A functional property of this sum is the following equality for the matrices above

$$exp(A \oplus B) = exp(A) \otimes exp(B)$$

For more aspects about these operations, refer Bellman (1960).

Phase type modeling for a multi state parallel system of two components having single repair facility is discussed in forthcoming section. Here operational and repair times of components of the system are followed phase type distribution, instead of other typical distribution like exponential distribution.

8.2 Two component parallel system with single repair facility

Assumptions

- A parallel system which contains two different components with single repair facility is considered.
- The life time of components is independently and identically distributed and it follows continuous PH distribution
- The failed components are repaired and the components are 'as good as new' after the repair action is completed. (That is repair action is considered to be perfect.)
- The repair time also follows continuous PH distribution.

We consider a system with two independent components . When a failure occurs for a component, it goes to the repair channel. Let X_k be the life time and Y_k be the repair time for component k, where $k = 1, 2$. We assume that X_k follows a continuous PH distribution $(\alpha(k), T(k))$ with order m_k and repair time Y_k follows a continuous PH distribution $(\delta(k), U(k))$ with order n_k. We denote the absorbtion rate vectors of the life time is $T^0(k)$ $(k = 1, 2)$ and absorbtion rate vector of repair time is $U^0(k)$

First of all we can describe states of the components. In terms of these states of the components we can describe the states of the system. We have to define states of the component k, for $k = 1, 2$. We denote w_k as state of the component k when component k is operational, r_k when it is in repair and qr_k when it is waiting for repair (ie, it is in queue for repairing). Following are the states of the system

$S_0 : \{w_1, w_2\}$, $S_1 : \{w_1, r_2\}$, $S_2 : \{r_1, w_2\}$, $S_3 : \{r_1, qr_2\}$, $S_4 : \{qr_1, r_2\}$

Infinitesimal generator for this two component parallel system can be written as

$$Q = \begin{pmatrix} Q_{00} & Q_{01} & Q_{02} & 0 & 0 \\ Q_{10} & Q_{11} & 0 & 0 & Q_{14} \\ Q_{20} & 0 & Q_{22} & Q_{23} & 0 \\ 0 & Q_{31} & 0 & Q_{33} & 0 \\ 0 & 0 & Q_{42} & 0 & Q_{44} \end{pmatrix},$$

where

$$Q_{00} = T(1) \oplus T(2), \quad Q_{01} = I_{m_1} \otimes T^0(2)\delta(2), \quad Q_{02} = T^0(1)\delta(1) \otimes I_{m_2}$$

$$Q_{10} = I_{m_1} \otimes U^0(2)\alpha(2), \quad Q_{11} = T_1 \oplus U_2, \quad Q_{14} = T^0(1)\delta(1) \otimes I_{n_2}$$

$$Q_{20} = U^0(1)\alpha(1) \otimes I_{m_2}, \quad Q_{22} = U(1) \oplus T(2), \quad Q_{23} = I_{n_1} \otimes T^0(2)\delta(2)$$

$$Q_{31} = U^0(1)\alpha(1) \otimes I_{n_2}, \quad Q_{33} = U(1) \otimes I_{n_2}$$

$$Q_{42} = I_{n_1} \otimes U^0(2)\alpha(2), \quad Q_{44} = I_{n_1} \otimes U(2)$$

If component 1 experiences the transition between its operational states, the transition probability matrix can be written as $T(1) \otimes I_{m_2}$, in which I_{m_2} is a identity matrix having the same order with matrix $T(2)$. Similarly if component 2 experiences the transition between its operational states, the transition probability matrix can be written as $I_{m_1} \otimes T(2)$. Hence the transition probability matrix of the system inside state space S_0 is $Q_{00} = T(1) \otimes I_{m_2} + I_{m_1} \otimes T(2) = T(1) \oplus T(2)$.

The transition probability matrix from state space S_0 to S_1 can be written as $Q_{01} = I_{m_1} \otimes T^0(2)\delta(2)$, which means that component 1 does not change and component 2 enters the absorbing state (ie, failed state) with the transition probability $T^0(2)$ and then enters repair phase of the system with the probability distribution $\delta(2)$. Similarly the transition probability matrix S_0 to S_2 can be written as $Q_{02} = T^0(1)\delta(1) \otimes I_{m_2}$.

The transition probability matrix S_1 to S_0 can be written as $Q_{10} = I_{m_1} \otimes U^0(2)\alpha(2)$, in which component 1 does not change and component 2 is repaired. The transition probability matrix inside the the state S_1 is $Q_{11} = T_1 \oplus U$. The transition probability matrix S_1 to S_4 is $Q_{14} = T^0(1)\delta(1) \otimes I_{n_2}$.

When two component parallel system attains steady state, according to the definition of stationary probability vector in continuous time Markov process, The stationary probability vector

$$\pi = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix}$$

satisfies the equation $\pi Q = 0$, subject to normalization condition $\pi e = 1$. We can use matrix analytical methods and suitable computer software to solve the above equations. After getting steady state probability vector $\pi_i, \;\; i = 0, 1, 2, 3, 4$, stationary availability can be easily solved. Stationary availability of two component parallel system is:

$$A = \sum_{i=0}^{2} \pi_i e$$

We shall apply the following proposition of PH distribution in the present contest [refer Neuts (1975), He (2014)].

Proposition 8.2.1. Assume that X_1 has a PH-distribution with PH-representation (α, T), X_2 has a PH-distribution with PH-representation (β, S), and X_1 and X_2 are independent. Then $X = max\{X1, X2\}$ has a PH-distribution with PH-representation (γ, U) where

$$\gamma = (\alpha \otimes \beta, (1 - \alpha e)\beta, (1 - \beta e)\alpha),$$

$$U = \begin{pmatrix} T \oplus S & T^0 \otimes I & I \otimes S^0 \\ 0 & S & 0 \\ 0 & 0 & T \end{pmatrix}$$

Proof. The proof is based on

$$P\{max\{X_1, X_2\} \leq t\} = P\{X_1 \leq t\} P\{X_2 \leq t\}$$

for independent random variables X_1 and X_2. It is easy to check that the above representation is a PH representation of a PH random variable, to be called X. The distribution function of X can be calculated routinely as follows, for $t > 0$,

$$P\{X \leq t\} = 1 - (\alpha \otimes \beta, (1-\alpha e)\beta, (1-\beta e)\alpha) exp(\begin{pmatrix} T \oplus S & T^0 \otimes I & I \otimes S^0 \\ 0 & S & 0 \\ 0 & 0 & T \end{pmatrix} t)e$$

$$= 1 - (\alpha \otimes \beta, (1-\alpha e)\beta, (1-\beta e)\alpha) \begin{pmatrix} exp(Tt)e \otimes (e - exp(St)e) + e \otimes exp(St)e \\ exp(St)e \\ exp(Tt)e \end{pmatrix}$$

$$= (1 - \alpha exp(Tt)e)(1 - \beta exp(St)e) = P\{X_1 \leq t\}P\{X_2 \leq t\} = P\{max\{X_1, X_2\} \leq t\}.$$

This completes the proof. □

Theorem 8.2.2. *The work time of two component, where k^{th} component's life time X_k follows a continuous PH distribution $(\alpha(k), T(k))$ with order m_k $(k = 1, 2)$, parallel system follows continuous phase type distribution $(\beta, S)$with order $m_1 m_2 + m_1 + m_2$ where*

$$\beta = (\alpha(1) \otimes \alpha(2), (1 - \alpha(1)e)\alpha(2), (1 - \alpha(2)e)\alpha(1)),$$

$$S = \begin{pmatrix} T(1) \oplus T(2) & I_{m_1} \otimes T^0(2) & T^0(1) \otimes I_{m_2} \\ 0 & T(1) & 0 \\ 0 & 0 & T(2) \end{pmatrix}$$

Proof. The system enters the failed state when both the components fail. According to property 1.9.1, phase type distribution are closed under minimum and maximum. That is the minimum and maximum of two independent phase type random variable is also a phase type random variable. The life time of two component parallel system is the maximum value of the life time of the components. The system life time follows phase type distribution (β, S) using proposition 8.2.1. □

This theorem states that two component parallel system has the life time distribution following the phase type distribution, PH(β, S).

According to property 1.9.2, Mean time between failure (MTBF) of system is obtained as $\mu_1 = -\beta S^{-1} e$.

Remark 8.2.3. A parallel system with more than two components can be dealt with on similar lines to evaluate the stationary characteristics and performance of the system. The dimensions of the matrices involved in such cases may be quite large, but the algorithmic methods suggested in this chapter could be easily extended for the purpose.

We consider here a numerical example of two component parallel system with operational times and repair times of components are governed by phase type distribution having the following matrices

$$\alpha(1) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \qquad T(1) = \begin{pmatrix} -0.02 & 0.02 & 0 \\ 0.01 & -0.08 & 0.07 \\ 0.005 & 0 & -0.01 \end{pmatrix}$$

and

$$\alpha(2) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \qquad T(2) = \begin{pmatrix} -0.2 & 0.1 & 0 \\ 0.03 & -0.05 & 0.02 \\ 0.02 & 0 & -0.02 \end{pmatrix}.$$

$$\delta(1) = \delta(2) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$U(1) = U(2) = \begin{pmatrix} -1.499 & 0 & 1.499 \\ 0 & -1.499 & 0 \\ 0 & 1.499 & -1.499 \end{pmatrix}$$

and

Absorption rate vectors of the operational times of the components are

$$T^0(1) = \begin{pmatrix} 0 \\ 0 \\ 0.095 \end{pmatrix}, \qquad T^0(2) = \begin{pmatrix} 0.1 \\ 0 \\ 0 \end{pmatrix}.$$

$$U^0(1) = U^0(2) = \begin{pmatrix} 0 \\ 1.499 \\ 0 \end{pmatrix}.$$

Using Mathematica software we can calculate stationary probability vectors as

$$\pi_0 = \begin{pmatrix} 0.1316 & 0.2634 & 0.2636 & 0.0325 & 0.0658 & 0.0659 & 0.0231 & 0.0461 & 0.0461 \end{pmatrix}$$
$$\pi_1 = \begin{pmatrix} 0.009 & 0.0087 & 0.0089 & 0.0022 & 0.0022 & 0.0022 & 0.0015 & 0.0015 & 0.0015 \end{pmatrix}$$
$$\pi_2 = \begin{pmatrix} 0.0016 & 0.0029 & 0.0029 & 0.0014 & 0.0029 & 0.0029 & 0.0015 & 0.0029 & 0.0028 \end{pmatrix}$$
$$\pi_3 = \begin{pmatrix} 0.0001 & 0 & 0 & 0.0003 & 0 & 0 & 0.0002 & 0 & 0 \end{pmatrix}$$
$$\pi_4 = \begin{pmatrix} 0.001 & 0.0003 & 0.0002 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Steady state availability is $A = 0.9979$.

We can evaluate the matrices β and S as

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$S=\begin{pmatrix}
-0.22 & 0.1 & 0 & 0.02 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0\\
0.03 & -0.07 & 0.02 & 0 & 0.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0.02 & 0 & -0.04 & 0 & 0 & 0.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0.01 & 0 & 0 & -0.28 & 0.1 & 0 & 0.07 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0\\
0. & 0.01 & 0 & 0.03 & -0.13 & 0.02 & 0 & 0.07 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0.01 & 0.02 & 0 & -0.1 & 0 & 0 & 0.07 & 0 & 0 & 0 & 0 & 0 & 0\\
0.005 & 0 & 0 & 0 & 0 & 0 & -0.3 & 0.1 & 0 & 0 & 0 & 0.1 & 0.095 & 0 & 0\\
0 & 0.005 & 0 & 0 & 0 & 0 & 0.03 & -0.15 & 0.02 & 0 & 0 & 0 & 0 & 0.095 & 0\\
0 & 0 & 0.005 & 0 & 0 & 0 & 0.02 & 0 & -0.12 & 0 & 0 & 0 & 0 & 0 & 0.095\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.02 & 0.02 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & -0.08 & 0.07 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.005 & 0 & -0.01 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0.1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.03 & -0.05 & 0.02\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02 & 0 & -0.02
\end{pmatrix}$$

Then Mean time between failures MTBF = 301 hours.

8.3 Conclusion

Phase type distribution is applied in this chapter to study the availability and MTBF of a repairable parallel system. Phase type distribution can represent as matrix form, which helps the computation by applying suitable softwares and matrix analytic theory. Its good analytic characteristics helps to solve the multi state reliability problems. In the study of real life situation, modeling becomes more qualitative by using this distribution.

EPILOGUE

In recent years, reliability analysis of multi state system has witnessed a remarkable development in the field of engineering. Many works have been created in the study of multi state system in reliability engineering. In this work, different methods were considered for reliability analysis of various multi state repairable systems. Probability concepts are incorporated in the evaluation of performance measures of multi state system. The main performance measures used in this thesis are availability, expected steady state performance and expected steady state performance deficiency. In the first chapter, preliminary ideas on continuous time Markov chain, Regenerative process, semi Markov process, binary state system, multi state system, phase type distribution etc with proper literature review were extensively presented. Definition, properties, performance measures and types of multi state system were comprehensively explained in the second chapter.

In chapter 3 we discussed basic ideas of applying stochastic process method for reliability assessment of a multi state system. Steady state reliability of different structures of independent components of a system is analyzed based on continuous time Markov chain. Traditional method such as stochastic process method is not usually effective in the application of reliability evaluation of complex multi state system. The combined Markov process technique and universal generating function (UGF) technique which is used in chapter 4 diminishes the dimension of the system of equations. In this work the technique is used for multi state system which is composed of statistically independent repairable components.

The major disadvantage of UGF technique is that theoretically it can be applied only for random variables and so this technique works with only steady state performance probability distribution when we consider reliability analysis of multi state system. In order to extend the application of UGF technique to dynamic reliability analysis of MSS, we demonstrate a special transform for a discrete state continuous time Markov chain which is known as Lz transform in chapter 5. Short term reliability evaluation of a power generating system using this method is presented in this chapter. Component criticality and importance analysis of multi state system is a hopeful area of study in reliability theory. The Lz transform method can be applied for assessing various importance measures in multi state system. The Lz transform technique has an important role in short term risk evaluation of power stations.

Application of Markov chain model in which transition time between any states of the system or component distributed as exponential, is not practicable into real world problems of reliability analysis. The principle advantage of semi Markov model is that it allows non exponential distribution for transition between states of components or system in reliability analysis. Preventive maintenance is defined as the series of actions taken on systematically for abolishing accumulative deterioration when the system is still in operating states. In chapter 6 we considered modeling of a multi state system with a periodic inspection during a specific period and hence it takes place preventive maintenance. A semi Markov reliability model for a power generating system is presented in that chapter. With the help of semi Markov process, we can develop algorithm of the complex periodic inspection policies for multi state system by increasing the maintenance effectively. Also we have to determine maintenance policies that maximizes the availability of multi state system while minimizing the cost of working of the system.

A multi state system reliability model with common cause failures, based on MRGP is presented in chapter 7. We can solve many other real life problems in multi state system reliability theory using Markov regenerative process. In chapter 8 we considered phase type modeling for the evaluation of a repairable parallel system using phase type distributions because of its versatility. we can develop various models for handling of maintenance and reliability of repairable multi state system.

In this work we considered only multi state systems with independent components. Modeling of multi state system with dependent components is an interesting problem in reliability analysis and it has many practical application in real life situation. Furthermore we have primarily learned parallel structure of multi state system in this work. Many problems have to be solved in reliability analysis of multi state series system, series- parallel system, k out of n systems etc. Optimization problems are important problem in multi state reliability theory. The combination of different types of multi state system with various norms and restrictions can form a lot of different stimulating optimization problems. For example, including economical indices connected with various levels of performance of system gives a vast variety of models in which design maintenance activity, warranty policy etc are optimized. Since computers are an integral part of our daily life, software reliability research has an important role in reliability analysis of MSS. More software reliability models needed to be developed as future research.

As a comparatively new discipline multi state system reliability has a lot of phenomena to be fulfilled. In the future, many auspicious multi state reliability research direction can be concentrated on component criticality and importance analysis, assessment and fault-tolerant design of multi state network systems, condition based maintenance system based on multi state system reliability theory, modeling probabilistic risk evaluation of multi state system, system state allocation, state space optimization, modeling of system degradation, optimization and estimation of reliability of complex multi state system and so on.

BIBLIOGRAPHY

[1] Abouammoh, A. M. and Al-Kadi (1991). On meassures of importance for components in multi state coherent systems, *Microelectronics Reliability*, **31**, 109-122.

[2] Abouammoh, A. M. and Al-Kadi (1995). Multistate coherent system of order k, *Microelectronics Reliability*, **35**, 1415-1421.

[3] Ascher, H. and Feingold, H. (1984). *Repairable systems: reliability modelling, inference misconceptions and their causes*, Dekker, New York.

[4] Assaf, D. And Levikson, B. (1982). Closure of phase type distributions under operations in reliability theory, *Annals of probability*, **10**, 265-269.

[5] Aven, T (1985). Reliability evaluation of multi state systems with multistate components, *IEEE Transactions On Reliability*, **34**, 473-479.

[6] Aven, T. (1985). Some considerations on reliability theory and its applications, *Reliability Engineering and System Safety*, **41**, 259-266.

[7] Aven, T. (1993). On performance measures for multi state monotone systems, *Reliability Engineering and System Safety*, **34**, 473-479.

[8] Aven, T. and Jensen, U. (1999). *Stochastic models in reliability.* Springer, New York.

[9] Barlow , R. E. And Proschan, F. (1965). *Mathematical theory of reliability*, SIAM, New York.

[10] Barlow , R. E. and Wu, A.S. (1978). Coherent systems with multistate elements, *Mathematics of Operations Research*, **3**, 275-281.

[11] Barlow , R. E. and Proschan, F. (1975). *Statistical theory of Reliability and life testing*, Holt, Rinehart and Winston, New York.

[12] Bellman, R. (1960). *Introduction to matrix analysis*, Mc. Graw-Hill, New York.

[13] Billinton, R. and Allan, R. (1996). *Reliability evaluation of power systems*, Plenum Press, New York.

[14] Billinton, R. and Li, W.Y. (1991). Hybrid approach for reliability evaluation of composite generation and transmission system using Monte Carlo simulation and enumeration technique, *IEEE Proceedings -C Generation, Transmission and Distribution*, **138(3)**, 233-241.

[15] Birnbaum, Z. W. (1969). On the importance of different components in a multi component system, *in Multi variate analysis II, Ed. P. R. Krishnai*, Academic Press, 591-592, New York.

[16] Block, H. W. and Savits, T. H. (1982). A decomposition for multi state monotone systems, *J. Applied Probability*, **19**, 391-402.

[17] Block, H. W. and Savits, T. H. (1984). Continuous multi state structure functions, *Operations Research*, **32**, 703-714.

[18] Boedigheimer, R. A. and Kapur, K.C. (1994). Customer-driven reliability models for multi-state coherent systems, *IEEE Transactions on Reliability*, **43(1)**, 46-50.

[19] Braun, M. (1993). *Differential equations and their applications*, Narosa Publishing House, New Delhi.

[20] Brunelle, R.D. and Kapur, K. C. (1999). Review and classification of reliability measures for multi state and continuum models. *IIE transactions*, **31**, 1171-1180.

[21] Bryson, M. C. and Siddiqui, M.M. (1969). Some criteria for aging , *J. American Statistical Association*, **64**, 1472-1483.

[22] Bueno, V.C. (1989). On the importance of components for multi state monotone systems, *Statistics and Probability Letters*, **7**, 51-59.

[23] Chacko, V.M. and Manoharan, M. (2011a). Joint importance measures for multi state reliability systems. *Opsearch (Springer)*, **48(3)**, 257-278.

[24] Chacko, V.M. and Manoharan, M. (2011b). Joint importance measures in network system,*Reliability: Theory and Applications*, **2**, 129 -139.

[25] Chakravarthy, S. (1983). Reliability analysis of a parallel system with an exponential life times and phase type repairs, *OR spectrum*, **5**, 229-240.

[26] Choi, H., Kulkarni, V. G. and Trivedi, K. S. (1994). Markov Re-generative stochastic petri nets, *Performance Evaluation*, **20**, 337-357.

[27] Cinlar, E. (1975), *Introduction to stochastic processes*, Prentice-Hall, Engle wood Cliffs, NJ, USA.

[28] Cocozza-Thivent, C. (1997). *Processus stochastiques et fiabilite des syst‘emes*, Springer, Berlin.

[29] Coddington, E. and Levinson, N. (1955). *Theory of ordinary differential equation.* Mc. Graw-hill Publishers, New York.

[30] Cox, D. R. (1955). *The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables*, Louvain, Belgium, 173-206.

[31] Deshpande, J. V. , Kochar, S. C. And Singh, H. (1986). Aspects of positive aging, *J.Applied Probability*, **23**, 748-758.

[32] Ebeling, C. E. (1997). *An introduction to reliability and maintainability engineering*, Mc. Graw- Hill, Boston.

[33] Ebrahimi , E. (1991). How to compare two systems. *J. Applied Probability*, **28**, 940-946.

[34] El- Neweihi, E. and Proschan, F. (1984). Degradable System: A survey of multi state reliability theory, *Communications in Statistics - Theory and Methods*, **13**, 403-432.

[35] El- Neweihi, E., Proschan, F. and Sethuraman, J. (1978). Multistate Coherent Systems, *J. Applied Probability*, **15**, 675-688.

[36] El- Neweihi, E., Proschan, F. and Sethuraman, J. (1988). Optimal allocation of multi state elements, *Handbook of Statistics*, **7**, 427-432.

[37] Endreinyi, J. (1979). *Reliability modelling in electric power systems*, John Wiley and Sons, New York.

[38] Erlang, A. K. (1917-18). Solution of some problems in the theory of probabilities of significance in automatic telephone exchanges, *The Post office Engineer's Journal*, **10**, 189-197.

[39] Eryilmaz, S. (2015). Dynamic assessment of multi state system using phase type modelling, *Reliability Engineering and System Safety*, **140**, 71-77.

[40] Esary, J. D. and Proschan, F. (1962). Coherent Structures of non identical components, *Technometrics*, **5 (2)**, 191-209.

[41] Fleming, K. N. (1974). *Reliability model for common mode failures in redundant safety systems*, Technical Report GA-A-13284, General Atomic Co., San Diego, Calif. (USA).

[42] Franciszek , G. (2007). Applications of semi- Markov process in reliability, *Reliability: Theory and Applications*, December- Special Issue, 3-4.

[43] Frenkel, I., Lisnianski, A. And Khavatskin, L. (2012). Availability assessment for aging refrigeration system by using Lz transform, *J. Reliability and Statistical Studies*, **5**, 33-43.

[44] Fricks, R., Telek, M., Puliafito, A. and Trivedi.K.S. (1997). Markov renewal theory applied to performability evaluation, *in: K. Bagchi, G. Zorbist (Eds), State of the art in performance modeling and simulation, Modelling and Simulation of Advanced Computer Systems: Applications and Systems*, Gordon and Breach publishers, New York , NJ, EUA, 193-236.

[45] Fricks, R.,Yin, L. and Trivedi, K. (2002). Application of semi Markov process and CTMC to evaluation of UPS system availability *in: RAMS 2002*, 584-591.

[46] Gnedenko, B. and Ushakov, I. (1995). *Probabilistic reliability engineering*, John Wiley and sons, New York.

[47] Griffith, W. S. (1980). Multistate reliability models, *J. Applied probability*, **17**, 735-744.

[48] Gurler, U. and Kaya, A. (2002). A maintenance policy for a system with multi state components: an approximate solution, *Reliability Engineering and system safety*, **76**, 117-127.

[49] He, Q. M. (2014). *Fundamentals of matrix- analytical methods*, Springer, New York.

[50] Hirsch, W.M., Meisner, M. and Boll, C. (1968). Cannibalization in multi component systems and theory of reliability, *Naval Research Logistics*, **15(3)**, 331-360.

[51] Huang, C. and Yuan, J. (2010). A two stage preventive maintenance policy for a multi state deterioration system. *Reliability Engineering and System safety*, **95(11)**, 1255-1260.

[52] Hudson, J. C. and Kapur, K. C. (1982). Reliability Theory for multi state systems with multi state components, *Microel, Reliab*, **22(1)**, 1-7.

[53] Issac, W.S., Nouralfath, M. and Daoud, A.K. (2010). Performance evaluation of multi state degraded systems with minimal repairs and imperfect preventive maintenance, *Reliability Engineering and System safety*, **95(2)**, 65-69.

[54] Kolowrocki, K. (2004). *Reliability of large systems*, Elsevier, New York.

[55] Korczak, E. (1997). Reliability analysis of non repaired multi state system, *Advances in safety and Reliability, Proceedings of the ESREL'97 International Conference on Safety and Reliability, 17-20 June, Lisbon, Portugal*, **vol.3**, 2213-2220.

[56] Kulakarni, V. G. (1995). *Modelling and analysis of stochastic systems*, Chapman and Hall, London, U.K.

[57] Levitin, G. (2005). *Universal generating function in reliability analysis and optimization*, Springer, London , UK.

[58] Levitin, G. (2007). Block diagram method for analyzing multi state systems with uncovered failures. *Reliability Engineering and System Safety*, **92(6)**, 727-734.

[59] Levitin, G. and Lisnianski, A. (1998). Structure optimization of power system with bridge topology, *Electric Power Systems Research*, **45**, 201-208.

[60] Levitin, G., Lisnianski, A. , Ben-Haim, H. and Elmakis , D. (1998). Redundancy optimization for series- parallel multi state Systems, *IEEE Transactions on Reliability*, **47(2)**, 165-172.

[61] Levitin, G., Lisnianski, A., and Elmakis, D. (1997). Structure optimization of power system with different redundant elements. *Electric Power Systems Research*, **43(1)**, 19-27.

[62] Levitin, G., Lisnianski, A. and Ben-Haim, H. (1998). Redundancy optimization for series-parallel multi state systems. *IEEE Transactions on Reliability*, **47(2)**, 165-172.

[63] Levitin, G. and Lisnianski, A. (1999). Optimal multi stage modernization of power system subject to reliability and capacity requirements, *Electrical Power Sytems Research*, **50**, 183-190.

[64] Levitin, G., Xing, L., Ben-Haim, H. and Dai, Y. (2011). Multi state systems with selective propagated failures and imperfect individual and group protections, *Reliability Engineering and System Safety*, **96**, 1657-1666.

[65] Li, C. , Chen, X. , Yi, X. and Tao, J. (2010). Heterogeneous redundancy optimization for multi state series -parallel sytem subject to common cause failures. *Reliability Engineering and system safety*, **95(3)**, 202-207.

[66] Li, W. and Zuo, M.J. .(2008). Reliability evaluation of multi state weighted k-out of n- systems. *Reliability Engineering and System Safety*, **93(1)**, 160-167.

[67] Limnios, N. and Oprisan, G. (2001). *Semi-Markov processes and reliability, Statistics for industry and technology*, Springer, UK.

[68] Lindqvist, B. H. (1987) . Monotone and associated Markov chains with application to reliability theory, *J. Applied Probability*, **24**, 679-695.

[69] Lisnianski, A. (2012). *Lz Transform for a Discrete State Continuous Time Markov Process and Its application to multi state system Reliability, Recent advances in system reliability: signatures, multi-state systems and statistical inference*, Springer, London, UK.

[70] Lisnianski, A. and Ben-Haim, H. (2013). Short term reliability for power stations by using Lz Transform , *J. Modern Power System and Clean Energy*, **1(2)**, 110-117.

[71] Lisnianski, A. and Ding, Y. (2009). Redundancy analysis for repairable multi state system by using stochastic processes methods and universal generating function technique, *Reliability Engineering and System Safety*, **94(12)**, 1788-1795.

[72] Lisnianski, A., Frenkel, I. and Ding, Y. (2010). *Multi state system reliability analysis and optimization for engineers and industrial managers*, Springer, London.

[73] Lisnianski, A. and Levitin, G. (2003). *Multistate system reliability-assessment, optimization and application*, World Scientific, Singapore.

[74] Lisnianski, A., Levitin, G. and Ben-Haim, H. (2000). Structure optimization of multi state system with time redundancy, *Reliability Engineering and System Safety*, **67**, 103-112.

[75] Lisnianski, A., Levitin, G., Ben-Haim, H. and Elmakis, D.(1996). Power system structure optimization subject to reliability constrains, *Electric Power Systems Research*, **39 (2)**, 145-152.

[76] Malefaki, S., Limnios, N. And Dersin, P. (2014). Reliability of maintained systems under semi Markov setting, *Reliability Engineering and System Safety*, **131**, 282-290.

[77] Manoharan, M., Harshinder singh and Neeraj Misra (1992). Preservation of phase-type distributions under Poisson shock models, *Advanced Applied Probability*, **24**, 92-103.

[78] Montoro-Cazorola, D., Perez-Ocon, R. and Segovia, M. C. (2007). Survival probabilities model for shock and wear models governed by phase type distributions, *Quality technology and Quantitative Management*, **4(1)**, 85-94.

[79] Montoro-Cazorola, D., Perez-Ocon, R. and Segovia, M.C. (2009). Shock and wear models under policy N using phase type distributions, *Applied Mathematical Modeling*, **33(1)**, 543-554.

[80] Murchland, J. (1975). Fundamental concepts and relations for reliability analysis of multistate systems. Reliability and Fault tree analysis. Theoretical and applied aspects of system reliability, *SIAM*, 581-618.

[81] Natvig, B. (1993). Strict and exact bounds for the availabilities in a fixed time interval for multi state monotone system. *Scandinavian Journal of Statistics*, **20**, 171-174.

[82] Natvig, B. and Streller, A . (1984). The steady state behaviour of multi state monotone systems, *J.Applied Probability*, **21**, 826-835.

[83] Natvig, B. (1982). Two suggestions of how to define a multi state coherent system, *Advanced Applied probability*, **14**, 434-455.

[84] Natvig, B. (1986). Improved upper bounds for the availabilities in a fixed time interval for multi state monotone systems. *Advanced Applied Probability*, **18**, 577-579.

[85] Neuts, M. F. (1975). Probability distributions of phase type, *Liber Amicorum Professor Emeritus H.* Florin, University of Louvain, Belgium, 173-206.

[86] Neuts, M. F. (1981). *Matrix geometric solution in stochastic models- An algorithmic approach*, Johns Hopkins University Press, Baltimore.

[87] Neuts, M. F. and Meier, K. S. (1981). On the use of phase type distributions in reliability modelling of systems with two components, *OR Spektrum*, **2 (4)**, 227-234.

[88] Neuts, M.F., Perez-Ocon, R. and Torres -Castro, I.(2000). Repairable Models with operating and repair times governed by phase type distributions, *Applied Probability*, **32**, 468-479.

[89] Nourelfath, M. And Ait-Kadi, D. (2004). A combined approach to solve the redundancy optimization problem for multi state systems under repair policies. *Reliability Engineering and System Safety*, **95 (11)**, 1255-1260.

[90] Nourelfath, M. and Ait-Kadi, D. (2007). Optimization of series-parallel multi state systems under maintenance policies. *Reliability Engineering and System Safety*, **92(12)**, 1620-1626.

[91] Perez-Ocon, R. and Ruiz - Castro, J. E. (2004). Two models for a repairable two systems with phase type sojourn time distribution, *Reliability Engineering and System Safety*, **84**, 253-260.

[92] Perman, M., Senegacnik, A. and Tuma, M. (1997). Semi-Markov models with an application to power plant reliability analysis, *IEEE Transactions on Reliability*, **46 (4)**, 526-532.

[93] Pyke, R. and Schaufele, R. (1966). The existence and uniqueness of stationary measures for Markov renewal sequences, *Annals of Mathematical Statistics*, **37**, 1439-1462.

[94] Qin, J., Niu, Y. And Li, Z. (2016). A combined method for reliability analysis of multi state system of minor repairable components, *Maintenance and Reliability*, **18(1)**, 80-88.

[95] Rausand, M. and Hoyland, A. (2004). *System reliability theory- Models, statistical methods and application*, Wiley Int., Canada.

[96] Ross, S. M. (1996). *Stochastic processes*, John Wiley.

[97] Ross, S. M. (1979). Multivariate state component systems, *Annals of Probability*, **7**, 379-383.

[98] Sahner, R., Trivedi, K.S. and Puliafito, A. (1995). *Performace and reliability analysis if computer systems: An example based approach using the SHARPE software package*, Kluwer Academic publishers, Dordrecht, The Netherlands.

[99] Segovia, M.C. and Labeau, P.E. (2013). Reliability of a multistate system subject to shocks using phase type distributions, *Applied Mathematical Modeling*, **37**, 4883-4904.

[100] Tian, Z., Levitin, G. and Zuo, M. .J. (2009). A Joint reliability redundancy optimization approach for multi state series -parallel systems. *Reliability Engineering and system safety*, **94(10)**, 1568-1576.

[101] Trivedi, K.S. and Bobbio, A. (2017). *Reliability and availability engineering, modeling, analysis and applications*, Cambridge University Press, UK.

[102] Ushakov, I. (1986). A Universal Generating Function, *Soviet Journal of Computer and System Science*, **24**, 37-49.

[103] Ushakov, I (2000). The method of generalised Generating Sequences, *European Journal of Operational Resarch*, **125**, 316-323.

[104] Vaurio, J.K. (1984). Reliability and availability equations for multi State components. *Reliability Engineering*, **7(1)**, 1-19.

[105] Vinayak, R. and Dharmaraja, S. (2012). Semi Markov modelling approach for deteriorating systems with preventive maintenance, *International Journal of performability Engineering*, **8(5)**, 515-526.

[106] Wereley, N. and Walker, B. (1988). Approximate Semi Markov chain reliability models, *in:27th IEEE Conference on Decision and Control*, **volume 3**, 2322-2329.

[107] Wu, J., Sheng Adam Ng, T. and Xie, M. (2010). Analysis of maintenance policies for finite life cycle multi state systems. *Computers and industrial Engineering*, **59(4)**, 638-646.

[108] Xie, W. (1999). *Markov regenerative process in SHARPE*, Masters Thesis, Duke University, Department of Electrical and Computer Engineering, Durham, NC, USA.

[109] Xu, D. , Sun, K. and Li, H. (2008). Parallel machine scheduling with almost periodic maintenance and non-preemptive jobs to minimize makespan. *Computers and Operations Research* , **35**, 1344-1349.

[110] Xue, J. and Yan, .K. (1995). Dynamic reliability analysis of coherent multi state Systems, *IEEE Transaction on Reliability*, **44(4)**, 683-688.

[111] Yu, H., Yang, J. And Mo, H. (2014). Reliability analysis of repairable multi state system with common bus performance sharing, *Reliability Engineering and System Safety*, **132**, 90-96.

[112] Zio, E. , Marella, M. and Podofillini, L.(2007). A Monte Carlo simulation approach to the availability assessment of multi state systems with operational dependencies, *Reliability Engineering and Sytem Safety*, **92**, 871-882.

[113] Zio, E., Podofillini, L. and Levitin, .G. (2004). Estimation of the importance measures of multi State elements by Monte Carlo simulation. *Reliability Engineering and System Safety*, **86(3)**, 191-204.

[114] Zuo, M. J. and Tian, Z. G. (2006). Performance evaluation for generalised multi State k- out of n- systems. *IEEE Transactions on Reliability*, **55(2)**, 319-327.

LIST OF PUBLISHED WORKS

(a) **Published:**

(1) Manoharan, M. and Vidhya, G. Nair (2017). Evaluation of system performance measure of multi state degraded system with minimal repair, *Reliability: Theory and Applications*, **1(44)(vol.12)**, 76-83. ISSN 1932-2321. $http://gnedenko-forum.org/Journal/2017/012017/RTA\_1\_2017-09.pdf$

(2) Vidhya, G. Nair and Manoharan, M. (2018a). Dynamic Multi State System Reliability Analysis of Power Generating Systems using Lz-transform. *ProbStat Forum*, **11**, 81-90, ISSN 0974-3235.
$http://probstat.org.in/PSF-2018-08.pdf$.

(3) Vidhya, G. Nair and Manoharan, M. (2018b). Reliability analysis of a multi state system with common cause failures using Markov Regenera-

tive Process. *Reliability: Theory and Applications*, **3(50)(vol.13)**, 82-88, ISSN 1932-2321.

(b) **Presentations in Conferences/Seminars:**

(1) Evaluation of System Performance Measures of Multi State Degraded System with Minimal Repair, *Second International Conference on Statistics for Twenty-first Century-2016 (ICSTC-2016)* organized by Department of Statistics University of Kerala, Trivandrum, Kerala, India, December 21-23, 2016.

(2) Dynamic MSS Reliability Analysis of Power Generating Systems using Lz transforms, *National Conference on Advances in Statistical Sciences* in conjunction with the Annual Conference of the Kerala Statistical Association (KSA), held in the Department of Statistical Sciences, Kannur University, Kerala, February 17-18, 2017.

(3) Reliability Analysis of Periodically Maintained Systems using Semi-Markov Process based UGF Technique, *International Conference on Statistics for Twenty-first Century-2017 (ICSTC-2017)* organized by Department of Statistics, University of Kerala, Trivandrum, Kerala, India, December 14-16, 2017.

(4) Analysis of a Multi State System with Common Cause Failures using Markov Regenerative Process, *International Conference on Theory and Applications of Statistics and Information Sciences (TASIS-2018)* con-

ducted by the Department of Statistics Bharathiar University, Coimbatore, Tamilnadu, India in conjunction with Annual Convention of ISPS and in association with IBS, January 5-7, 2018.

(5) Reliability analysis of a repairable parallel system based on phase type distribution, *National Seminar on Recent Trends in Statistics (NSRTS)* conducted by the Department of Statistics, University of Calicut, March 13-15, 2018.