Ph. D Thesis **MATHEMATICS**

A STUDY ON STRENGTH OF STRONG FUZZY GRAPHS AND EXTRA STRONG k- PATH DOMINATION IN STRONG FUZZY GRAPHS

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DEPARTMENT OF MATHEMATICS

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CERTIFICATE

I hereby certify that the thesis entitled **A STUDY ON STRENGTH OF STRONG FUZZY GRAPHS AND EXTRA STRONG k- PATH DOMINATION IN STRONG FUZZY GRAPHS** is a bonafide work carried out by **Smt. Chithra K. P.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Raji Pilakkat

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DECLARATION

I hereby declare that the thesis, entitled "**A STUDY ON STRENGTH OF STRONG FUZZY GRAPHS AND EXTRA STRONG k- PATH DOMINATION IN STRONG FUZZY GRAPHS**" is based on the original work done by me under the supervision of **Dr. Raji Pilakkat**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Contents

List of Symbols

$A(G)$	the adjacency matrix of G
C_n	the cycle of length n.
deg v	degree of a vertex v
$dS_k(v)$	the extra strong $k-$ path degree of a vertex v in a fuzzy graph G.
$dN_k(v)$	the extra strong $k-$ path neighbourhood degree of a vertex v in a fuzzy graph G.
$d_k(v, S)$	the minimum length of the extra strong paths from v to u where $v \in V \setminus S$ and $u \in S$.
$ESmk - DS(G)$	the set of all minimal extra strong k- path dominating sets of a fuzzy graph G.
$ES\,\gamma_{S_k}(G)$	fuzzy extra strong k- path lower domination number.

$ES\,\Gamma_{S_k}(G)$	fuzzy extra strong k- path upper domination number.
$ES\ PN_k[u,S]$	fuzzy extra strong k- path private neighbour.
$G(V,\mu,\sigma)$	a fuzzy graph.
$G_1 < G_2 < \ldots < G_m$	a sequence of m n-linked fuzzy graphs.
$G_1 \square G_2$	the merartesian product of two fuzzy graphs $G_1$and G_2.
$G_1 \otimes G_2$	the tensor product of two fuzzy graphs $G_1$and G_2.
$G_1 \odot G_2$	the corona of two fuzzy graphs $G_1$and G_2.
$G_1 \vee G_2$	the join of two fuzzy graphs $G_1$and G_2.
$G_1[G_2]$	the composition of two fuzzy graphs $G_1$and G_2.
$G_1 \boxtimes G_2$	the normal product of two fuzzy graphs $G_1$and G_2.
$H(V,\mu,\sigma)$	the partial fuzzy subgraph of a fuzzy graph.
$L(G)$	the line graph of a fuzzy graph $G(V,\mu,\sigma)$.
$M(G)(V_M,\mu_M,\sigma_M)$	the fuzzy middle graph a fuzzy graph $G(V,\mu,\sigma)$
P_n	a Path of length n.
$S(G)(V_s,\mu_s,\sigma_s)$	the shadow graph of a fuzzy graph $G(V,\mu,\sigma)$.

$split(G)(V_{split}, \mu_{split}, \sigma_{split})$	the split graph of a fuzzy graph $G(V, \mu, \sigma)$
$sd(G)(V_{sd}, \mu_{sd}, \sigma_{sd})$	the subdivision graph of a fuzzy graph $G(V, \mu, \sigma)$
$T(G)(V_T, \mu_T, \sigma_T)$	the total graph of a fuzzy graph $G(V, \mu, \sigma)$.
$V(G)$	the vertex set of the graph G.
W_n	the fuzzy wheel graph.
$\delta_{S_k}(G)$	minimum of extra strong $k-$ path degrees of vertices of a fuzzy graph G.
Δ_{S_k}	maximum of extra strong $k-$ path degrees of vertices of a fuzzy graph G.
$\delta_{N_k}(G)$	the minimum extra strong $k-$ path neighbourhood degree of a fuzzy graph.
Δ_{N_k}	the maximum extra strong $k-$ path neighbourhood degree of a fuzzy graph.

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Chapter 0

Introduction

Graph theory has tremendous applications in many real life problems and many areas of science such as chemistry, computer networks, computational neuro - science, condensed matter physics etc. By using the principles of graph theory many problems in the field of economics, linguistics, artificial intelligence, pattern recognition, network topologies etc. can be modeled and analysed.

Graphs do not model all the systems properly due to the uncertainty or haziness of the parameters of systems. For example, a social network may be represented as a graph where vertices represent accounts (persons, institutions, etc.) and edges represent the relation between the accounts. If the relations among accounts are to be measured as good or bad according to the frequency of contacts among the accounts. This and many other problems motivated to define fuzzy graphs. Azriel Rosenfeld was first introduced the concept of fuzzy graphs. Crisp graph and fuzzy graph are structurally similar. But fuzzy graph has a separate importance, when there is an uncertainty on vertices and/or edges comes.

M-strong fuzzy graphs [4] were introduced by Bhutani and Battou. Bhutani and Rosenfeld consider strong arcs in fuzzy graphs [5] for their work. Mathew and Sunitha have introduced different types of arcs in fuzzy graphs and studied their properties [30].

Ore and Berge studied the domination set in graphs. Due to the diversity of applications of domination theory to real situation or location problem, the research in this field grows rapidly. Domination in Fuzzy graphs is discussed by A. Somasundram and S. Somasundram through their paper Domination in fuzzy graphs -1 [50].

In this thesis we consider strong fuzzy graphs which were introduced by Mordeson J. N. and Peng [34]. Sheeba M. B. [48] defined the strength of fuzzy graphs which are connected. We extend this definition to arbitrary fuzzy graphs as the maximum of strength of all connected components of a fuzzy graph. Also, in our work we have made an attempt to introduce the concept of extra strong $k-$ path domination in strong fuzzy graphs.

Outline of the Thesis

Apart from this introductory chapter, we have presented our work in six chapters.

In *Chapter 1*, we describe the basic concepts, facts, elementary results and some of the operations of crisp graphs and fuzzy graphs. In this chapter we familiarise the concept of strength of fuzzy graphs and give some theorems that explain the strength of certain fuzzy graphs such as fuzzy path, fuzzy cycle etc. which are needed in the subsequent discussion.

In *Chapter 2*, we first derive an algorithm for finding the strength of a fuzzy path in a fuzzy graph $G(V, \mu, \sigma)$ and then the length of the path joining two vertices with minimum length and maximum strength. All of these algorithms are illustrated through examples. Apart from this, we define properly linked fuzzy graphs and derive strength of such graphs when each part of it is a strong fuzzy complete graph. Also in this chapter we find the strength of a strong fuzzy complete bipartite graph, strong fuzzy diamond graph, strong fuzzy butterfly graph and strong fuzzy bull graph.

In *Chapter 3*, we discuss join of some strong fuzzy graphs, corona of some strong fuzzy graphs, subdivision graph, middle graph, total graph, split graph and shadow graph of some strong fuzzy graphs. There are five sections in this chapter. In the first section, 'strength of join of fuzzy graphs' we find the strength of join of (1) two complete fuzzy graphs, (2) two fuzzy fan graphs, (3) two fuzzy star graphs, (4) two strong fuzzy paths, (5) strong fuzzy wheel graph which is the join of a fuzzy cycle and a fuzzy trivial graph. In the next section 'Corona of strong fuzzy graphs' we find the strength of corona of (1) a fuzzy trivial graph and a strong fuzzy graph which is not a fuzzy null graph, (2) two fuzzy null graphs $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ with $|U| = 1$ and $|V| > 1$, (3) two strong fuzzy paths, (4) two strong fuzzy butterfly graphs. In the section 'Fuzzy subdivision graph of strong fuzzy graphs' we find the strength of subdivision graph of (1) strong fuzzy path, (2) strong fuzzy butterfly graph, (3) strong fuzzy Bull graph, (4) strong fuzzy star graph, (5) strong fuzzy diamond graph and (6) fuzzy complete graph. In the next section 'Fuzzy middle graph' we find strength of middle graph of (1)complete fuzzy graph, (2)strong fuzzy star graph and (3) strong fuzzy diamond graph. In the next section 'total fuzzy graph' we find the strength of total graph of fuzzy null graph and a fuzzy complete graph. In the next section 'Fuzzy split graph' we find the strength of split graph of a strong fuzzy path and a fuzzy complete graph. In the next section 'Fuzzy shadow graph' we find the strength of a strong fuzzy path and a fuzzy complete graph.

In *Chapter 4*, the strengths of Cartesian product, tensor product, composition and normal product of certain strong fuzzy graphs are determined.

First of all, we prove the Cartesian product of two fuzzy paths, each has P_2 as its underlying crisp graph is a fuzzy cycle and its strength is 2. Also we find the strength of Cartesian product of two fuzzy paths with respective crisp graphs P_m and P_n for all values of m and n and strength of Cartesian product of two strong fuzzy graphs with underlying crisp graphs P_2 and C_n. We define the fuzzy book, and fuzzy pages and find the strength of fuzzy book. The strength of Cartesian product of a strong fuzzy path on 2 vertices and a strong fuzzy butterfly graph is also find.

In the next section, we determine the strength of tensor product of a strong fuzzy path on two vertices and a strong fuzzy path on n vertices. Also we find the strength of tensor product of a strong fuzzy path on two vertices and a fuzzy star graph, a strong fuzzy cycle. The strength of tensor product of two fuzzy complete graphs is also find here.

The third section discusses the strength of composition of strong fuzzy paths P_m and P_n for all values of m and n and prove that the strength of composition of two strong fuzzy paths on 2 and n vertices is not equal to that of the strength of composition of two strong fuzzy paths on n and 2 vertices respectively. We derive the strength of composition of a strong fuzzy path on two vertices and a strong fuzzy star graph and that of strong fuzzy Bull graph, and a strong fuzzy cycle.

The fourth section deals with the normal product of some strong fuzzy graphs and determine the strength of normal product of two strong fuzzy graphs with their respective underlying crisp graphs, (1) the paths P_2 and $P_n, n > 1$, (2) the complete graphs K_n and K_m, (3) the paths P_2 and the star graph S_n, (4) the star graphs S_m and S_n. This section also introduces a new concept called fuzzy merger graph. Using this concept, we derive the strength of normal product of a strong fuzzy path on two vertices and a strong fuzzy butterfly graph is 2.

In *Chapter 5* we find the strengths of line graphs of some strong fuzzy graphs which include strong fuzzy butterfly graph, strong fuzzy star graph, strong fuzzy bull graph and strong fuzzy diamond graph.

A path P in a fuzzy graph $G(V, \mu, \sigma)$ with all its edges have weight equal to w where $w = \min\ \{\sigma(uv) : \sigma(uv) > 0 \text{ in } G\}$ is called a weakest path. A weakest path which is not a proper subpath of any other weakest path in the fuzzy graph G is called a maximal weakest path in G. We find strength of line graph a strong fuzzy path and strength of line graph of strong fuzzy cycle.

Chapter 6 introduces extra strong $k-$ path domination in a strong fuzzy graph $G(V, \mu, \sigma)$, fuzzy extra strong $k-$ path neighbour of a vertex, for a subset X of V, the open and closed extra strong $k-$ path neighbourhood of X, fuzzy extra strong $k-$ path isolated vertex, fuzzy extra strong $k-$ path neighbourhood degree, minimal and maximal fuzzy extra strong $k-$ path dominating set and fuzzy extra strong $k-$ path domination number. Also we give an algorithm for finding an extra strong $k-$ path minimal dominating set of a fuzzy graph and find extra strong $k-$ path domination number of certain strong fuzzy graphs.

Fuzzy extra strong $k-$ path private neighbour, fuzzy extra strong $k-$ path independent set and fuzzy extra strong $k-$ path minimal (and maximal) redundant and irredundant set are introduced and discussed with some of its properties.

Chapter 1

Preliminaries

Graph theory is most accepted because of its tremendous applications in various fields of Mathematics and other subjects. The publications of last thirty years show that Graph Theory is the fastest growing area among all the subjects in all disciplines. Many problems can be described by using a mathematical structure consisting of a set of points together with lines joining certain pair of points; such a diagram is termed as a graph [7].

The purpose of this chapter is to list the terminology and notation that we shall use in this work. Much of the terms used are standard graph theoretic terminology, a few terms will be introduced later when their turn comes.

A (undirected) graph [39] $G(V(G), E(G))$ consists of a nonempty set $V(G)$ and a collection $E(G)$ of unordered pair of elements of $V(G)$. If there is no ambiguity we simply write $G(V, E)$ or just G instead of $G(V(G), E(G))$ and if $e = (u, v)$, where $e \in E$ and $u, v \in V$, we simply write $e = uv$. An element, indicated by a point, of V is called a vertex [7]. An element, a line joining the points representing ends, of E is called an edge [7], V is the vertex set and E is the edge set of G [39].

1.1 Basics of Graph Theory

Let $G(V, E)$ be the given graph. The order [7] of G is the number of vertices of G and the size [7] of G is the number of edges of G. The vertices u and v are said to be adjacent if $e = uv$ is an edge of G and the edge e is said to incident with (incident to or incident at) u and v. The end vertices of the edge e [7] are u and v. Then the vertex v is called a neighbour of u. The set of all neighbours of the vertex u in a graph G is denoted by $N(u)$ [7]. Adjacent [7] edges have a common vertex. An edge with identical ends is called a loop [7] and an edge with distinct ends is called a link [7]. Two or more links with the same pair of ends are said to be parallel edges or multiple edges and graph having multiple edges is a multigraph [7]. A graph having a set of vertices connected by edges, where the edges have a direction associated with them is a directed graph (digraph) [7]. If edges have no orientation in a graph then that graph is an undirected graph. A simple graph is an undirected graph having no multiple edges and loops [7]. A graph $G(V, E)$ is finite [7] if both V and E are finite. A graph with a single vertex is called a trivial graph [7] and other graphs are nontrivial.

Through out the thesis, we consider only finite, simple, undirected graphs.

The degree [20] of a vertex v in the graph G is the number of edges incident to v and is denoted by $deg\ v$. A vertex of degree one is called an end vertex or a pendant vertex [32] and a vertex adjacent to a pendant vertex is called a support vertex [7]. A pendant edge is the edge incident with a pendant vertex. A vertex v is isolated [7] if $deg\ v = 0$. By an empty graph [7] we mean a graph with no edges. The minimum degree of vertices in G is denoted by $\delta(G)$ and maximum degree of vertices in G by $\Delta(G)$ [7]. If both $\delta(G)$ and $\Delta(G)$ is equal to r then G is said to be $r-$ regular or regular of degree r [7]. A simple graph G is said to be complete [40] if every pair of distinct vertices of G are adjacent in G. By K_n we mean a complete graph on n vertices. If the vertex set of a graph can be partitioned into two subsets, X and Y so that every edge has one end in X and other end in Y is called a bipartite graph ; such a partition (X, Y) is called a bipartition of the bipartite graph [7]. A simple bipartite graph is complete [7] if each vertex of X is adjacent to all vertices of Y. A complete bipartite graph with $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. When $m = 1$, K_{mn} is called a star graph [1]. The Wagner graph is the graph which is formed by adding to an octagon four edges joining its diagonally opposite pairs of vertices [27]. A planar undirected graph with 4 vertices and 5 edges is called a diamond graph [52]. It consists of a complete graph K_4 minus one edge(http:// en.m.wikepedia.org).

Let G be a simple graph of order n, where $V(G) = \{u_1, u_2, \ldots, u_n\}$. The

$n \times n$ zero-one matrix $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } u_i u_j \in E(G), \\ 0 & \text{if } u_i u_j \notin E(G). \end{cases}$$

is the adjacency matrix [9] of G

Note that A is a symmetric matrix, i.e, row i of A is identical to column i of A for every integer i with $1 \leq i \leq n$. It is observed that $\sum\limits_{j=1}^{n} a_{ij} = \sum\limits_{k=1}^{n} a_{kj} = deg(v_i)$.

Let G be a simple graph [58] of order n, where $V(G) = \{u_i : i = 1, 2, \ldots, n\}$ and $E(G) = \{e_j : j = 1, 2, \ldots, n\}$. The $n \times m$ matrix $M(G) = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 0 & \text{if } u_i \text{ is not an end of } e_j, \\ 1 & \text{if } u_i \text{ is an end of the non-loop } e_j, \\ 2 & \text{if } i \text{ is an end of the loop } e_j. \end{cases}$$

is the incidence matrix [32] of G. It is observed that $\sum\limits_{j=1}^{n} m_{ij} = deg(u_i)$ and $\sum\limits_{i=1}^{m} m_{ij} = 2$.

A walk [7] in a graph G is an alternating sequence of vertices and edges, such as $W = u_0 e_1 u_1 e_2 \ldots e_n u_n$, beginning and ending with vertices in which $e_i = u_{i-1} u_i$; u_0 is the origin and u_n is the terminus of W. The walk W is said to join u_0 and u_n; it is also referred to as a $u_0 - u_n$ walk. The length [7] of a walk is the number of edges in it. A walk is called a trail [7] if all the edges appearing in the walk are distinct. It is called a path [7] if all its vertices are distinct. Thus a path in G is automatically a trail in G. When writing a path, we usually omit the edges. A cycle [7] is a closed trail in which all the vertices are distinct. A cycle of length n is denoted by C_n and a path with n vertices is denoted by P_n. Note that P_n has length $(n-1)$ [11]. A butterfly graph is constructed by joining two cycles C_3 with a common vertex [13]. A bull graph consists of a triangle with two pendent edges at two distinct vertices of the triangle [17].

If there exist at least one path joining any two vertices of a graph G then it is said to be connected [22]. Otherwise, it is a disconnected graph [17]. For any two vertices u_i and u_j connected by a path in a graph G, the distance [11] between u_i and u_j, denoted by $d(u_i, u_j)$, is the length of a shortest $u_i - u_j$ path.

A graph K is called a subgraph [7] of G if $V(K) \subseteq V(G)$, and $E(K) \subseteq E(G)$. In this case G is a supergraph of K. Given any two graphs G and K, K is an induced subgraph [9] of G if $V(K) \subseteq V(G)$, only adjacent vertices in K are adjacent in G. In this case if $V(K) = S$, we write $K = G[S]$ or $K =< S >$. A subgraph K of G is a spanning subgraph [14] of G, if $V(K) = V(G)$. A maximal complete subgraph of a graph is a clique [6] of the graph. That is if Q is a clique in G, then no subgraph of G which contains Q properly is complete.

If e is an edge of a graph G, then $G - e$ is the graph in which it is obtained from G by deleting the edge e [59]. More generally, if F is any set of edges in G, then $G - F$ is the graph obtained from G by deleting all the edges in F [59]. Similarly, if u is a vertex of a graph G, then the graph obtained from G by deleting the vertex u and all edges incident with it is denoted by $G - u$ [59].

More generally, if S is any set of vertices in $G, G - S$ is the graph obtained from G by deleting all the vertices in S, and all edges incident with at least one of the vertices of S [59].

A component [22] of a graph G is a connected subgraph not properly contained in any other connected subgraph. A vertex v in a connected graph G is a cut vertex [9] if $G - v$ is disconnected. A connected graph that has no cut vertices is called a block [7]. A block of G containing exactly one cut vertex of G is called an end-block [9] of G.

1.2 Operations on graphs

This section deals with some of the operations on graphs that are used in subsequent chapters. $G_1 \cup G_2$ is the union [58] of two graphs G_1 and G_2 with vertex set is the union of $V(G_1)$ and $V(G_2)$ and edge set is the union of $E(G_1)$ and $E(G_2)$. The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with vertex set same as that of $G_1 \cup G_2$ and edge set $E(G_1) \cup E(G_2) \cup \{u_i u_j : u_i \in V(G_1)$ and $u_j \in V(G_2)\}$ is called the join [18] of graphs G_1 and G_2. The corona [18] of two graphs G_1 and G_2 is the graph $G = G_1 \odot G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where i^{th} vertex of the copy of G_1 is adjacent to every vertex in i^{th} copy of G_2. The middle graph [32] of the graph G is the graph $M(G) = (V(G) \cup E(G), E'(G))$, where $uv \in E'$ if and only if either u is a vertex of G and v is an edge containing u, or u and v are edges having a vertex in common.

The line graph [58] $L(G)$ of a graph G, is the graph with vertex set is the edge set of $G, E(G)$ and edge set is $\{ef : e, f \in E(G)$ and e, f have a vertex in common$\}$. The Cartesian product [26] $G = G_1 \Box G_2$ of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is the graph G whose vertex set $V_1 \times V_2$. Let (u_1, v_1) and (u_2, v_2) be two vertices of G. They are adjacent in $G_1 \Box G_2$, if and only if $u_1u_2 \in E_1$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E_2$. The tensor product (or direct product) [8] $G = G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph G whose vertex set is $V(G_1) \times V(G_2)$. Let (u_1, v_1) and (u_2, v_2) be two vertices being adjacent in $G_1 \otimes G_2$, if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$. The strong (or normal) product [40] $G_1 \boxtimes G_2$ of two simple graphs G_1 and G_2 is the graph with $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$, where (u_1, v_1) and (u_2, v_2) are adjacent in $G1 \boxtimes G_2$ if either

1. $u_1 = u_2$ and v_1 is adjacent to v_2, or
2. u_1 is adjacent to u_2 and $v_1 = v_2$, or
3. u_1 is adjacent to u_2 and v_1 is adjacent to v_2.

The composition (lexico graphic product) [47] $G_1[G_2]$ of two graphs G_1 and G_2, is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1[G_2]$, whenever $u_1u_2 \in E(G_1)$, or $u_1 = u_2$ and $v_1v_2 \in E(G_2)$.

Through out this thesis we consider the product of two graphs with disjoint vertex sets.

1.3 Fuzzy Relations

In this section we give some definitions in fuzzy set theory. A classical crisp set is normally defined as a collection X of objects that can be finite, countable, or uncountable.

A fuzzy subset [25] of a set X is a function $\mu : X \longrightarrow [0,1]$, where $[0,1]$ denotes the set $\{t \in \mathcal{R} : 0 \leq t \leq 1\}$ [60]. Let μ be a fuzzy subset of X then the support of μ, $Supp(\mu) = \{x \in X : \mu(x) > 0\}$ [35]. Let μ, ν be two fuzzy subsets of X. Then

1. $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x), \forall x \in X$.
2. $\mu \subset \nu$ if $\mu(x) \leq \nu(x), \forall x \in X$ and there exists at least one $x \in X$ such that $\mu(x) < \nu(x)$.
3. $\mu = \nu$ if $\mu(x) = \nu(x)$, for all $x \in X$.

Let X and Y be any two subsets and μ, ν be fuzzy subsets of X and Y respectively. Then a fuzzy relation σ from the fuzzy subset μ into the fuzzy subset ν is a fuzzy subset σ of $X \times Y$ such that $\sigma(uv) \leq \mu(x) \wedge \nu(x)$ for all $u \in X$ and $v \in Y$. Also let $\sigma : X \times Y \longrightarrow [0,1]$ be a fuzzy relation from a fuzzy subset μ of X into a fuzzy subset ν of Y and $\rho : Y \times Z \longrightarrow [0,1]$ be a fuzzy relation from a fuzzy subset ν of Y into a fuzzy subset η of Z. Define $\sigma \circ \rho : X \times Z \longrightarrow [0,1]$ by $\sigma \circ \rho(x,z) = \vee\{\sigma(x,y) \wedge \rho(y,z) \,|\, y \in Y\}$ for all $x \in X, z \in Z$. Then $\sigma \circ \rho$ is called the composition of σ with ρ [35].

Note that $\sigma \circ \rho$ is a fuzzy relation from a fuzzy subset μ of X into a fuzzy subset η of Z. The composition operation, $\sigma \circ \rho$ can be computed similar to matrix multiplication, where the addition and multiplication are replaced by $\vee$ and $\wedge$ respectively. Composition being associative, we use the notation σ^2 to denote the composition $\sigma \circ \sigma$, σ^k to denote $\sigma^{k-1} \circ \sigma$, $k > 1$. Define $\sigma^\infty(x, y) = \vee\{\sigma^k(x, y)|k = 1, 2, \ldots\}$ [35].

1.4 Fuzzy graphs

A fuzzy graph [35] $G(V, \mu, \sigma)$ is a non empty set V together with a pair of functions $\mu : V \longrightarrow [0, 1]$ and $\sigma : V \times V \longrightarrow [0, 1]$ such that for all u, v in V, $\sigma(u, v) \leq \mu(u) \wedge \mu(v)$. We call μ the fuzzy vertex set of G and σ the fuzzy edge set of G, respectively.The fuzzy graph $K(V, \nu, \tau)$ is called a partial fuzzy subgraph [43] of $G(V, \mu, \sigma)$ if $\nu \subset \mu$ and $\tau \subset \sigma$. Similarly, the fuzzy graph $K(U, \nu, \tau)$ is called a fuzzy subgraph [16] of $G(V, \mu, \sigma)$ induced by U if $U \subset V, \nu(u) = \mu(u)$ for all $u \in U$ and $\tau(u, v) = \sigma(u, v)$ for all $u, v \in U$. A vertex u of a fuzzy graph $G(V, \mu, \sigma)$ is said to be isolated vertex [50] if $\sigma(u, v) < \mu(u) \wedge \mu(v)$ for all $v \in V \setminus \{u\}$. Through out this Thesis the edge between two vertices u and v in a fuzzy graph is denoted by uv rather than (u, v).

The fuzzy graph [23] $G(V, \mu, \sigma)$ with $\sigma(u, v) = 0$ for all $u, v \in V$ is called a fuzzy null graph. A fuzzy trivial graph [30] is a fuzzy null graph on a single vertex.

The underlying crisp graph of a fuzzy graph $G(V, \mu, \sigma)$ is denoted by $G(V, E)$. A sequence of distinct vertices $P = u_0, u_1, u_2, \ldots, u_n$ such that $\sigma(u_{i-1}u_i) > 0$, $1 \leq i \leq n$ is called a path [42] P in a fuzzy graph $G(V, \mu, \sigma)$. Here length of the path P is $n \geq 1$. The consecutive pairs (u_{i-1}, u_i) are called edges of the fuzzy path. The strength of P [5] is defined as $\bigwedge_{i=1}^{n} \sigma(u_{i-1}u_i)$. That is weight of the weakest edge of the fuzzy path P is called the strength of P. A single vertex u may also be considered as a fuzzy path. In this case the fuzzy path is of length 0, and its strength is defined to be $\mu(u)$. A partial fuzzy subgraph $H(V, \mu, \sigma)$ is said to be connected [51] if $\sigma^{\infty}(uv) = \vee\{\sigma^k(v_{i-1}v_i) : k = 1, 2, \ldots, n\} > 0$ where $\mu(u) > 0$, and $\mu(v) > 0$ $\forall u, v \in V$.

A fuzzy cycle is the one in which its underlying crisp graph is a cycle and there exist more than one edge uv such that $\sigma(uv) = \wedge\{\sigma(u_iu_j) : \sigma(u_iu_j) > 0\}$. Maximal connected partial fuzzy subgraphs are called components [31]. In fact, u and v are connected if, and only if, $\sigma^{\infty}(uv) > 0$. A fuzzy graph G is connected [51] if, and only if, $\sigma^{\infty}(uv) > 0$ for all $u, v \in V$.

A fuzzy graph G is a forest if the underlying crisp graph is a forest and a tree if the underlying crisp graph is connected forest. A fuzzy graph $G(V, \mu, \sigma)$ is called a complete fuzzy graph [3] if $\sigma(uv) = \mu(u) \wedge \mu(v)$, for all $u, v \in V$. A fuzzy graph $G(V, \mu, \sigma)$ is said to be a strong fuzzy graph if $\sigma(uv) = \mu(u) \wedge \mu(v)$, for all $uv \in E$, the edge set of G, the crisp graph which we call the edge set of G itself. A fuzzy graph $G(V, \mu, \sigma)$ is regular if, and only if (i) its underlying crisp graph is an odd cycle and σ is a constant function, (ii) its underlying crisp graph is an even cycle and either σ is a constant function or alternate edges have same weights [49]. A fuzzy star graph [57] $G(\mu, \sigma)$ consists of two vertex sets V and U with $|V| = 1$ and $|U| > 1$, such that for $v \in V$ and $u_i \in U$, $\sigma(v, u_i) > 0$ and $\sigma(u_i, u_{i+1}) = 0, 1 \leq i \leq n$.

Let u and v be two distinct vertices of $G(V, \mu, \sigma)$, a fuzzy graph with underlying crisp graph $G(V, E)$. Let its order and size be n and m respectively. If there exists at least one path between u and v of length less than or equal to k then the connectedness of strength k between u and v [49] is defined as the maximum of the strength of all paths between them of length less than or equal to k. Otherwise it is defined as zero. The $n \times n$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} \sigma(v_i v_j) & \text{if } i \neq j, \\ \mu(v_i) & \text{if } i = j. \end{cases}$$

is called the weight matrix of G.

An example of a connected fuzzy graph is depicted as in Figure 1.1. The connectedness of strength 2 between the vertices v_1 and v_4 is 0.2. The connectedness of strength 3 between the vertices v_1 and v_4 is 0.5.

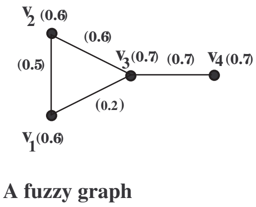


Figure 1.1: A fuzzy graph G.

The weight matrix A of the fuzzy graph in Figure 1.1 is

$$A = \begin{bmatrix} 0.6 & 0.5 & 0.2 & 0.0 \\ 0.5 & 0.6 & 0.6 & 0.0 \\ 0.2 & 0.6 & 0.7 & 0.7 \\ 0.0 & 0.0 & 0.7 & 0.7 \end{bmatrix}$$

Also let A be an $n \times n$ weight matrix of the fuzzy graph G. For all $i \geq n$, the least positive integer n such that $A^n = A^i$ is called the strength of G. The strength of the fuzzy graph G in Figure 1.1 is 3.

A path $P = v_i v_{i+1} \ldots v_j$ of a fuzzy graph $G(V, \mu, \sigma)$ is said to connect the vertices v_i and v_j of G strongly [48] if its strength is maximum among all the paths between v_i and v_j. Such paths are called strong paths. Any strong path between two distinct vertices v_i and v_j in G with minimum length is called an extra strong path [48] between them. The maximum length of extra strong paths between every pair of distinct vertices in G is called the strength of connectivity of the graph G [48]. Strength of connectivity of the graph G is proved to be the same as strength of G.

Theorem 1.4.1. [48] *The strength of*

(i) a strong fuzzy path on n vertices is its length $(n-1)$.

(ii) a complete fuzzy graph is one.

(iii) a regular fuzzy graph on n vertices is $[\frac{n}{2}]$.

(iv) a fuzzy star graph is 2.

Theorem 1.4.2. *[48] The strength of a fuzzy cycle G with underlying crisp graph a cycle on n vertices and l weakest edges, which altogether form a subpath in G is $n-l$ if $l \leq [\frac{n+1}{2}]$ and $[\frac{n}{2}]$ if $l > [\frac{n+1}{2}]$.*

Theorem 1.4.3. [48] *Let G be a fuzzy cycle with underlying crisp graph a cycle of length n, having l weakest edges which do not altogether form a subpath. If $l > [\frac{n}{2}]-1$ then the strength of the graph is $[\frac{n}{2}]$ and if $l = [\frac{n}{2}]-1$ then the strength of the graph is $[\frac{n+1}{2}]$.*

Theorem 1.4.4. [48] *In a fuzzy cycle of length n suppose there are $l < [\frac{n}{2}]-1$ weakest edges which do not altogether form a subpath. Let s denote the maximum length of the subpath which does not contain any weakest edge. If $s \leq [\frac{n}{2}]$ then the strength of the graph is $[\frac{n}{2}]$ and if $s > [\frac{n}{2}]$ then the strength of the graph is s.*

Chapter 2

Strength of certain fuzzy graphs

In this chapter we derive an algorithm for finding the strength of fuzzy graphs. Strength of strong fuzzy complete bipartite graph have been determined. A new concept named properly linked fuzzy graphs is introduced in this chapter. Also a fuzzy merger graph is defined and find a relation connecting fuzzy merger graph and a $1-$ linked fuzzy graph. Further strength of such graphs are determined. Also strength of strong fuzzy Wagner graph has been determined.

Some results of this chapter are included in the following paper Chithra K. P., Raji Pilakkat, International Journal of Pure and Applied Mathematics, 106(3) 2016, 883-892

2.1 Algorithm for finding strength of fuzzy graphs

In this section we consider only those fuzzy graphs whose underline graphs are connected. Graph theory and graph algorithms are inseparably interwined subjects. Bhattacharya and Suraweera [2] have given an algorithm for finding $\sigma^{\infty}(u, v)$ using maximum spanning tree algorithm of fuzzy graphs. Here we give an algorithm for finding the strength of a fuzzy graph in a direct method. The concept of strength of connectivity between two vertices of a fuzzy graph introduced by Bhattacharya and Suraweera [2] was further studied by Sheeba M. B. [48] by introducing two new terminologies extra strong paths and strength of fuzzy graphs. Though theoretical approach is the strong clear cut method, it is some times difficult and tedious to find the strength for arbitrary graphs. So we tried for an algorithmic approach to find the strength of the fuzzy graphs and have succeeded. This section discusses an algorithmic approach to find the strength of fuzzy graphs.

Algorithm 2.1.1. Algorithm for finding the strength of a path in a fuzzy graph G.

Let G be a fuzzy graph and v_i, v_j be two vertices of G. Let $P = x_1x_2 \dots x_n$, where $v_i = x_1$ and $v_j = x_n$ be a $v_i - v_j$ path with $\sigma_k = \sigma(x_kx_{k+1}), k = 1, 2, \dots, (n-1)$. Then the minimum value of σ_k, for $k = 1, 2, \dots, (n-1)$ is the strength of P.

Input : $\sigma_1 = \sigma(x_1x_2)$.

Step 1. For $k = 2$, find σ_k and compare σ_1 and σ_k.

If $\sigma_k > \sigma_1$ then ignore the value of σ_k.

If $\sigma_k \leq \sigma_1$ then $\sigma_1 = \sigma_k$.

Step 2. Repeat Step 1 for $k = 3, 4, \ldots, (n-1)$.

Output : The strength of $P = \sigma_1$.

Illustration:

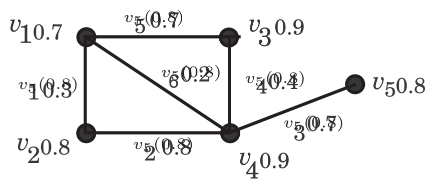


Figure 2.1: Fuzzy graph G.

Consider the fuzzy graph G in Figure 2.1. There are 3 paths joining v_1 and v_5; $v_1v_4v_5$, $v_1v_3v_4v_5$ and $v_1v_2v_4v_5$ in G. Let $P_1 = v_1v_2v_4v_5$. Then $\sigma_1 = \sigma(v_1v_2) = 0.3, \sigma_2 = \sigma(v_2v_4) = 0.8, \sigma_3 = \sigma(v_4v_5) = 0.7$. Here $\sigma_2 > \sigma_1$. Therefore ignore σ_2. Since $\sigma_3 > \sigma_1$ ignore σ_3. So $\sigma_1 = 0.3$ is the strength of P_1.

In a similar manner we can find the strength of the path of $v_1v_4v_5 = 0.2$ and that of $v_1v_3v_4v_5$ is equal to 0.4.

Algorithm 2.1.2. Algorithm for finding $k_{v_iv_j}$, the length of the path joining two vertices v_i and v_j with minimum length and with maximum strength.

Let G be a fuzzy graph with underlying crisp graph $G(V, E)$ having vertex set $\{v_1, v_2, \ldots, v_n\}$. For the vertices v_i, v_j, $k_{v_i v_j}$ denotes the minimum length of all the $v_i - v_j$ paths having maximum strength.

Input : All the paths joining v_i and v_j in G.

Step 1. Name the paths between v_i and v_j as $P_1, P_2, \ldots, P_n$.

Step 2. Find the strength S_1 of P_1 using Algorithm 2.1.1.

Step 3. For $k = 2$, find the strength S_k of the k^{th} path, P_k by Algorithm 2.1.1 and compare it with S_1.

If $S_1 < S_k$ then rename P_k by P_1.

If $S_1 > S_k$ then ignore the path P_k and repeat the step with P_{k+1} instead of P_k.

If $S_1 = S_k$ then rename P_k by P_1 if length of $P_k <$ length of P_1. Otherwise ignore P_k.

Step 4. Repeat Step(3) with $k = k+1, k+2, \ldots, n$ to get the path P_1 with minimum length and with maximum strength between v_i and v_j.

Step 5. The length of the path P_1 is $k_{v_i v_j}$.

Illustration:

For the fuzzy graph G in Figure 2.1, name the paths $v_1v_2v_4v_5$, $v_1v_4v_5$, $v_1v_3v_4v_5$ as P_1, P_2 and P_3 respectively. By Algorithm(1), $S_1 =$ strength of $P_1 = 0.3$, $S_2 =$ strength of $P_2 = 0.2$ and $S_3 =$ strength of $P_3 = 0.4$. Here $S_1 > S_2$ so ignore the path P_2 and compare the strength of P_1 and P_3. Since $S_3 > S_1$, ignore the path P_1. Then the length of $P_3(= 3)$ is $k_{v_1 v_5}$.

By the same algorithm we have, $k_{v_1v_2} = 3, k_{v_1v_3} = 1, k_{v_1v_4} = 2, k_{v_2v_3} = 2, k_{v_2v_4} = 1, k_{v_2v_5} = 2, k_{v_3v_4} = 1, k_{v_3v_5} = 2$, and $k_{v_4v_5} = 1$.

Algorithm 2.1.3. Algorithm for finding the strength of a fuzzy graph.

Let G be a fuzzy graph with underlying crisp graph G^* having vertex set $\{v_1, v_2, \ldots, v_n\}$. For $m \geq 1, 1 \leq i < i+m \leq n$, let $k_{v_iv_{i+m}}$ denote the minimum length of all the paths joining v_i and v_{i+m} having maximum strength. Let $m_{v_1v_2} = k_{v_1v_2}$ and define for $1 \leq i < i+1 < \ldots < i+m \leq n$, $m_{v_iv_{i+1}\ldots v_{i+m}}$ recursively as $m_{v_iv_{i+1}\ldots v_{i+m}} = max\{m_{v_iv_{i+1}\ldots v_{i+m-1}}, k_{v_iv_{i+m}}\}$ for $m \geq 2$ and $m_{v_{i+m}v_{i+m+1}} = max\{m_{v_{i+m-1}v_{i+m}\ldots v_n}, k_{v_{i+m}v_{i+m+1}}\}$, for $m = 1$. Then strength of G is $m_{v_{n-1}v_n}$.

Input : A fuzzy graph G with vertex set $\{v_1, v_2, \ldots v_n\}$.

Step 1. Choose the vertices v_1 and v_2 and find $k_{v_1v_2} = m_{v_1v_2}$.

Step 2. For $i = 1$, $m = 2$ find $k_{v_iv_{i+m}}$ and $m_{v_iv_{i+1}\ldots v_{i+m}} =$
$max\{m_{v_iv_{i+1}\ldots v_{i+m-1}}, k_{v_iv_{i+m}}\}$.
Do the same for $i = 1, m = 3, 4, \ldots (n-i)$ successively and find $m_{v_iv_{i+1}\ldots v_{i+m}}$.

Step 3. For $i = 1, m = 1$ find $k_{v_{i+m}v_{i+m+1}}$ and $m_{v_{i+m}v_{i+m+1}} = max\{m_{v_{i+m-1}v_{i+m}v_n}, k_{v_{i+m}v_{i+m+1}}\}$.

Step 4. For the value $i = 2$ perform Step(2) for $m = 2, 3, \ldots, (n-i)$ and then Step(3) for $m = 1$ successively to find $m_{v_{i+m}v_{i+m+1}}$.

Step 5. Repeat Step(4) for $i = 3, 4, \ldots, (n-2)$ to find $m_{v_{n-1}v_n}$ which is the strength of G.

Illustration:

For the graph G in Figure 2.1, $k_{v_1v_2} = 3 = m_{v_1v_2}$.

$i = 1, m = 2, k_{v_1v_3} = 1$ and $m_{v_1v_2v_3} = max\{m_{v_1v_2}, k_{v_1v_3}\} = 3$,

$i = 1, m = 3, k_{v_1v_4} = 2$ and $m_{v_1v_2v_3v_4} = max\{m_{v_1v_2v_3}, k_{v_1v_4}\} = 3$,

$i = 1, m = 4, k_{v_1v_5} = 3$ and $m_{v_1v_2v_3v_4v_5} = max\{m_{v_1v_2v_3v_4,k_{v_1v_4}}\} = 3$,

$i = 1, m = 1, k_{v_2v_3} = 2$ and $m_{v_2v_3} = max\{m_{v_1v_2v_3v_4v_5}, k_{v_2v_3}\} = 3$,

$i = 2, m = 2, k_{v_2v_4} = 1$ and $m_{v_2v_3v_4} = max\{m_{v_2v_3}, k_{v_2v_4}\} = 3$,

$i = 2, m = 3, k_{v_2v_5} = 2$ and $m_{v_2v_3v_4v_5} = max\{m_{v_2v_3v_4}, k_{v_2v_5}\} = 3$,

$i = 2, m = 1, k_{v_3v_4} = 1$ and $m_{v_3v_4} = max\{m_{v_2v_3v_4v_5}, k_{v_3v_4}\} = 3$,

$i = 3, m = 2, k_{v_3v_5} = 2$ and $m_{v_3v_4v_5} = max\{m_{v_3v_4}, k_{v_3v_5}\} = 3$,

$i = 3, m = 1, k_{v_4v_5} = 1$ and $m_{v_4v_5} = max\{m_{v_3v_4v_5}, k_{v_4v_5}\} = 3$. Then the strength of G is 3.

2.2 Strength of strong fuzzy complete bipartite graph

We start this section with very simple but very useful and strong result which states that if two vertices are adjacent in a strong fuzzy graph then the path (edge) uv is the extra strong path connecting them. Therefore the length of the extra strong path joining two adjacent vertices in a strong fuzzy graph is one. Two vertices in a fuzzy graph are said to be adjacent if the weight of the edge determined by them is positive that is they are adjacent in the underlying crisp graph. We also determine the strength of strong fuzzy complete bipartite graphs.

Theorem 2.2.1. *Let G be a strong fuzzy graph. If u and v are two adjacent vertices of G then the length of the extra strong path joining u and v is one.*

Proof. Suppose that u and v are adjacent in G. Since G is a strong fuzzy graph, the edge uv, has strength $\mu(u) \wedge \mu(v)$. All the other paths joining u and v have strength less than or equal to $\mu(u) \wedge \mu(v)$. Hence the edge uv is the unique extra strong path joining u and v. Hence the result. □

Corollary 2.2.1. If G is a strong fuzzy complete graph then its strength is one.

Remark 2.2.1. G is a strong fuzzy graph. Suppose u and v are two adjacent vertices of G. Then the path (edge) uv is the only extra strong path joining u and v. So for the computation of strength of a fuzzy graph we need to consider only its distinct non-adjacent vertices.

Theorem 2.2.2. *Strength of a strong fuzzy complete bipartite graph [44] G is two if $|V(G)| > 2$.*

Proof. Let $G(V, \mu, \sigma)$ be a strong fuzzy complete bipartite graph with K_{mn} as its underlying crisp graph. Suppose that $m + n > 2$. Let $U = \{u_i : i = 1, , \ldots, m\}$ and $V = \{v_j : j = 1, 2, \ldots, n\}$ be the bipartite sets. Also let u and v be any two distinct non-adjacent vertices of G. If u and $v \in U$, then all the $u - v$ paths in G pass through atleast one vertex in V. Let w be a vertex in V with $\mu(w) \geq \mu(v_i), \forall v_i \in V$. Then strength of any $u - v$ path is less than or equal to that of the path uwv in G. Therefore uwv is one of the extra strong paths joining u and v and which is of length 2. Similar is the case when both $u, v \in V$. Hence the theorem. □

2.3 Properly linked fuzzy graphs

This section deals with properly linked fuzzy graphs. We give certain examples for it. Also, we find out the strength of properly linked fuzzy graphs.

Definition 2.3.1. A finite sequence of distinct fuzzy graphs [36] $G_1, G_2, \ldots, G_m$ with the property that $V(G_i) \cap V(G_j)$ is nonempty if and only if $|j - i| \leq 1, 1 \leq i, j \leq m$ is called a properly linked sequence or simply properly linked . It is $n-$ linked if the crisp graph induced by $< V(G_i) \cap V(G_j) >$ is K_n, a complete graph on n vertices, if $|j - i| = 1, 1 \leq i, j \leq m$.

Definition 2.3.2. A fuzzy graph G is said to be properly linked ($n-$ linked) if there exists a finite sequence of properly linked partial fuzzy subgraphs $G_1, G_2, \ldots,$ G_m, where $m > 1$, such that $G = G_1 \cup G_2 \cup \ldots \cup G_m$. In this case we say that $G_1, G_2, \ldots, G_m$ are parts of G.

Notation 2.3.1. If a fuzzy graph G is a union of a sequence of m, $n-$ linked fuzzy graphs $G_1, G_2, \ldots, G_m$, for some n then we write $G = G_1 < G_2 < \ldots < G_m$.

Lemma 2.3.1. Let $G = G_1 < G_2 < \ldots < G_m$ be a $1-$linked strong fuzzy graph with $G_1, G_2, \ldots, G_m$ (where $m > 1$) as its parts. Let $G_1, G_2, \ldots, G_m$ be complete strong fuzzy graphs. For, $i = 1, 2, \ldots, n-1$, let $V(G_i) \cap V(G_{i+1}) = \{v_i\}$. Let u, v be any two distinct vertices of G. For $k < m$, if $u \in V(G_1) \setminus \{v_1\}$ and $v \in V(G_{k+1}) \setminus \{v_k\}$ then the length of extra strong path joining u and v in G is $k+1$.

Proof. This result is proved by induction on k. When $k = 1$, $u \in V(G_1) \setminus \{v_1\}$ and $v \in V(G_2) \setminus \{v_1\}$. Therefore all the $u - v$ paths pass through v_1. Since G is complete, the extra strong path joining u and v_1 and the same for v_1 and v are respectively uv_1 and v_1v. Therefore the length of the extra strong path joining u and v is 2.

Now, let us assume the result is true for every $k \leq n < m-1$. Let $u \in V(G_1) \setminus \{v_1\}$ and $v \in V(G_{n+2}) \setminus \{v_{n+1}\}$. Note that every $u - v$ path must pass through the vertex v_{n+1}. By induction hypothesis the length of every extra strong path joining u and v_{n+1} is $n+1$. Since v_{n+1} and v lie in G_{n+2}, the only extra strong path joining v_{n+1} and v is the edge $v_{n+1}v$. Hence the length of the extra strong path joining u and v is $n+2$. In fact there is only one extra strong path joining u and v. Hence the lemma holds by induction. $\square$

The following theorem is an immediate consequence of Lemma 2.3.1.

Theorem 2.3.1. *Let G be a $1-$ linked fuzzy graph with m (where $m > 1$) complete fuzzy graphs as its parts. Then the strength of G is m, the diameter of G.*

Theorem 2.3.1 can be used to find the strength of certain fuzzy graphs. For example strong fuzzy butterfly graph, strong fuzzy bull graph.

Definition 2.3.3. A strong fuzzy butterfly graph is a strong fuzzy graph with its underlying crisp graph is a butterfly graph [55].

Corollary 2.3.1. The strength of a strong fuzzy butterfly graph is two.

Proof. A strong fuzzy butterfly graph is a properly linked fuzzy graph with two complete fuzzy triangles as its parts. So, by Theorem 2.3.1, the strength of a fuzzy butterfly graph is 2. □

Definition 2.3.4. A strong fuzzy bull graph is a strong fuzzy graph with its underlying crisp graph is a bull graph [10].

Corollary 2.3.2. The strength of a strong fuzzy bull graph is 3.

Proof. A strong fuzzy bull graph is $1-$ linked by a sequence of 3 complete strong fuzzy graphs G_1, G_2 and G_3, where G_1 and G_2 are fuzzy paths on two vertices and G_3 a strong fuzzy triangle graph which is a complete strong fuzzy graph. So the strength of a strong fuzzy bull graph is three. □

Theorem 2.3.1 can be generalized as follows.

Notation 2.3.2. If $P_1 = u_1u_2 \ldots u_n$ and $P_2 = u_nu_{n+1} \ldots u_m$ are two paths in a fuzzy graph G then $P_1 + P_2$ denote the path $u_1u_2 \ldots u_nu_{n+1} \ldots u_m$.

Theorem 2.3.2. *Let $G(V, \mu, \sigma)$ be a properly linked fuzzy graph with the complete fuzzy graphs $G_1, G_2, \ldots, G_m$ as its parts, where $m > 1$. Suppose for $i =$*

$1, 2, \ldots, m-1$, $< V(G_i) \cap V(G_{i+1}) >= K_{n_i}$, *a complete fuzzy graph on* n_i *vertices. Then the strength of* G *is the diameter* m *of* G *[20].*

Proof. Let $V(G_i) \cap V(G_{i+1}) = \{u_{i1}, u_{i2}, \ldots, u_{in_i}\}$, for $i = 1, 2, \ldots,$
$m - 1$. Let u and v be any two distinct non-adjacent vertices of G.
We prove the theorem in two steps.

Step 1: In this step we prove that if $u \in V(G_1) \setminus \{u_{11}, u_{12}, \ldots, u_{1n_1}\}$ and $v \in V(G_{k+1}) \setminus \{u_{k1}, u_{k2}, \ldots, u_{kn_k}\}$ then the length of the extra strong $u - v$ path is $k + 1$, where $1 \leq k \leq m - 1$.

We prove this result by induction on k. Assume that $k = 1$.

Then every $u - v$ path lies completely in $G_1 \cup G_2$. When $m = 2$ it is obvious. Otherwise, any $u - v$ path have at least 4 subpaths P_1, P_2, P_3, P_4, and can be written as $P_1 + P_2 + P_3 + P_4$ where P_1 is a path from u to some vertex u_{2i} of $\{u_{21}, u_{22}, \ldots, u_{2n_2}\}$ in $G_1 \cup G_2$, P_2 is a path from u_{2i} to some vertex w in $G_3 \cup G_4 \cup \ldots \cup G_m$, P_3 is a path from w to some vertex u_{2j} of $\{u_{21}, u_{22}, \ldots, u_{2n_2}\}$ in $G_3 \cup G_4 \cup \ldots \cup G_m$ and P_4 is a path from u_{2j} to v in $G_1 \cup G_2$. Such paths, obviously have strength $\leq$ that of the path $P_1 + u_{2i}u_{2j} + P_4$. Thus we can conclude that all the extra strong paths joining u and v lie completely in $G_1 \cup G_2$.

Since G_1 and G_2 are complete fuzzy graphs, both u and v are adjacent to all the vertices of $\{u_{11}, u_{12}, \ldots, u_{1n_1}\}$. If $\mu(u_{1k}) = \bigvee_{i=1}^{m} \mu(u_{1i})$, then $uu_{1k}v$ is an extra strong path joining u and v in $G_1 \cup G_2$ and is of length 2.

Assume that the result is true for $k \leq n \leq m - 2$. To prove the result for $k = n+1$, let $u \in V(G_1) \setminus \{u_{11}, u_{12}, \ldots, u_{1n_1}\}$ and $v \in V(G_{n+2}) \setminus \{u_{n+11}, u_{n+12} \ldots, u_{n+1n_{n+1}}\}$. Then as above we prove that every extra strong path joining u and v lies completely in $G_1 \cup G_2 \cup \ldots \cup G_{n+2}$. When $n = m-2$, it is obvious. For $n < m-2$, if the result is not true then there exist a $u-v$ path in G which passes through a vertex of $V(G_{n+3}) \setminus \{u_{n+21}, u_{n+22}, \ldots, u_{n+2n_{n+2}}\}$. Then it must pass through at least one vertex of the set $\{u_{n+21}, u_{n+22}, \ldots, u_{n+2n_{n+2}}\}$. Any such path have at least four subpaths P_1, P_2, P_3, P_4 where P_1 is a path from u to some vertex u_{n+2i} of the set $\{u_{n+21}, u_{n+22}, \ldots, u_{n+2n_{n+2}}\}$ in $G_1 \cup G_2 \cup \ldots \cup G_{n+2}$, P_2 is a path from u_{n+2i} to a vertex z in $G_1 \cup G_{n+3} \cup G_{n+4} \cup \ldots \cup G_m$, P_3 is a path from z to some vertex u_{n+2j} of $\{u_{n+21}, u_{n+22}, \ldots, u_{n+2n_{n+2}}\}$ in $G_{n+3} \cup G_{n+4} \cup \ldots \cup G_m$ and P_4 is a path from u_{n+2j} to v in $G_1 \cup G_2 \cup \ldots \cup G_{n+2}$.

Clearly the path $P_1 + u_{n+2i}u_{n+2j} + P_4$ has strength greater than or equal to all such paths. Thus any extra strong path can be written as sum of two paths P, Q where P is an extra strong path from u to $w \in \{u_{n+11}, u_{n+12}, \ldots, u_{n+1n_{n+1}}\}$ in $G_1 \cup G_2 \cup \ldots \cup G_{n+2}$, where $\mu(w) = \bigvee_{i=1}^{n+1} \mu(u_{n+1,i})$ and Q is the edge wv of G_{n+2}, since G_{n+2} is complete. Now by induction hypothesis the length of P is $n+1$. Therefore the length of extra strong path joining u and v is $n+2$.

Step 2: Let u and v be any two vertices of G. Suppose u and v belong to the same part G_i of G. Then u and v are adjacent because G_i is complete. Hence the edge uv is the only extra strong $u-v$ path in G. Otherwise u belongs to some G_i and v belongs to some G_j of G, where G_i and G_j are two distinct parts of G. Without loss of generality assume that $i < j$. Then by Step 1 we can conclude that the length of extra strong path is $j-i+1 \leq m$. In particular when $i=1$ and $j=m$, the length of extra strong $u-v$ path is m. Therefore $\mathscr{S}(G)=m$. Hence the theorem. □

Remark 2.3.1. Theorem 2.3.2 need not be true when at least one part of a properly linked fuzzy graph fails to be a complete fuzzy graph. For example, the strong fuzzy graph G in Figure 2.2 is a $2-$linked fuzzy graph of strength 3 with parts G_1 and G_2, where G_1 is a complete fuzzy graph but G_2 is not a complete fuzzy graph.

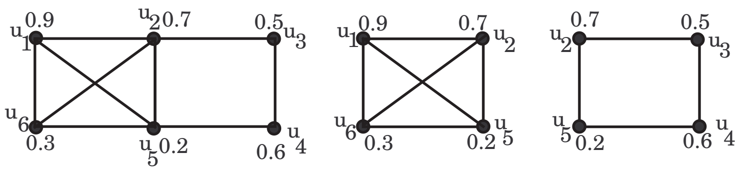


Figure 2.2: A 2-linked fuzzy graph G, its parts G_1 and G_2.

Definition 2.3.5. A fuzzy diamond graph [24] is a fuzzy graph with the underlying crisp graph is a diamond graph.

From the definition of a strong fuzzy diamond graph, which is a $2-$ linked fuzzy graph having two parts and each is complete. Therefore we have the following Corollary.

Corollary 2.3.3. The strength of a strong fuzzy diamond graph is 2.

Definition 2.3.6. Let $G_1, G_2, \ldots, G_n$ be n simple graphs with vertex sets $V_1, V_2, \ldots, V_n$ respectively. For $i \neq j$ if $|V_i \cap V_j| \geq 1$, let $Z_{ij} = V_i \cap V_j$. Let $V = \bigcup\limits_{k=1}^{n} V_k$

and $Z = \cup Z_{ij}$, where the union is taken for those $i \neq j$ for which $|V_i \cap V_j| \geq 1$. For such i and j, form a single vertex z_{ij} by merging the vertices of $V_i \cap V_j$. Let $U = (V \setminus Z) \cup \{z_{ij}\}$. The simple graph with vertex set U and edge set E is called the merger graph of $G_1, G_2, \ldots, G_n$; where, for $u \neq v \in U$, $uv \in E$ provided,

1. $u, v \in V \setminus Z$ and are adjacent in $\overset{n}{\underset{i=1}{\cup}} G_i$.
2. $u = z_{ij}$ for some i and j, $v \in V \setminus Z$ and v is adjacent to at least one vertex in Z_{ij}.
3. $u = z_{ij}$, $v = z_{kl}$ for some i, j, k and l and at least one vertex of Z_{ij} is adjacent to at least one vertex of Z_{kl}.

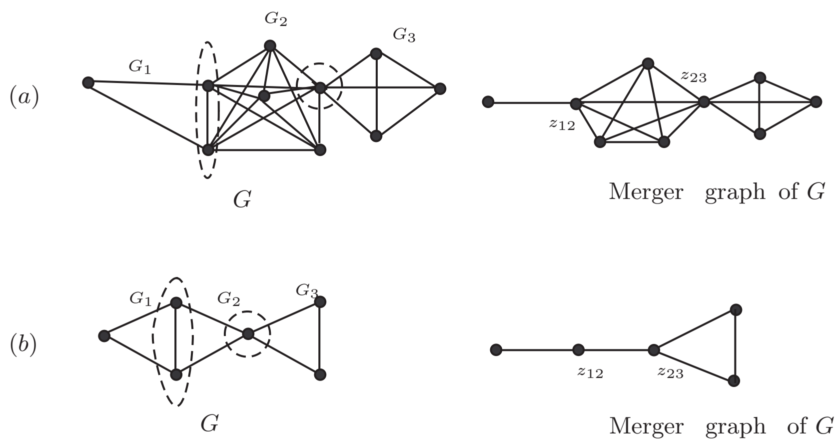


Figure 2.3: Fuzzy graph G and its merger graph.

Note 2.3.1. If $V_i \cap V_j = \phi, \forall i, j$ then the merger graph of $G_1, G_2, \ldots, G_n$ is just $G_1 \cup G_2 \cup \ldots \cup G_n$.

Definition 2.3.7. Let $G_i(V_i, \mu_i, \sigma_i), i = 1, 2, \ldots, n$ be fuzzy graphs with underlying crisp graph $G_i(V_i, E_i)$. The fuzzy merger graph $G(U, \mu_{mer}, \sigma_{mer})$ is a fuzzy graph with its underlying crisp graph $G(U, E)$ is a merger graph of $G_i(V_i, E_i), i = 1, 2, \ldots, n$ where U and E are as in Definition 2.3.6 and the membership functions μ_{mer} and σ_{mer} are given by

$$\mu_{mer}(u) = \begin{cases} \mu_i(u) & \text{if } u \in V_i \setminus Z \text{ for some } i, \\ \bigwedge\limits_{v \in V_i \cap V_j} (\mu_i(v) \wedge \mu_j(v)) & \text{if } u = w_{ij} \text{ for some i, j.} \end{cases}$$

$$\sigma_{mer}(uv) = \begin{cases} \sigma_i(uv) & \text{if } u, v \in V_i \text{ for some i ,} \\ \mu_{mer}(u) \wedge \mu_{mer}(v) & \text{otherwise.} \end{cases}$$

Remark 2.3.2. Let $G_i(V_i, \mu_i, \sigma_i), i = 1, 2, \ldots, n$ be n fuzzy graphs such that $V_i \cap V_j \neq \phi$ if and only if $|j - i| = 1$. Then the merger graph of these fuzzy graphs $G_i, i = 1, 2, \ldots, n$ is a $1-$ linked fuzzy graph. Thus if each G_i is a complete strong fuzzy graph and $V_i \cap V_j \neq \phi$ if and only if $|j - i| = 1$ then, their merger fuzzy graph has strength equal to its diameter by Theorem 2.3.1 which is also equal to the strength of $\overset{n}{\underset{i=1}{\cup}} G_i$. This result need not be true if $G_i's$ are not complete. For example, the merger graph of G in Figure 2.2 is the fuzzy butterfly graph which is of strength 2 but the strength of G is 3.

2.4 Strong fuzzy Wagner graph

A strong fuzzy Wagner graph is a strong fuzzy graph with its underlying crisp graph is a Wagner graph (https://en.wikipedia.org/wiki/$Wagner_graph$).

Theorem 2.4.1. *Let G be a strong fuzzy Wagner graph. Then* $2 \leq \mathscr{S}(G) \leq 4$.

Proof. From Figure 2.4 it is clear that to prove the theorem it is enough to prove $\mathscr{S}(G)$ never be greater than 4.

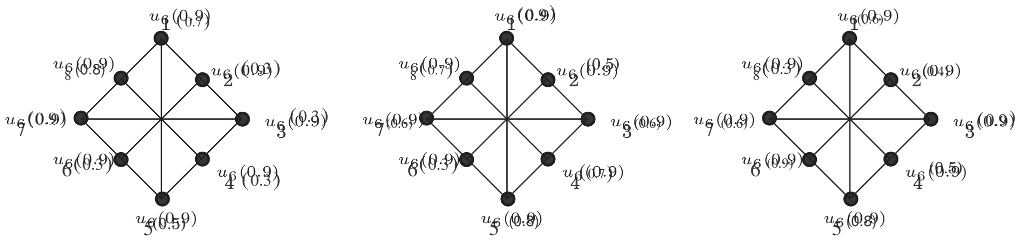


Figure 2.4: Strong fuzzy Wagnergraphs with strengths $2, 3$ and 4.

Let u, v be two nonadjacent vertices of G. Without loss of generality assume, $u = u_1$. Then $v \in \{u_3, u_4, u_6, u_7\}$. If possible assume, length of an extra strong $u - v$ path is greater than 4 then there exists at least one path joining u and v of length greater than 4. But all those paths must pass either through both u_2, u_8 or through u_4, u_6. Since G is a strong fuzzy graph, these paths never be extra strong. □

Chapter 3

Strength of Fuzzy graphs derived from certain known Fuzzy graphs

In this chapter we determine the strength of join and corona of certain strong fuzzy graphs and that of middle graph and total graph of certain strong fuzzy graphs.

Definition 3.0.1. [34] Let $G_i(V_i, \mu_i, \sigma_i), i = 1, 2$ be two connected fuzzy graphs with underlying crisp graphs $G_i(V_i, E_i), i = 1, 2$ respectively. Then the union of two fuzzy graphs denoted by $G_1 \cup G_2$ of G_1 and G_2 is $G(V, \mu, \sigma)$ with underlying crisp graph $G(V, E)$ is the union of $G_i(V_i, E_i), i = 1, 2$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ and

$$\mu(u) = \begin{cases} \mu_1(u) & \text{if } u \in V_1 \setminus V_2, \\ \mu_2(u) & \text{if } u \in V_2 \setminus V_1. \end{cases}$$

$$\sigma(uv) = \begin{cases} \sigma_1(uv) & \text{if } uv \in E_1 \setminus E_2, \\ \sigma_2(uv) & \text{if } uv \in E_2 \setminus E_1. \end{cases}$$

3.1 Strength of join of fuzzy graphs

In this section we concentrate our study on the strength of join of certain known fuzzy graphs.

Definition 3.1.1. [34] Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two connected fuzzy graphs with underlying crisp graph $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ respectively, where $V_1 \cap V_2 = \phi$. Then the join $G(V, \mu, \sigma)$ of G_1 and G_2 is the fuzzy graph with the underlying crisp graph $G(V, E)$ is the join of $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E'$ where E' is the set of all edges joining the vertices in V_1 with vertices in V_2, also the membership functions μ and σ are defined as follows.

$$\mu(u) = \begin{cases} \mu_1(u) & \text{if } u \in V_1, \\ \mu_2(u) & \text{if } u \in V_2. \end{cases}$$

$$\sigma(uv) = \begin{cases} \sigma_1(uv) & \text{if } u, v \in V_1, \\ \sigma_2(uv) & \text{if } u, v \in V_2, \\ \mu_1(u) \wedge \mu_2(v) & \text{if } u \in V_1 \text{ and } v \in V_2. \end{cases}$$

Example 3.1.1. 1. A fuzzy complete $2-$ partite graph is the join of two fuzzy null graphs.

2. A fuzzy wheel graph is the join of a fuzzy cycle and a trivial fuzzy graph.

3. A fuzzy star graph is the join of a fuzzy null graph and a fuzzy trivial graph.

Remark 3.1.1. From the definition of join G of two fuzzy graphs G_1 and G_2, both G_1 and G_2 can be considered as maximal partial fuzzy subgraphs of G.

The join of two complete fuzzy graphs is again a complete fuzzy graph. Hence we have the following Theorem.

Theorem 3.1.1. *The strength of join of two complete fuzzy graphs is one.*

Definition 3.1.2. A fuzzy fan graph F_{mn} [41] is defined as the join of a fuzzy null graph on $m \geq 1$ vertices and a fuzzy path on $n \geq 1$ vertices.

Figure 3.1: Fuzzy fan graph F_{24}.

Let $G(V, \mu, \sigma)$ be the join of two graphs $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ where G_1 a fuzzy null graph on m vertices with vertex set $V_1 = \{u_1, u_2, \ldots, u_m\}$ and G_2

a strong fuzzy path on n vertices with vertex set $V_2 = \{v_1, v_2, \ldots, v_n\}$. If $n = 1$ then for any m, G is a fuzzy star graph. Therefore its strength is 1 if $m = 1$ and 2 if $m > 1$. If $n = 2$ and $m = 1$ then G is complete. Therefore its strength is one. If $n = 2$ and $m > 1$ then $\mathscr{S}(G)$ is 2. For if u and v are two nonadjacent vertices in G then they are vertices of G_1. If w is a vertex of G_2 with maximum weight among the vertex of G_2 then uwv is an extra strong path of G.

Now consider the cases for $n \geq 3$. First of all suppose that $m = 1$. In this case $V_1 = \{u_1\}$. If $\mu_1(u_1) < \bigwedge_{j=1}^{n} \mu_2(v_j)$ then clearly for any u and $v \in V(G_2)$, this $u - v$ path of G_2 is the extra strong $u - v$ path of G.

Otherwise $\mu_2(v_i) < \mu_1(u_1)$. Then we have two cases. Let u and v be two nonadjacent vertices of G. Let l be the maximum of length of all subpaths of G_2 of strength $> \mu_1(u_1)$ if such a path exists, otherwise let l be zero. Then $\mathscr{S}(G) = 2 \vee l$.

The preceding discussion may be generalized as follows.

Theorem 3.1.2. *Let $G(V, \mu, \sigma)$ be the join of two fuzzy graphs $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$, where G_1 is a fuzzy null graph on $m \geq 2$ vertices with vertex set $V_1 = \{u_1, u_2, \ldots, u_m\}$ and G_2 a strong fuzzy path on n ≥ 3 vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$. If there exists a path in G_2, whose strength is $> \bigvee_{i=1}^{m} \mu_1(u_i)$ then let l be the maximum of length of all such subpaths of G_2. Then $\mathscr{S}(G) = l \vee 2$.*

Proof. Let u and v be two nonadjacent vertices of G. If both of them are in V_1 then all $u - v$ paths must pass through at least one vertex of V_2, since all v_j in

V_2 are adjacent to both u and v. Then uv_jv is an extra strong $u-v$ path in G where $\mu(v_j) \geq \overset{n}{\underset{i=1}{\vee}} \mu_2(v_i)$.

If both u and v are in V_2 then we have the following cases.

Case 1. $l = 0$.

In this case every subpath of G_2 has strength $\leq \overset{n}{\underset{i=1}{\vee}} \mu_1(u_i)$. Let u_j be the vertex in V_1 such that $\mu_1(u_j) \geq \overset{m}{\underset{i=1}{\vee}} \mu_1(u_i)$ then uu_jv is an extra strong $u-v$ path in G.

Case 2. $l = 1$.

As u, v are nonadjacent vertices of G_2, the $u-v$ path of G_2 contains a vertex of weight $\leq \overset{n}{\underset{i=1}{\vee}} \mu_1(u_i)$. Let u_j be a vertex of G_1 such that $\mu_1(u_j) \geq$ strength of the $u-v$ path in G. Then uu_jv is an extra strong $u-v$ path in G_2.

If $l > 1$ then clearly $\mathscr{S}(G) = l$. □

Corollary 3.1.1. Let G_1, G_2 and G be fuzzy graphs as in Theorem 3.1.2. Then if $\overset{m}{\underset{i=1}{\vee}} \mu_1(u_i) < \overset{n}{\underset{j=1}{\wedge}} \mu_2(v_j)$ then $\mathscr{S}(G) = n - 1$.

Now consider the case where both G_1 and G_2 are fuzzy paths. That is $G(V, \mu, \sigma)$ is the join of two strong fuzzy graphs say, $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ with underlying crisp graphs P_m with vertex set $V_1 = \{u_1, u_2, \ldots, u_m\}$ and P_n with vertex set $V_2 = \{v_1, v_2, \ldots, v_n\}$. The case $n = 1$ and the case $m = 1$ are included in Theorem 3.1.2. So we suppose that both $m, n \geq 2$. For $n = m = 2$, G is a complete strong fuzzy graph on 4 vertices so its strength is 1.

When $m = 2$ and $n > 2$, $\mathscr{S}(G) = 2 \vee l$ where l is the maximum of length of all subpaths of G_2 having strength $> \overset{n}{\underset{i=1}{\vee}} \mu_1(u_i)$ if there exists such a path in G_2. The following theorem determines $\mathscr{S}(G)$ in all other cases.

Theorem 3.1.3. *Let $G(V, \mu, \sigma)$ be the join of two strong fuzzy graphs $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ with underlying crisp graphs P_m and P_n with vertex sets $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$; $m \geq n > 2$. Let l_1 be the maximum of length of all subpaths of G_1 of strength $> \overset{n}{\underset{j=1}{\vee}} \mu_2(v_j)$ if such a path exists otherwise let $l_1 = 0$. Also let l_2 be the maximum of length of all subpaths of G_2 of strength $> \overset{m}{\underset{i=1}{\vee}} \mu_1(u_i)$ if such a path exists otherwise let $l_2 = 0$. $\mathscr{S}(G) = l_1 \vee l_2 \vee 2$.*

Proof. Let u and v be two nonadjacent vertices of G. Then either u and $v \in V_1$ or u and $v \in V_2$.

Case 1. $l_1 = l_2 = 0$.

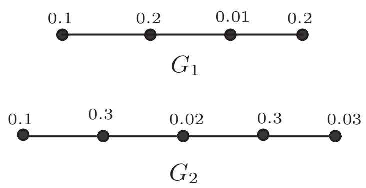


Figure 3.2: Example for two fuzzy paths having $l_1 = l_2 = 0$.

Let u and $v \in V_1$. Then an extra strong $u - v$ path is uv_jv where v_j is such that $\mu_2(v_j) = \overset{n}{\underset{i=1}{\vee}} \mu_2(v_i)$. Similarly if u and $v \in V_2$ then uu_iv is an extra strong $u-v$ path in G, where u_i is such that $\mu_1(u_i) = \overset{m}{\underset{j=1}{\vee}} \mu_1(u_j)$. So in this case $\mathscr{S}(G) = 2$.

Case 2. $l_1 = 1$ and $l_2 = 0$ or $l_1 = 0$ and $l_2 = 1$.

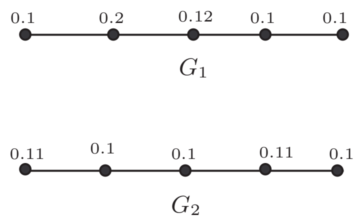


Figure 3.3: Example for two fuzzy paths having $l_1 = 1$ and $l_2 = 0$.

Consider the case $l_1 = 1$ and $l_2 = 0$. Let u and $v \in V_2$. Since $l_2 = 0$, the strength of $u-v$ path of $G_2 \leq \overset{m}{\underset{i=1}{\vee}} \mu_1(u_i)$. Let u_j be a vertex of G_1 such that $\mu_1(u_j) = \overset{m}{\underset{i=1}{\vee}} \mu_1(u_i)$. Then uu_jv is an extra strong $u-v$ path in G of length 2. Let u and $v \in V_1$. Since $l_1 = 1$, uu_1v is an extra strong $u-v$ path in G_1 where $v_i = \overset{n}{\underset{j=1}{\vee}} \mu_2(v_j)$. Thus the strength of G is 2.

Similarly we can prove the case when $l_1 = 0$ and $l_2 = 1$.

Case 3. $l_1 > 1$ or $l_2 > 1$.

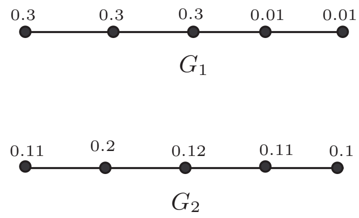


Figure 3.4: Example for two fuzzy paths having $l_1 > 1$ and $l_2 = 0$.

First of all we consider the case, $l_1 > 1$. In this case $l_2 = 0$. Therefore if $u, v \in V_2$, as in case 2, uu_jv is an extra strong path in G, where $\mu_1(u_j) = \bigvee_{i=1}^{m} \mu_1(u_i)$. Now let $u, v \in V_1$. Since $l_1 > 1$ either strength of the $u - v$ path in G_1 is $> \bigvee_{j=1}^{n} \mu_2(v_j)$ or there exists a vertex $v_i \in V_2$ such that $\mu_2(v_i) \geq$ strength of the $u - v$ path in G_1. In the first case the $u - v$ path of G_1 is the only an extra strong $u - v$ path in G. In the second case the path uv_iv in G is an extra strong $u - v$ path. From this it follows that $\mathscr{S}(G) = l_1$. Similarly we can prove that $\mathscr{S}(G) = l_2$ if $l_2 > 1$. Therefore $\mathscr{S}(G) = l_1 \vee l_2$. □

3.1.1 Fuzzy wheel graph

Definition 3.1.3. A fuzzy wheel graph W_n is the join of the fuzzy cycle C_{n-1} and a fuzzy trivial graph.

Some results of this chapter are included in the following paper Chithra K. P., Raji Pilakkat, International Journal of Pure and Applied Mathematics, 106(3) 2016, 883-892

Definition 3.1.4. A vertex h of the wheel graph W_n is said to be a fuzzy hub if it is adjacent to all the other vertices of W_n.

Definition 3.1.5. A strong fuzzy wheel graph is a fuzzy wheel graph which is also a strong fuzzy graph.

Theorem 3.1.4. *For $n \geq 4$, let $W_n = C_{n-1} \vee K_1$ be a strong fuzzy wheel graph with fuzzy hub h and $u_1u_2 \ldots u_{n-1}u_1$ the fuzzy cycle C_{n-1}. If $\mu(h) < \bigvee_{i=1}^{n-1} \mu(u_i)$ then the strength of W_n is the strength of C_{n-1}.*

Proof. Choose two distinct non-adjacent vertices u and v of W_n. Clearly $u, v \in V(C_{n-1})$. Since $\mu(h) < \bigvee_{i=1}^{n-1} \mu(u_i)$, all paths joining u and v, through h have strength $\mu(h)$, which is less than the strength of any path joining u_i and u_j in C_{n-1}. Therefore the length of extra strong paths joining u and v in W_n and those in C_{n-1} are one and the same. Hence the result. □

Theorem 3.1.5. *Let W_n be as in Theorem 3.1.4. If $\mu(h) \geq \bigvee_{i=1}^{n-1} \mu(u_i)$ then the strength of W_n is one when $n = 4$ and two when $n > 4$.*

Proof. When $n = 4$, W_n is a complete fuzzy graph. Therefore the strength of W_n is one [48].

Now suppose that $n > 4$. Let u and v be any two distinct non-adjacent vertices of W_n. Therefore both belong to $V(C_{n-1})$. Clearly uhv is an extra strong $u - v$ path in W_n. So strength of W_n is 2. □

The only remaining case is that some vertices of C_{n-1} have weight greater than $\mu(h)$ and some have weight less than or equal to $\mu(h)$. In this case we have the following result.

Theorem 3.1.6. *Let W_n be as in Theorem 3.1.4. Suppose that $\mu(h) \leq \mu(u_i)$ for some but not all the vertices $u_i, i = 1, 2, \ldots, n-1$. Let P be one of the maximal paths of C_{n-1} with the property that each edge of which has strength greater than $\mu(h)$. Let l be the length of P. Then $\mathscr{S}(W_n) = l \vee 2$.*

Proof. Let u, v be any two distinct non-adjacent vertices of W_n. Then u and v are vertices of C_{n-1}. Let P_1 and P_2 be two paths in C_{n-1} having u and v as the end vertices.

Suppose that $l \leq 1$. Then both P_1 and P_2 have strength less than or equal to $\mu(h)$. Therefore uhv is an extra strong path in W_n. Hence in this case $\mathscr{S}(W_n) = 2$.

Now suppose $l \geq 2$. If both the paths P_1 and P_2 have strength less than or equal to $\mu(h)$ then uhv is an extra strong path joining u and v and which is of length 2. If exactly one of the paths P_1 and P_2 say P_1 has strength greater than $\mu(h)$, then the extra strong path joining u and v in W_n is the path P_1. Since each edge of which has strength greater than $\mu(h)$, the length of $P_1 \leq$ length of $P = l$. In particular if u and v are the end vertices of P, then P itself is an extra strong path joining u and v. Hence the theorem. $\square$

3.2 Corona of strong fuzzy graphs

Definition 3.2.1. [21] Let $G_1(U, \mu_1, \sigma_1)$and $G_2(V, \mu_2, \sigma_2)$ be two fuzzy graphs with the respective underlying crisp graphs $G_1(U, E_1)$ and $G_2(V, E_2)$ where $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. Then the corona $G(W, \mu, \sigma)$ of G_1 and G_2 is a fuzzy graph with the underlying crisp graph is the corona $G = G_1 \odot G_2$ of $G_1(U, E_1)$ and $G_2(V, E_2)$ with vertex set $W = U \cup (\cup_{i=1}^{n} V_i)$, where $V_i = \{v_{1i}, v_{2i}, \ldots, v_{ni}\}$, $i = 1, 2, \ldots m$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, the vertex v_{ji} represent the vertex v_j of G_2 in the i^{th} copy of G_2 corresponding to the vertex u_i of G_1. The fuzzy subset μ and the fuzzy relation σ on W are defined as

$$\mu(w) = \begin{cases} \mu_1(u_i) & \text{if } w = u_i \in U, \\ \mu_2(v_j) & \text{if } w = v_{ji}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \end{cases}$$

and

$$\sigma(uv) = \begin{cases} \sigma_1(uv) & \textit{if} \quad u, v \in U, \\ \sigma_2(uv) & \textit{if there exists an } i \textit{ such that } u = v_{j_1 i} \\ & \text{and} \quad v = v_{j_2 i} \text{ for some } j_1, j_2 \text{ with } 1 \leq j_1 \neq j_2 \leq m, \\ \mu_1(u) \wedge \mu_2(v) & \text{if } v = v_{ji} \text{ for some } i \text{ and } u = u_i \\ & \text{or} \quad u = v_{ji} \text{ for some } i \text{ and } v = u_i. \end{cases}$$

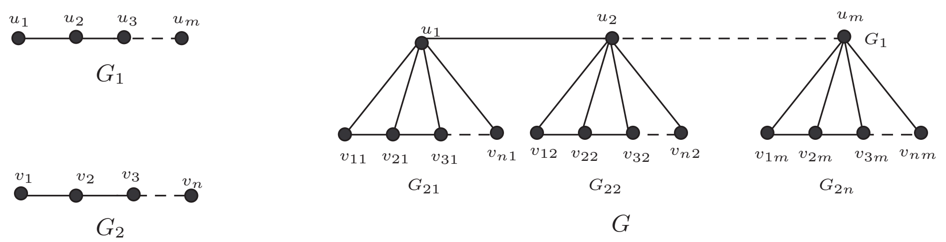


Figure 3.5: Two fuzzy graphs G_1 and G_2 and their corona.

As in the case of join, if G is the corona of G_1 and G_2 then both G_1 and G_2 can be considered as partial fuzzy subgraphs of G.

If G_1 is complete and G_2 is a trivial fuzzy graph then,

$$\mathscr{S}(G) = \begin{cases} 1 & \text{if } G_1 \text{ is trivial,} \\ 3 & \text{otherwise.} \end{cases}$$

Notation 3.2.1. Suppose G_1 and G_2 are fuzzy graphs as in Definition 3.2.1. The copy of G_2 in G corresponding to the vertex u_i of G_1 in the corona G of G_1 and G_2 is denoted by G_{2i}.

Let $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ be two fuzzy null graphs and $G(W, \mu, \sigma)$ be the corona of G_1 and G_2. If $|V| = 1$ then G is the union of paths on two vertices. Therefore, by Theorem 1.4.1 strength of G is one. If $|V| > 1$ then G is the union of strong fuzzy star graphs with at least three vertices. So its strength is 2 by Theorem 1.4.1.

The corona of a fuzzy trivial graph and a non - null strong fuzzy graph is their join. Hence the Theorem. Hence by the discussions which precedes Theorem 3.2.1.

Theorem 3.2.1. *Let $G_1(U, \mu_1, \sigma_1)$ be a fuzzy trivial graph with vertex set $\{u\}$ and $G_2(V, \mu_2, \sigma_2)$ be not a fuzzy null graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let $G(W, \mu, \sigma)$ be the corona of G_1 and G_2. Let l be the maximum of length of all subpaths of G_2 of strength $> \mu_1(u)$ if such a path exists. Otherwise let l be zero. Then $\mathscr{S}(G) = l \vee 2$.*

Definition 3.2.2. Let $G(V, \mu, \sigma)$ be a fuzzy graph. A path P with ends u and v in G is said to be a critical extra strong path if P is an extra strong $u - v$ path with length is equal to $\mathscr{S}(G)$.

Note 3.2.1. A fuzzy graph G may contain more than one critical extra strong paths.

Notation 3.2.2. The minimum of strength of all critical extra strong paths of a fuzzy graph G is denoted by $\sigma_0(G)$ or simply by σ_0 if there is no confusion.

Proposition 3.2.1. Let $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ be two strong fuzzy graphs. Let $G(W, \mu, \sigma)$ be the corona of G_1 and G_2 then $\mathscr{S}(G) \geq \mathscr{S}(G_1)$.

Proof. Let $u, v \in V(G)$. If u and v are in $V(G_1)$ then all the $u - v$ paths lie completely in the partial fuzzy subgraph G_1 of G. So length of an extra strong $u - v$ path in G is equal to that in G_1. So by definition of strength of a fuzzy graph, $\mathscr{S}(G) \geq \mathscr{S}(G_1)$. Hence the proposition. □

The following Theorems deal with only those fuzzy graphs whose underlying crisp graphs are connected.

Definition 3.2.3. Let G be a fuzzy graph of strength $\mathscr{S}(G)$. Any extra strong path of length $\mathscr{S}(G) - 1$ is called a minus critical extra strong path .

Theorem 3.2.2. *Let $G(W, \mu, \sigma)$ be the corona of two strong fuzzy graphs $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ on the vertex sets $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1\}$ respectively. Let $\mu_2(v_1) < \sigma_0$. Suppose $\mathscr{S}(G_1) \geq 4$ and G_1 contains no minus critical extra strong path. Then $\mathscr{S}(G) = \mathscr{S}(G_1)$ provided, the ends of every critical path in G_1 is also connected by a path of length $\leq \mathscr{S}(G_1) - 2$.*

Proof. Let u, v be two nonadjacent vertices of G.

If u and $v \in V(G_1)$ then length of the extra strong $u{-}v$ path in G is $\leq \mathscr{S}(G_1)$. If $u = v_{1i}$ and $v = v_{1j}, i \neq j$ then all the $u - v$ paths must pass through both u_i and u_j and all such paths in G have strength $\mu_2(v_1)$. The hypothesis of the theorem imply that there is a $u{-}v$ path in G_1 of length $\leq \mathscr{S}(G_1){-}2$. Therefore, the length of the extra strong $u - v$ path in G is $\leq \mathscr{S}(G_1) - 2 + 2 = \mathscr{S}(G_1)$.

Similarly, if $u \in V(G_1)$ and $v \in V(G_{2i})$ then the length of the extra strong $u - v$ path is $\leq \mathscr{S}(G_1)$.

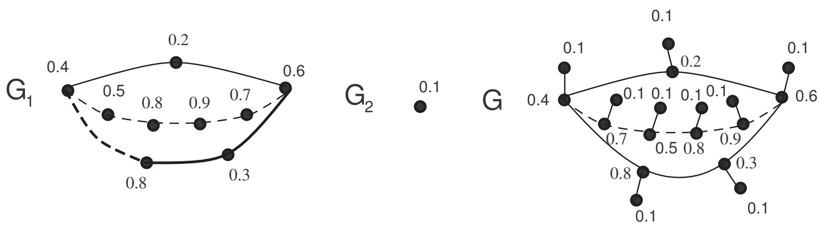


Figure 3.6: Corona G of two strong fuzzy graphs G_1 and G_2 with $\mathscr{S}(G_1) = \mathscr{S}(G) = 6$.

□

The following three theorems can be proved in the same way as the previous one was proved.

Theorem 3.2.3. *Let $G(W, \mu, \sigma)$ be the corona of two strong fuzzy graphs $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ on the vertex sets $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1\}$ respectively. Let $\mu_2(v_1) < \sigma_0$. Suppose $\mathscr{S}(G_1) \geq 4$ and G_1 contains a minus critical extra strong path. Then $\mathscr{S}(G) = \mathscr{S}(G_1)$ provided the ends of every critical path in G_1 is connected by a path of length $\leq \mathscr{S}(G_1) - 2$ and the ends of every minus critical path in G_1 is also connected by a path of length $\leq \mathscr{S}(G_1) - 1$.*

Theorem 3.2.4. *Let $G(W, \mu, \sigma)$ be the corona of two strong fuzzy graphs $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ on the vertex sets $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1\}$ respectively. Suppose there exists no minus extra strong path in G_1. Let $\mu_2(v_1) < \sigma_0$ and $\mathscr{S}(G_1) \geq 4$. Then $\mathscr{S}(G) = \mathscr{S}(G_1) + 1$ provided the ends of every critical path is also connected by a path of length $\mathscr{S}(G_1) - 1$ and there exists a critical* *path in G whose ends are connected by paths of length $\mathscr{S}(G)$ and $\mathscr{S}(G)-1$ only in G_1.*

Theorem 3.2.5. *Let $G(W,\mu,\sigma)$ be the corona of two strong fuzzy graphs $G_1(U,\mu_1,\sigma_1)$ and $G_2(V,\mu_2,\sigma_2)$ on the vertex sets $U=\{u_1,\ldots,u_n\}$ and $V=\{v_1\}$ respectively. Suppose there exists a minus extra strong path in G_1. If $\mu_2(v_1)<\sigma_0$ and $\mathscr{S}(G_1)\geq 4$ then $\mathscr{S}(G)=\mathscr{S}(G_1)+1$ provided that the ends of every critical extra strong path in G_1 is connected by a path of length $\leq \mathscr{S}(G_1)-1$ and either there exists a minus critical extra strong path whose ends are connected only by paths of length $\mathscr{S}(G)-1$ or there exists a critical extra strong path whose ends are connected only by paths of length $\geq \mathscr{S}(G)-1$.*

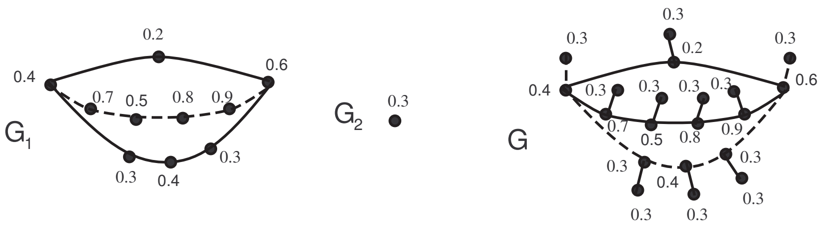


Figure 3.7: Corona of two fuzzy graphs G_1 and G_2 with $\mathscr{S}(G_1)=5$, $\mathscr{S}(G)=6$.

Theorem 3.2.6. *Let $G(W,\mu,\sigma)$ be the corona of two strong fuzzy graphs $G_1(U,\mu_1,\sigma_1)$ and $G_2(V,\mu_2,\sigma_2)$ on the vertex sets $U=\{u_1,\ldots,u_n\}$ with $|U|>1$ and $V=\{v_1\}$ respectively. Suppose that there exists a critical extra strong path P in G_1 such that either its ends are joined by only one path in G_1 or every other paths in G_1 which joins the ends of P with strength $\geq \mu_2(v_1)$ is of length $\geq$ that of P. Then $\mathscr{S}(G)=\mathscr{S}(G_1)+2$.*

Proof. Let u, v be two nonadjacent vertices of G. If u and $v \in V(G_1)$ then length of the extra strong $u-v$ path in G is less than or equal to $\mathscr{S}(G_1)$. Now suppose that there exists a critical extra strong path P in G_1 such that its ends are joined by only one path in G_1. Let u_i and u_j be the ends of P. Then there exists only one path in G joining v_{1i} and v_{1j}. Therefore it itself is an extra strong $v_{1i} - v_{1j}$ path in G and its length is $\mathscr{S}(G_1)+2$. Therefore in this case $\mathscr{S}(G) = \mathscr{S}(G_1)+2$.

In the second case also, we suppose that u_i and u_j are the ends of P. Then the extra strong $v_{1i} - v_{1j}$ path in G is the path obtained by adding the edges $v_{1i}u_i$ and u_jv_{1j} at the ends u_i and u_j of the path P to P respectively. Therefore in this case also $\mathscr{S}(G) = \mathscr{S}(G_1) + 2$. □

Theorem 3.2.7. *Let $G(W, \mu, \sigma)$, $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ be as in Theorem 3.2.6. If there exists a critical path P in G_1 of strength $\leq \mu_2(v_1)$ then $\mathscr{S}(G) = \mathscr{S}(G_1) + 2$.*

Proof. Suppose there exists a critical path P of strength $\leq \mu_2(v_1)$ in G_1 with ends u_i and u_j. Then the $v_{1i} - v_{1j}$ path in G obtained by adding the edges $v_{1i}u_1$ and u_jv_{1j} at u_i and u_j respectively of P to P is an extra strong $v_{1i} - v_{1j}$ path of length $\mathscr{S}(G_1) + 2$. So we can conclude that $\mathscr{S}(G) = \mathscr{S}(G_1) + 2$. □

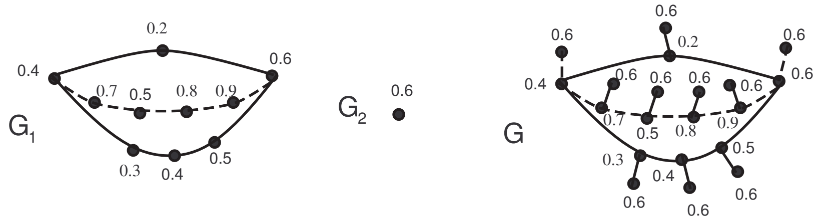


Figure 3.8: Corona of two fuzzy graphs G_1 and G_2 with $\mathscr{S}(G_1) = 5$ and $\mathscr{S}(G) = 7$.

Theorem 3.2.8. *Let $G_1(U, \mu_1, \sigma_1)$ be a strong fuzzy graph. Suppose that the underlying crisp graph is a path with vertex set $U = \{u_1, u_2, \ldots, u_n\}$. Let $G_2(V, \mu_2, \sigma_2)$ be a fuzzy null graph with vertex $V = \{v_1, v_2, \ldots, v_m\}$ with $|V| > 1$. Let $G(W, \mu, \sigma)$ be the corona of G_1 and G_2. Then $\mathscr{S}(G) = \mathscr{S}(G_1) + 2$.*

Proof. When $|V| = 1$, the result follows by Theorem 3.2.6. When $|V| > 1$, let u and v be two nonadjacent vertices of G. If both of them are in G_1 then the length of the extra strong $u - v$ path is $\leq \mathscr{S}(G_1)$. If u and v are in the same copy of G_2, say G_{2i} of G then uu_iv is the only extra strong path joining them.

If u and v are in different copies of G_2 in G say $u \in G_{2i}$ and $v \in G_{2j}, i \neq j$ then every $u - v$ path in G is a union of the edge uu_i, $u_i - u_j$ path in the partial fuzzy subgraph of G_1 of G and the edge u_jv. So length of an extra strong $u - v$ path is equal to (length of the $u_i - u_j$ path in G_1)+2. If $u_i = u_1$ and $u_j = u_n$ then the length of the extra strong $u - v$ path in $G = \mathscr{S}(G_1) + 2$. If $u \in V(G_{2i})$ and $v \in V(G_1)$ then the length of the extra strong $u - v$ path is clearly $\leq \mathscr{S}(G_1) + 2$. (See Figure 3.9).

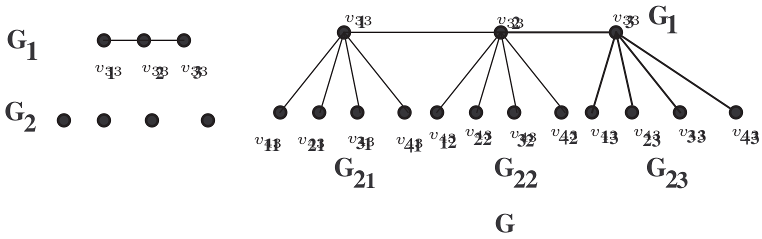


Figure 3.9: Two fuzzy graphs G_1 and G_2 and their corona.

□

Suppose $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ are two strong fuzzy paths on n and m vertices respectively. Let $G(W, \mu, \sigma)$ be the corona of G_1 and G_2. Then G is a path on 2 vertices if $n = m = 1$. So in this case $\mathscr{S}(G) = 1$. When $n = 2$ and $m = 1$, G is a path on 4 vertices. So in this case $\mathscr{S}(G) = 3$. When $n = 1$ and $m = 2$, G is a fuzzy cycle on 3 vertices. Hence $\mathscr{S}(G) = 1$. When $n = 2, m = 2$, G is a $1-$ linked fuzzy graph with 3 parts. Therefore its strength is 3. Theorem 3.2.9 gives the general case.

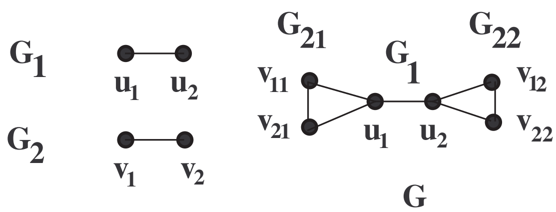


Figure 3.10: Corona G of two fuzzy graphs G_1 and G_2 with $|G_1| = |G_2| = 2$.

Theorem 3.2.9. *Let $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ be two strong fuzzy paths with P_n and P_m be their respective crisp graphs, where n and $m \geq 3$. For each vertex u_i of G_1, let l_i be the maximum length of subpaths of G_{2i} whose strength $> \mu_1(u_i)$ and let $l = \overset{n}{\underset{i=1}{\vee}} l_i$. If there is no such path, let $l_i = 0$. Then the strength of the corona $G(W, \mu, \sigma)$ of G_1 and G_2 is $(n + 1) \vee l$.*

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$. Let u and v be two nonadjacent vertices of $G(V, \mu, \sigma)$. If u and v are in G_1 then the extra strong $u - v$ path in G lies in G_1 and hence its length $\leq \mathscr{S}(G_1) = n - 1$.

Let $u \in V(G_{2i})$ and $v \in V(G_1)$. Then all the $u - v$ paths must pass through u_i of G_1. Since u and u_i are adjacent the only extra strong $u - v$ path in G is the union of the edge uu_i and the path $u_i - v$ of G_1. So, the length of the extra strong $u - v$ path is $\leq \mathscr{S}(G_1) + 1 = n$.

Let $u \in V(G_{2i})$ and $v \in V(G_{2j}), i \neq j$. Then all the $u - v$ paths in G must pass through both u_i and u_j of G_1. So in this case the extra strong $u - v$ path in G is a union of the edge uu_i of G, the path $u_i - u_j$ of G_1 and the edge $u_j v$ of G. So, length of the extra strong $u - v$ path in G is $\leq \mathscr{S}(G_1) + 2 = n + 1$.

Let $u, v \in V(G_{2i})$. If $l_i \geq 2$, then the length of any extra strong $u - v$ path in G is $\leq l_i$. If u and v are the end vertices of a subpath of G_{2i} of length l_i such that if strength $> \mu_1(u_i)$ then the length of extra strong $u - v$ path is l_i. Otherwise, that is if $l_i \leq 1$, it is 2. Hence the Theorem. □

Theorem 3.2.10. *Let $G_1(U, \mu_1, \sigma_1)$ and $G_2(V, \mu_2, \sigma_2)$ be two strong fuzzy but-*

terfly graphs. Then the strength of corona $G(W, \mu, \sigma)$ of G_1 and G_2 is 4.

Proof. Let w be the central vertex of G_1 and w' be the central vertex of G_2. Let the vertices of G_1 be $\{u_1, u_2, u_3, u_4, w\}$ and vertices of G_2 be $\{v_1, v_2, v_3, v_4, w'\}$. Let us denote the copy of G_2 corresponding to w by G_{2w}. Let u, v be two nonadjacent vertices of G.

If u and v are in G_1 the length of extra strong $u - v$ path in G is ≤ 2.

Let $u = v_{ij}$ and $v = v_{ik}$ where $j \neq k$. Then, both u and v are adjacent to $u_i \in V(G_1)$ in G. Also all the $u-v$ paths either pass through u_i of G_1 or through the central vertex w' of G_{2i} in G. If $\mu_1(u_i) = \mu_2(w')$ then all the $u - v$ paths have same strength. So the length of the extra strong path is 2.

If $\mu_1(u_i) > \mu_2(w')$ then the extra strong path does not pass through w'. Therefore the extra strong $u - v$ path is uu_iv. If $\mu_1(u_i) < \mu_2(w')$ then the extra strong $u - v$ paths lie completely in G_{2i}. Therefore such paths have length 2. Now let us suppose that u and v be in two different copies of G_2. If $u \in G_{2i}$ and $v \in G_{2j}$ then the length of the extra strong $u - v$ path in G is 4. On the other hand if $u \in G_{2i}$ and $v \in G_{2w}$ then length of the extra strong $u-v$ path is 3. Also if u_i or u_j is w, the length of the extra strong $u - v$ path is 3. So $\mathscr{S}(G) = 4$. □

3.3 Fuzzy Subdivision graph

Definition 3.3.1. [?] Let $G(V, \mu, \sigma)$ be a fuzzy graph with underlying crisp graph $G(V, E)$. Then the subdivision graph of G, denoted by $sd(G)$, is the fuzzy graph $sd(G)(V_{sd}, \mu_{sd}, \sigma_{sd})$ with the underlying crisp graph is the subdivision graph of $G(V, E)$, where the vertex set $V_{sd} = V \cup E$ and the membership functions μ_{sd} and σ_{sd} are defined as

$$\mu_{sd}(u) = \begin{cases} \mu(u) & \text{if } u \in V, \\ \sigma(u) & \text{if } u \in E. \end{cases}$$

$$\sigma_{sd}(u, e) = \begin{cases} \mu_{sd}(u) \wedge \mu_{sd}(e) & \text{if } u \in V, e \in E \text{ and } u \text{ is one of the end vertices of } e \ in \text{ G}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.3.1. *Let G be a strong fuzzy path on n vertices. Then the strength $\mathscr{S}(sd(G))$ of the subdivision graph $sd(G)$ of G is $2\mathscr{S}(G)$.*

Proof. The subdivision graph of a strong fuzzy path on n vertices is a strong fuzzy path on $2n-1$ vertices. (See Figure 3.11). So strength of $sd(G)$ is $(2n-1)-1 = 2(n-1) = 2\mathscr{S}(G)$. □

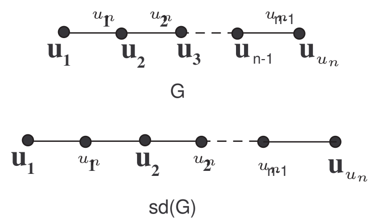


Figure 3.11: A strong fuzzy path and its subdivision graph.

Theorem 3.3.2. *Let $G(V, \mu, \sigma)$ be a strong fuzzy butterfly graph. Then the strength $\mathscr{S}(sd(G))$ of the subdivision graph of G is 6.*

Proof. Let the vertex set of G be $\{u_1, u_2, u_3, u_4, u_5\}$ with u_3 as the central vertex. Then G is a $1-$ linked fuzzy graph with two parts G_1 and G_2, where both G_1 and G_2 are fuzzy cycles on 3 vertices. Its subdivision graph is also a $1-$ linked fuzzy graph with two parts which are cycles on 6 vertices. (See Figure 3.12). Since each part $G_i, (i = 1, 2)$ of G has at least two weakest edges of G_i, $sd(G_i), i = 1, 2$ has at least 4 weakest edges in $sd(G_i)$.

Let u, v be any two vertices of $sd(G)$. If both u and $v \in V(sd(G_i)), i = 1, 2$ then any extra strong path joining u and v lie completely in $sd(G_i), i = 1$ or 2. So the strength of the $u - v$ path in G is 3 by Theorem 1.4.2. Since $u_3 \in V(G_1) \cap V(G_2)$, $u_3 \in V(sd(G_1)) \cap V(sd(G_2))$. If $u \in sd(G_1) \setminus \{u_3\}$ and $v \in sd(G_2) \setminus \{u_3\}$ then all the $u - v$ paths can be considered as the union of two paths P_1 of $sd(G_1)$ joining u to u_3 and P_2 of $sd(G_2)$ joining u_3 to v. Therefore, the length of any extra strong the $u - v$ path is less than or equal to the length of any extra strong

$u - u_3$ path in $sd(G_1)$ and $u_3 - v$ path in $sd(G_2)$ which is less than or equal to $3 + 3 = 6$. Also when u is the vertex $\in V(sd(G))$ corresponding to the edge u_1u_2 in G_1 and v is the vertex $\in V(sd(G))$ corresponding to the edge u_4u_5 in $V(G_2)$ the strength of the $u - v$ path is exactly 6.

Hence the theorem. □

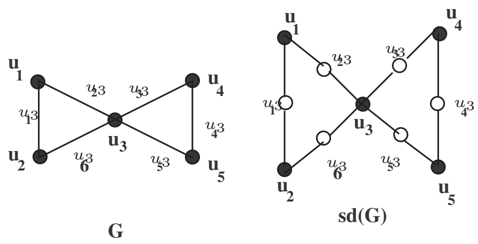


Figure 3.12: A strong fuzzy butterfly graph and its subdivision graph.

Theorem 3.3.3. *Let G be a strong fuzzy Bull graph then the strength $\mathscr{S}(sd(G))$ of the subdivision graph of G is* 6.

Proof. A fuzzy bull graph $G(V, \mu, \sigma)$ is a $1-$linked fuzzy graph with three parts. Let P, P' and P'' be its parts, where P and P'' are fuzzy paths on two vertices and P' is a fuzzy triangle. Then $sd(G)$ is also a $1-$ linked fuzzy graph with parts $G_1 = sd(P), G_2 = sd(P')$ and $G_3 = sd(P'')$. (See Figure 3.13).

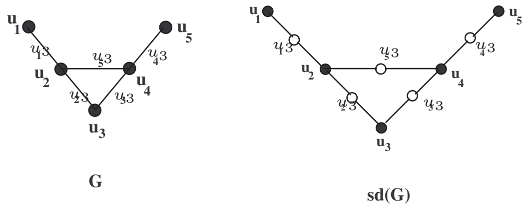


Figure 3.13: A strong fuzzy Bull graph G and its subdivision graph $sd(G)$.

Let u and v be any two non-adjacent vertices of $sd(G)$. If $u, v \in V(G_1)$ or $u, v \in V(G_3)$ then the length of any extra strong $u - v$ path in G is 2. Since both $sd(P)$ and $sd(P'')$ are paths on 3 vertices.

If $u, v \in V(G_3)$ then all the paths joining u and v lie completely in G_3. Since G_3 is the subdivision graph of the strong fuzzy triangle P'', it is a strong fuzzy cycle on 6 vertices. As P' contains at least 2 weakest edges, $sd(G_3)$ contains at least 4 weakest edges. Therefore by Theorem 1.4.2, the length of the extra strong $u - v$ path in G_3 is 3.

Let $\{w\} = V(G_1) \cap V(G_3)$ and $\{w'\} = V(G_2) \cap V(G_3)$. If $u \in V(G_1)$ and $v \in V(G_2)$ then all the $u - v$ paths pass through both w and w' in $sd(G)$. Since w and w' are adjacent in G, the extra strong path joining w and w' in $sd(G)$ is wew' where e is the vertex in $sd(G)$ corresponding to the edge ww' in G. So the length of the extra strong path joining u and v is $\leq 2 + 2 + 2 = 6$. When u and v are the pendant vertices of G then the extra strong $u - v$ path has length exactly 6. Therefore $\mathscr{S}(G) = 6$.

□

For a strong fuzzy tree G, the strength of G is the diameter of the underlying crisp graph of G. The subdivision graph of a fuzzy star graph is a fuzzy tree. It is immediate from the definition of fuzzy star graph and subdivision of a fuzzy graph the strength of the subdivision graph of a fuzzy star graph is 4.

Theorem 3.3.4. *The strength of the subdivision graph of a fuzzy star graph [56] G is 4.*

Note 3.3.1. Let G be a strong fuzzy cycle on n vertices with l weakest edges in G having weight w. Then the edges in $sd(G)$ incident with that vertices of $sd(G)$ corresponding to weakest edges of G are of weight w. Therefore in $sd(G)$, there are $2l$ weakest edges.

Theorem 3.3.5. *Let G be a strong fuzzy cycle of length n, which contains l weakest edges and which do not contain any weakest edge of G. Then the strength, $\mathscr{S}(sd(G))$, of the subdivision graph of G is $2\mathscr{S}(G)$.*

Proof. We have by Note 3.3.1, for a strong fuzzy cycle G of length n, if there are l weakest edges which altogether form a subpath in G then there are $2l$ weakest edges which altogether form a subpath in $sd(G)$.

By Theorem 1.4.2 if $2l \leq [\frac{2n+1}{2}]$ then $\mathscr{S}(sd(G)) = 2n - 2l = 2(n-l)$. If $2l \leq [\frac{2n+1}{2}]$ then $l \leq [\frac{n+1}{2}]$ so $\mathscr{S}(sd(G)) = 2\mathscr{S}(G)$. Also by Theorem 1.4.2 if $2l > [\frac{2n+1}{2}]$ then $\mathscr{S}(sd(G)) = [\frac{2n}{2}] = 2[\frac{n}{2}]$. We have $2l > [\frac{2n+1}{2}]$ implies $l \geq [\frac{n+1}{2}]$ so $\mathscr{S}(sd(G)) = 2\mathscr{S}(G)$.

Suppose there are l weakest edges which do not altogether form a subpath in G. Then the $2l$ weakest edges of $sd(G)$ also do not form a subpath in $sd(G)$. So by Theorem 1.4.3 if $2l > [\frac{2n}{2}] - 1$ then $\mathscr{S}(sd(G)) = [\frac{2n}{2}]$.

But if $2l > [\frac{2n}{2}] - 1$ then $l > [\frac{n}{2}] - 1$. Hence in this case $\mathscr{S}(sd(G)) = 2\mathscr{S}(G)$.

Similarly if $2l < [\frac{2n}{2}] - 1$ then $\mathscr{S}(sd(G)) = [\frac{2n}{2}]$. Also $2l < [\frac{2n}{2}] - 1$ implies $l \leq [\frac{n}{2}] - 1$. So $\mathscr{S}(sd(G)) = [\frac{2n}{2}] = 2\mathscr{S}(G)$.

Hence the proof. □

Theorem 3.3.6. *The strength of the subdivision graph of a strong fuzzy diamond graph is* 4.

Proof. Let G be a strong fuzzy diamond graph with vertex set $\{u_1, u_2, u_3, u_4\}$. Also let u and v be two non-adjacent vertices of $sd(G)$.

Case 1. $u, v \in V(G)$.

If u and v are adjacent in G and e be the edge joining u and v in G then the strength of the $u - v$ path is less than or equal to $\mu_{sd}(u) \wedge \mu_{sd}(v)$ and which is equal to $\sigma_{sd}(e)$. So the path uev is the extra strong path joining u and v in $sd(G)$ which is of length 2. If u and v are non-adjacent vertices in G then $u, v \in \{u_2, u_4\}$ as shown in Figure 3.14. Suppose that $u = u_2$ and $v = u_4$.

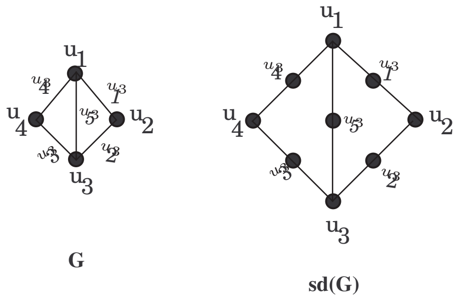


Figure 3.14: A strong fuzzy diamond graph G and its subdivision graph $sd(G)$.

Then any extra strong path joining u and v in $sd(G)$ pass either through u_1 or through u_3 depending up on their weights. Without loss of generality assume that $\mu(u_1) \geq \mu(u_3)$. Since u_1 is adjacent to both u and v and e_1 is the edge uu_1 and e_4 is the edge u_1v in G, so $ue_1u_1e_4v$ is an extra strong path in $sd(G)$ and it is of minimum length among all the other strong paths.

Case 2. $u, v \in E(G)$.

If u and v have a common vertex w in G then in $sd(G)$ the path uwv is an extra strong path since, the strength of all the paths joining u and v in $sd(G)$ have strength $\leq \mu_{sd}(u) \wedge \mu_{sd}(v)$ and $\mu_{sd}(w) \geq \mu_{sd}(u) \wedge \mu_{sd}(v)$. So the length of the extra strong path joining u and v is 2.

Otherwise, suppose u and v have no common vertex in G then u and $v \in \{e_1, e_3\}$ or $\{e_2, e_4\}$. (See Figure 3.14). Without loss of generality assume that u and $v \in \{e_1, e_3\}$. In this case all the $u - v$ paths have strength less than or equal to $\mu_{sd}(u) \wedge \mu_{sd}(v) = \mu_{sd}(u_1) \wedge \mu_{sd}(u_2) \wedge \mu_{sd}(u_3) \wedge \mu_{sd}(u_4)$. Therefore, the length of the extra strong path is the minimum distance between u and v which is 4.

Case 3. $u \in V(G)$ and $v \in E(G)$.

Without loss of generality assume that $u = u_1$ and $v = e_3$ where e_3 is the vertex in $sd(G)$ corresponding to the edge u_3u_4 in G. (See Figure 3.14). Then strength of each $u - v$ path in $sd(G) \leq \mu_{sd}(u) \wedge \mu_{sd}(v) = \mu(u_1) \wedge \mu(u_3) \wedge \mu(u_4)$. So the extra strong path joining u and v lies completely in the maximal partial fuzzy subgraph of $sd(G)$ with vertex set $\{u_1, u_3, u_4, e_3, e_4, e_5\}$, which is a strong fuzzy cycle on 6 vertices. So the length of the extra strong path joining u and v is 3.

□

Theorem 3.3.7. *Let G be a fuzzy complete graph. Then the strength $\mathscr{S}(sd)(G)$ is 3 for $n = 3$ and 4 for $n > 3$.*

Proof. When $n = 3$, $sd(G)$ is a strong fuzzy cycle on 6 vertices having at least 4 weakest edges. So the the result follows by Theorem 1.4.2.

Consider the case, $n > 3$. Let u, v be two non-adjacent vertices of $sd(G)$. If u and v are the vertices of G then the extra strong path joining u and v is uev where e is the edge uv in G and is of length 2.

If u and v are edges of G then, if they have a common vertex w in G then the path uwv in $sd(G)$ is of strength exactly equal to $\mu_{sd}(u) \wedge \mu_{sd}(v)$, which is an extra strong path joining them in $sd(G)$.

Suppose $u = u_j u_k$ and $v = u_l u_m$ are edges of G and have no vertex in common. The weight of the edges uu_j and uu_k are the same and the weight of the edges vu_l and vu_m are the same in $sd(G)$ and the extra strong path joining any two vertices u' and u'' of G in $sd(G)$ is $u'eu''$, where e is the edge joining u' and u'' in G. So all the $u-v$ paths must have same strength in $sd(G)$. Therefore, the length of the extra strong path joining u and v is the length of the shortest $u-v$ path in $sd(G)$, which is 4. □

3.4 Fuzzy middle graph

Definition 3.4.1. [29] Let $G(V, \mu, \sigma)$ be a fuzzy graph with its underlying crisp graph $G(V, E)$. The fuzzy middle graph of G is denoted by $M(G)(V_M, \mu_M, \sigma_M)$ with crisp graph $M(G)(V_M, E_M)$, where the vertex set $V_M = V \cup E$ and edge set $E_M = \{uv :$ either u and v are two adjacent edges of G or $u \in V$ and $v \in E$ with u as one end vertex of $v\}$,

$$\mu_M(u) = \begin{cases} \mu(u) & \text{if } u \in V, \\ \sigma(u) & \text{if } u \in E. \end{cases}$$

and

$$\sigma_M(uv) = \begin{cases} \sigma(u) \wedge \sigma(v) & \text{if } u, v \text{ are two adjacent edges of } G, \\ \sigma(v) & \text{if } u \in V \text{ and } v \in E \text{ with } u \text{ as one end vertex of } v. \end{cases}$$

From the definition of middle graph of a fuzzy graph it is clear that the middle graph of a strong fuzzy path on n vertices is a $1-$ linked fuzzy graph with n parts, each of which is a complete fuzzy graph. So by Lemma 2.3.2 the strength of middle graph of a strong fuzzy path on n vertices is n.

Theorem 3.4.1. *Let $G(V,\mu,\sigma)$ be a complete strong fuzzy graph with $M(G)(V_M, \mu_M,\sigma_M)$ its fuzzy middle graph. Then*

$$\mathscr{S}(M(G)) = \begin{cases} 0 & \text{if } |V| = 1, \\ 2 & \text{if } |V| \geq 2. \end{cases}$$

Proof. Let $G(V,\mu,\sigma)$ be a complete fuzzy graph with middle graph $M(G)(V_M,\mu_M, \sigma_M)$. Let $\{u_1,u_2,\ldots,u_n\}$ be the vertex set and $\{e_1,e_2,\ldots,e_{\frac{n(n-1)}{2}}\}$ be the edge set of G. If $|V| = 1$ then G and $M(G)$ are fuzzy trivial graphs. Hence $\mathscr{S}(M(G)) = \mathscr{S}(G) = 0$.

If $|V| = 2$ then $M(G)$ is a path on 3 vertices. Hence $\mathscr{S}(G) = 2$.

Suppose that $|V| > 2$. Let u,v be two non - adjacent vertices of $M(G)$.

Case 1. $u,v \in V(G)$.

Since all the vertices of G are adjacent, $e = uv$ is an edge of G and therefore it is a vertex of $M(G)$. By the definition of $M(G)$, e is adjacent to both u and v in $M(G)$ and $\mu_M(e) = \sigma(e) = \mu(u) \wedge \mu(v)$. As all the paths joining u and v

in $M(G)$ have strength less than or equal to $\mu(u) \wedge \mu(v)$, uev is an extra strong $u - v$ path in $M(G)$.

Case 2. $u, v \in E(G)$.

Suppose u_l, u_k, u_m and $u_j \in V(G)$ such that $u = u_l u_k$ and $v = u_m u_j$, in G. Then any path joining u and v have strength $\leq \mu(u_l) \wedge \mu(u_k) \wedge \mu(u_m) \wedge \mu(u_j) = \mu_\circ$(say). If w is the edge $u_m u_l$ or $u_m u_k$ or $u_j u_l$ or $u_j u_k$ of G then uwv is a path in $M(G)$ with strength $\mu_\circ$. So the length of any extra strong $u - v$ path in $M(G)$ is 2.

Case 3. $u \in V(G)$ and $v \in E(G)$.

Clearly all the $u - v$ paths in $M(G)$ must have strengths $\leq \mu_M(u) \wedge \mu_M(v)$. Let w be one of the end vertices of v. Since G is complete, u is adjacent to w. Therefore $e = uw$ is an edge of G. Hence it is a vertex of $M(G)$ adjacent to both u and v in $M(G)$. Thus uev is a $u - v$ path in $M(G)$ having strength $\mu_M(u) \wedge \mu_M(v)$. Therefore uev is an extra strong $u - v$ path in $M(G)$. Hence the theorem. □

Theorem 3.4.2. *Let $G(V, \mu, \sigma)$ be a strong fuzzy star graph and $M(G)(V_M, \mu_M, \sigma_M)$ be its fuzzy middle graph. Then*

$$\mathscr{S}(M(G)) = \begin{cases} 0 & \text{if } |V| = 1, \\ 2 & \text{if } |V| = 2, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. If $|V| = 1$ or 2 then G is a complete fuzzy graph. Therefore the result follows from Theorem 3.4.1. So suppose that $|V| \geq 3$. Let $V = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of G with u_n as the central vertex and $\{e_1, e_2, \ldots, e_{n-1}\}$ be the edge set of G with $e_k = u_k u_n$. Then $\{u_1, u_2, \ldots, u_n, e_1, e_2, \ldots, e_{n-1}\}$ be the vertex set of the middle graph $M(G)$ of G (See Figure 3.15).

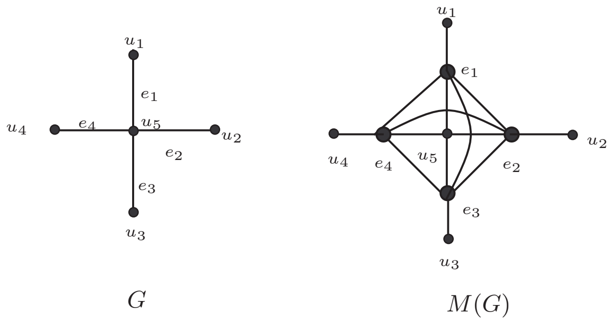


Figure 3.15: A strong fuzzy star graph and its middle graph $M(G)$.

Let u and v be two non-adjacent vertices of $M(G)$. Then we have the following three cases:

i Both u and v are pendant vertices of G.

ii One of u and v is a pendant vertex and the other is the central vertex of G .

iii One of u and v say u is a pendant vertex of G and other is a vertex of $M(G)$ which corresponds to an edge in G with u is not an end vertex.

In the first case, let us suppose that $u = u_i$ and $v = u_j$, where $1 \leq i \neq j \leq n-1$. Then all the $u-v$ paths pass through both the vertices e_i and e_j in $M(G)$. Since the middle graph of a strong fuzzy graph is strong fuzzy, $M(G)$ is a strong fuzzy graph. As e_i and e_j are the support vertices of u and v in $M(G)$ respectively, all $u-v$ paths in $M(G)$ must pass through these two vertices. Thus the $u-v$ path ue_ie_jv of $M(G)$ will be an extra strong $u-v$ path with length 3.

In the second case, without loss of generality assume $u = u_i$ and $v = u_n, i < n$. All the $u-v$ paths must pass through e_i. As e_i is adjacent to both u and v, ue_iv is an extra strong $u-v$ path with length 2.

In the last case, we suppose that $u = u_i$, where $i \neq n$ and $v = e_j = u_nu_j$, where $j \neq i$. Here, ue_iv is an extra strong $u-v$ path in G of length 2. Hence the Theorem. □

Theorem 3.4.3. *Let $G(V, \mu, \sigma)$ be a strong fuzzy diamond graph and $M(G)(V_M, \mu_M, \sigma_M)$ be its fuzzy middle graph. Then $\mathscr{S}(M(G)) = 3$.*

Proof. Let $V = \{u_1, u_2, u_3, u_4\}$ be the vertex set and $E = \{e_1, e_2, \ldots, e_5\}$ where $e_1 = u_1u_2, e_2 = u_2u_3, e_3 = u_3u_4, e_4 = u_4u_1, e_5 = u_1u_3$ be the edge set of G. Then $V_M = \{u_1, u_2, \ldots, u_4, e_1, e_2, \ldots, e_5\}$ (See Figure 3.16).

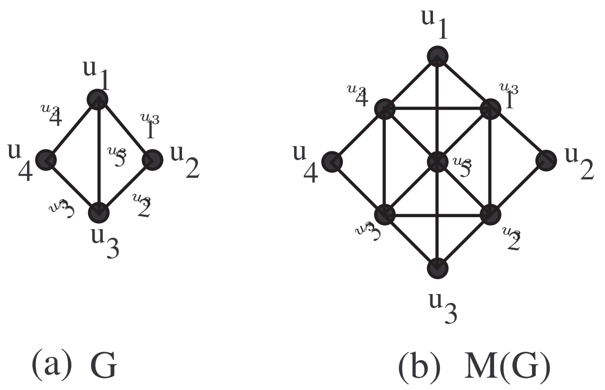


Figure 3.16: A fuzzy diamond graph G and its middle graph $M(G)$.

Let u, v be two nonadjacent vertices of $M(G)$. Then

Case 1. u and v are two adjacent vertices in G.

Suppose $u = u_1$ and $v = u_2$. Since $\sigma(e_1) = \mu(u_1) \wedge \mu(u_2) = \mu_M(e_1)$ and the vertex e_1 is adjacent to both u_1 and u_2 in $M(G)$, $u_1e_1u_2$ is the extra strong path joining u and v. So that the length of the extra strong $u_1 - u_2$ path is 2. In all other cases also the length of the extra strong $u - v$ path is 2.

Case 2. u and v are two non - adjacent vertices in G say $u = u_2$ and $v = u_4$.

Without loss of generality assume that $\mu(u_1) \leq \mu(u_3)$. If $\mu(u_4) \wedge \mu(u_2) \leq \mu(u_1)$ then the length of an extra strong path must be the minimum length of the path joining u and v, which is 3.

If $\mu(u_4) \wedge \mu(u_2) > \mu(u_1)$ then $u_2e_2e_3u_4$ is an extra strong $u - v$ path and is of length equal to 3.

Case 3. u and v are two non-adjacent edges in G.

Then $u, v \in \{e_1, e_3\}$ or $u, v \in \{e_2, e_4\}$ in $M(G)$. So all the $u - v$ paths have strength $= \mu_M(e_1) \wedge \mu_M(e_3) = \mu(u_1) \wedge \mu(u_2) \wedge \mu(u_3) \wedge \mu(u_4)$. Therefore each $e_1 e_i e_3, i = 2$ or 4 or 5 is an extra strong path and is of length 2.

Case 4. u is a vertex of G and v is an edge of G.

Because of the symmetry we need only to consider the case $u = u_1$ and $v = e_3$. As all extra strong $u - v$ path have strength $\leq \mu_M(u_1) \wedge \mu_M(e_3) = \mu(u_1) \wedge \mu(u_3) \wedge \mu(u_4)$ and since $\mu_M(e_5) = \sigma(e_5) = \mu(u_1) \wedge \mu(u_3)$, $u_1 e_5 u_3$ is an extra strong $u - v$ path of length 2. Hence the theorem. □

3.5 Total fuzzy graph, fuzzy split graph and fuzzy shadow graph

This section discusses the strength of total fuzzy graph, fuzzy split graph and fuzzy shadow graph.

3.5.1 Total fuzzy graph

The total graph of a graph $G(V, E)$ is the graph with vertex set $V \cup E$ and two vertices are adjacent, whenever they are either adjacent or incident in G [53].

Definition 3.5.1. [28] Let $G(V, \mu, \sigma)$ be a fuzzy graph with underlying crisp graph $G(V, E)$. Then the total fuzzy graph of G, denoted by $T(G)$ is the fuzzy graph $T(G)(V_T, \mu_T, \sigma_T)$ with the underlying crisp graph is the total graph of $G(V, E)$, where the vertex set $V_T = V \cup E$ and the membership functions μ_T and σ_T are defined as

$$\mu_T(u) = \begin{cases} \mu(u) & \text{if } u \in V, \\ \sigma(u) & \text{if } u \in E. \end{cases}$$

and for $u, v \in V_T$,

$$\sigma_T(uv) = \begin{cases} \sigma(uv) & \text{if } u, v \in V, \\ \sigma(u) \wedge \sigma(v) & \text{if } u, v \in E \text{ and have a common vertex,} \\ \mu(u) \wedge \sigma(v) & \text{if } u \in V, v \in E \text{ and } u \text{ is a vertex incident with} E, \\ 0 & \text{otherwise.} \end{cases}$$

If G is a trivial or a null fuzzy graph then $T(G)$ is G and hence $\mathscr{S}(T(G)) = \mathscr{S}(G) = 0$. If G is a strong fuzzy path on 2 vertices then $T(G)$ is a complete strong fuzzy graph on 3 vertices. Hence $\mathscr{S}(T(G)) = 1$ by Theorem 1.4.1.

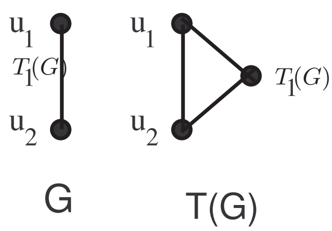


Figure 3.17: A fuzzy path on 2 vertices and its total fuzzy graph.

Theorem 3.5.1. *Let $G(V, \mu, \sigma)$ be a strong fuzzy graph with its underlying crisp graph $G(V, E)$, a path on $n > 2$ vertices. Then the strength of $T(G)$ is $n - 1$.*

Proof. Let us suppose that $V = \{u_1, u_2, \ldots .u_n\}$ and $E = \{e_1, e_2, \ldots, e_{n-1}\}$ where $e_i = u_i u_{i+1}, i = 1, 2, \ldots, n - 1$.

Let u, v be two non-adjacent vertices of $T(G)$ (See Figure 3.18).

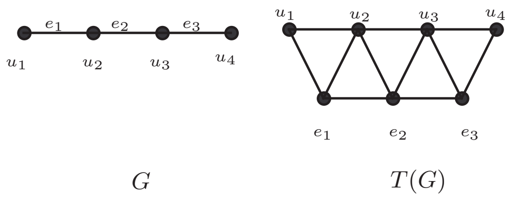


Figure 3.18: A fuzzy path on 4 vertices and its total fuzzy graph.

Case 1. $u, v \in \{u_1, u_2, \ldots, u_n\}$ or $u, v \in \{e_1, e_2, \ldots, e_{n-1}\}$.

First of all suppose that $u, v \in \{u_1, u_2, \ldots, u_n\}$. Let us suppose that $u = u_i$ and $v = u_j, i < j$. Clearly any extra strong $u - v$ path lie in the maximal partial fuzzy subgraph G' with vertex set $\{u_i, e_i, u_{i+1}, e_{i+1}, \ldots, u_j, e_j\}$. In this case there is only one extra strong $u - v$ path, namely $u_i u_{i+1} \ldots u_j$ If possible, let k be the least positive integer $\geq i$ such that e_k belongs to the vertex set of an extra strong $u - v$ path P in $T(G)$. Let $e_k e_{k+1} \ldots e_{k+h}$ be the maximal subpath of P which lies in the path $e_1 e_2 \ldots e_{n-1}$ beginning at e_k.

Then $P_1 = u_k u_{k+1} e_k e_{k+1} \ldots e_{k+h} u_{k+h+1}$ or $P_2 = u_k u_{k+1} e_k \ldots e_{k+h} u_{k+h}$ or

$P_3 = u_k e_k \ldots e_{k+h} u_{k+h+1}$ or $P_4 = u_k e_k \ldots e_{k+h} u_{k+h}$ is a subpath of P. Then by replacing the subpaths P_3 by $u_k u_{k+1} \ldots u_{k+h+1}$ and P_2 and P_4 by the path $u_k u_{k+1} \ldots u_{k+h}$, we get a $u - v$ path of strength $\geq$ that of P and length less than or equal to that of P; a contradiction. So we can conclude that the extra strong $u - v$ path in this case is $u_i u_{i+1} \ldots u_{j-1} u_j$. Therefore the length of the extra strong $u - v$ path is less than or equal to $n - 1$.

Similarly we can prove that if $u, v \in \{e_1, e_2, \ldots, e_{n-1}\}$ then the length of the extra strong $u - v$ path is $\leq n - 2$.

Case 2. $u \in \{u_1, u_2, \ldots, u_n\}$ and $v \in \{e_1, e_2, \ldots, e_{n-1}\}$.

Let $u = u_i$ and $v = e_j, i < j$. In this case every extra strong $u - v$ path must be a subpath of the maximal partial fuzzy subgraph G'' of $T(G)$ with vertex set $u_i, u_{i+1}, \ldots, u_j, e_i, e_{i+1}, \ldots, e_j$. Let us denote the path $u_1 u_2 \ldots u_n$ of $T(G)$ by P_1 and the path $e_1 e_2 \ldots e_{n-1}$ of $T(G)$ by P_2. Also let P be an extra strong $u - v$ path in $T(G)$ which lies in G''. Suppose k is the least positive integer such that $e_k \in V(P)$. Then $i \leq k \leq j$. By case 1 we can conclude that P is $u_i u_{i+1} \ldots u_k e_k \ldots e_j$. Its length is clearly $k - i + j - k + 1 = j - i + 1 \leq n - 1$ and equal to $n - 1$ if $i = 1$ and $j = n$. □

3.5.2 Fuzzy split graph

Definition 3.5.2. [7] For a graph G and a vertex v of G, the neighbourhood set $N(v)$ is defined as the set of all vertices of G which are adjacent to v in G.

Definition 3.5.3. [46] For a graph G the split graph $split(G)$ is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v') = N(v)$.

Unless otherwise specified we denote the vertex corresponding to the vertex v of G in $split(G)$ by v' and the set of all such vertices v' by V'.

Hence the $split(G)$ has the vertex set $V_{split} = V \cup V'$ and the edge set $E_{split} = \{uv : u, v \in V$ and u, v are adjacent in G or $u \in V, v = w' \in V'$ such that u, w are adjacent in $G\}$.

Definition 3.5.4. The fuzzy split graph $split(G)(V_{split}, \mu_{split}, \sigma_{split})$ of a fuzzy graph $G(V, \mu, \sigma)$ is a fuzzy graph with underlying crisp graph $split(G)$ where $\mu_{split}(u) = \mu_{split}(u') = \mu(u)$ for $u \in V$ and $u' \in V'$.

$$\sigma_{split}(uv) = \begin{cases} \sigma(uv) & \text{if } u \text{ and } v \text{ are adjacent in } V, \\ \mu(u) \wedge \mu(v) & \text{if } v = w' \in V' \text{ such that } u \text{ and } w, \\ & \text{are adjacent in } G, \\ 0 & \text{othewise.} \end{cases}$$

For $|V| = 1$, $split(G)$ is a null fuzzy graph on 2 vertices. Therefore its strength is 0.

Theorem 3.5.2. *Let $G(V, \mu, \sigma)$ be a strong fuzzy path on $n > 1$ vertices. Then the strength of $split(G)$, the fuzzy split graph of G is*

$$\mathscr{S}(split(G)) = \begin{cases} n-1 & \text{if } n > 3, \\ 3 & \text{if } n = 2, 3. \end{cases}$$

Proof. When $n = 2$, the split graph of G is a fuzzy path on 4 vertices. Hence strength of $split(G)$ is 3. (See Figure 3.19 (a)).

When $n = 3$, let $V = \{v_1, v_2, v_3\}$ be the vertex set of G as shown in Figure 3.19 (b) and u, v be two nonadjacent vertices of fuzzy split graph of G. If $u = v_1$ and $v = v_3$ then uv_2v and $uv_2'v$ are the only extra strong $u - v$ paths in $split(G)$. If $u = v_1$ and if v is either v_1' or v_3', the respective extra strong $u - v$ paths are $v_1v_2v_1'$ $v_1v_2v_3'$, which is of length 2. If $u = v_i'$ and $v = v_{i+1}', i = 1, 2$, all the $u - v$ paths have length 3. In the case $u = v_1'$ and $v = v_3'$ there is only one $u - v$ path, which is of length 2. Therefore in this case the strength of fuzzy split graph of G is 3.

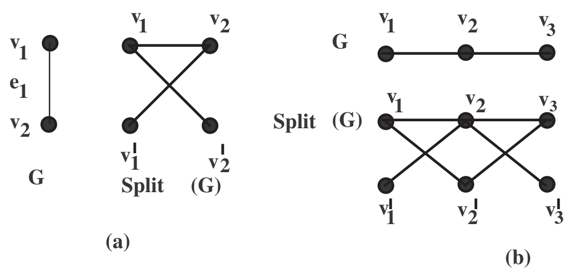


Figure 3.19: (a) A fuzzy path on 2 vertices and its split graph, (b) A fuzzy path on 3 vertices and its split graph.

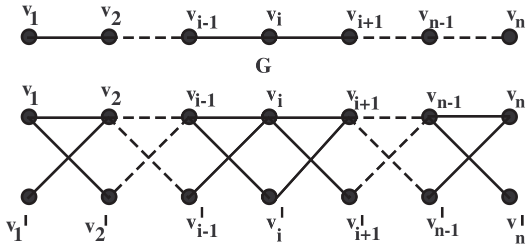


Figure 3.20: A fuzzy path on n vertices and its split graph.

When $n > 3$ we proceed as follows. Let u and v be two nonadjacent vertices of $split(G)$.

Case 1. $u, v \in \{v_1, v_2, \ldots, v_n\}$.

Let $u = v_i$ and $v = v_j, i < j$. Then every extra strong $u - v$ path must be a subpath P of the maximal partial fuzzy subgraph G' of $split(G)$ with vertex set $v_i, v_{i+1}, \ldots, v_j, v'_{i+1}, \ldots, v'_{j-1}$. First of all note that if a $u - v$ path P in $split(G)$ passes through v_{i-1} then it must pass through v'_i and v_{i+1}. In fact $v_i v_{i-1}$, $v_{i-1} v'_i$ and $v'_i v_{i+1}$ are edges of P. In this case by deleting the vertices v_j for $j < i$ and v'_j for $j \leq i$ of P and adding the edge $v_i v_{i+1}$ to P we get new $u - v$ path with less length and, strength not less than that of P, a contradiction. If P passes through v'_i then $v_i v_{i-1}$, $v_{i-1} v'_i$ and $v'_i v_{i+1}$ are edges of P. As above by deleting these edges of P and adding the edge $v_i v_{i+1}$, we get a $u - v$ path with strength not less than that of P but length strictly less than that of P, a contradiction.

Similarly we can prove that P can't pass through any of the vertices $v_k, k > j$

or $v'_k, k \geq j$.

Also from the definition of $split(G)$ the path P must pass through either v_k or $v'_k; k = i+1, \ldots, j-1$. Since $\mu_{split}(v_k) = \mu_{split}(v'_k)$ for $k = 1, 2, \ldots, n$, the path P is of the form $u_i u_{i+1} \ldots u_{j-1} u_j$ where $u_k = v_k$ or $v'_k, k = i+1, \ldots, j-1$. In such a way that if some $u_k = v_k$ then $u_{k+1} = v_{k+1}$ or v'_{k+1} and if some $u_k = v'_k$ then $u_{k+1} = v_{k+1}$. Clearly length of P is $j - i$.

If $u = v_1$ and $v = v_n$ then the length of the extra strong $u - v$ path is equal to $n - 1$. (See Figure 3.19).

Case 2. $u, v \in \{v'_1, v'_2, \ldots, v'_n\}$.

As u is adjacent to v_{i-1} and v_{i+1} and v is adjacent to v_{j-1} and v_{j+1} only and any path from u to v traverse through v_k or v'_k. Then there exist an extra strong path P in the maximal partial fuzzy subgraph of G with vertex set $v'_i, v_{i+1}, v'_{i+1}, \ldots, v_{j-1}, v'_j$.

Clearly P contains either v_k or v'_k but not both for $i \leq k \leq j$. Therefore length of $P = j - i$.

Case 3. $u \in \{v_1, v_2, \ldots, v_n\}$ and $v \in \{v'_1, v'_2, \ldots, v'_n\}$.

Let $u = v_i$ and $v = v'_j$ with $i \leq j$. If $i = j$ then $v = v'_i$. In this case, since v'_i is adjacent to only v_{i+1} and v_{i-1} all the extra strong $u - v$ path must pass through either v_{i-1} or v_{i+1}. If $\mu(v_{i-1}) \geq \mu(v_{i+1}$ then $uv_{i-1}v$ is an extra strong path in G, otherwise $uv_{i+1}v$ is an extra strong path in G.

As in the proof of Case (2) we can conclude that the length of the extra strong path joining u and v is j-i. $\square$

Proposition 3.5.1. Let $G(V, \mu, \sigma)$ be a strong fuzzy complete graph on 3 vertices. If there exists a vertex in G whose weight is strictly less than the weight of the other two vertices then $\mathscr{S}(split(G))$ is 3. Otherwise it is 2.

Proof. Consider a strong fuzzy complete graph with vertex set $\{v_1, v_2, v_3\}$. Let u, v be two nonadjacent vertices of $split(G)$.

Suppose $u = v_i$ and $v = v_i', 1 \leq i \leq 3$. Since v_i is adjacent to all vertices except v_i' and since $\mu(v_j) = \mu_{split}(v_j), \forall j$, in $split(G)$, uv_kv is an extra strong $u - v$ path, where $v_k, k \neq i$ is a vertex of G with $\mu(v_k) \geq max\{\mu(v_j) : j \neq i\}$. Now suppose, $u = v_i'$ and $v = v_j', 1 \leq i \neq j \leq 3$.

Let v_k be the vertex distinct from v_i and v_j. If $\mu(v_k) \geq \mu(v_i) \wedge \mu(v_j)$ then uv_kv is an extra strong $u - v$ path in $split(G)$. Otherwise, that is if $\mu(v_k) < \mu(v_i) \wedge \mu(v_j)$ then uv_jv_iv is an extra strong $u - v$ path in $split(G)$.

If at least two vertices of G have the minimum weight then all edges of $split(G)$ have the same weight. Therefore $\mathscr{S}(split(G)) = 2$.

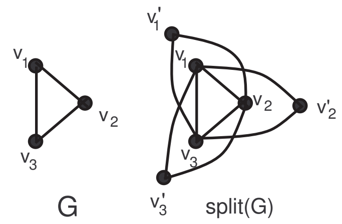


Figure 3.21: A graph G on 3 vertices and its split graph.

□

We generalize Proposition 3.5.1 as follows:

Theorem 3.5.3. *Let $G(V, \mu, \sigma)$ be a strong fuzzy complete graph on $n \geq 3$. If there exist two vertices u, v in G such that $\mu(u) \wedge \mu(v)$ is greater than the strength of all other vertices in G. Then the strength of split(G) is 3. Otherwise it is 2.*

3.5.3 Fuzzy shadow graph

Definition 3.5.5. [54] The shadow graph of a connected graph $G(V, E)$ is constructed by taking two copies of G say $G'(V', E')$ and $G''(V'', E'')$ and by joining each vertex v' of G' to those vertices in G'' which are neighbours of v'', where v' and v'' represent the same vertex v of G.

Definition 3.5.6. The fuzzy shadow graph $S(G)(V_s, \mu_s, \sigma_s)$ of a fuzzy graph $G(V, \mu, \sigma)$ with underlying crisp graph $G(V, E)$ is defined as a fuzzy graph with its underlying crisp graph is the shadow graph of $G(V, E)$ with vertex set $V_S =$

$V' \cup V''$ where V' and V'' are the vertex sets corresponding to the two copies of $G'(V', E')$ and $G''(V'', E'')$ of $G(V, E)$. For each $v \in V$, the vertices $v' \in V'$ and $v'' \in V''$ corresponding to v have weight $\mu(v)$ that is, $\mu_S(v') = \mu_S(v'') = \mu(v)$ For $v' \in V'$ and a neighbour w'' of v'' in V'', $\sigma_S(v'w'') = \mu(v) \wedge \mu(w)$ and for two adjacent vertices u',v' in V' and for two adjacent vertices u'', v'' in V'', $\sigma_S(u'v') = \sigma_S(u''v'') = \sigma(uv)$, where $u, v \in V$, and σ_S is zero in all the other cases.

Theorem 3.5.4. *Let $G(V, \mu, \sigma)$ be a strong fuzzy path on n vertices. Then the strength $\mathscr{S}(S(G))$ of the shadow graph $S(G)$ of G is*

$$\mathscr{S}(S(G)) = \begin{cases} n-1 & \text{if } n \geq 3, \\ 2 & \text{if } n = 2. \end{cases}$$

Proof. For $n = 2$ the shadow graph of G is a fuzzy cycle on 4 vertices. Hence by Theorem 1.4.2 its strength is 2. Let u and v be two non-adjacent vertices of $S(G)$. The underlying crisp graph of $S(G)$ has vertex set $V(G') \cup V(G'')$, where G' and G'' are two copies of G with vertex set $V(G') = \{v'_1, v'_2, \ldots, v'_n\}$, $V(G'') = \{v''_1, v''_2, \ldots, v''_n\}$. (See Figure 3.22).

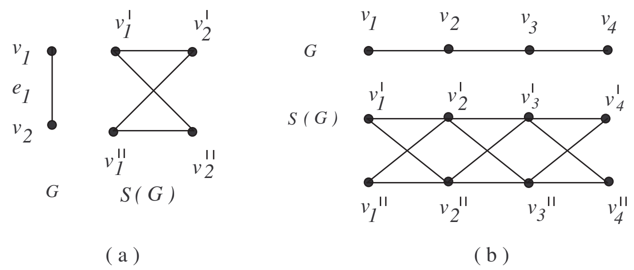


Figure 3.22: (a) Fuzzy path on 2 vertices and its shadow graph (b) Fuzzy path on 4 vertices and its shadow graph.

Case 1. $u, v \in \{v'_1, v'_2, \ldots, v'_n\}$.

Let $u = v'_i$ and $v = v'_j, i < j$. Then all the extra strong $u - v$ paths must be a subpath of the maximal partial fuzzy subgraph G' of $S(G)$ with vertex set $v'_i, v'_{i+1}, \ldots, v'_j, v''_{i+1}, \ldots, v''_j$.

If an extra strong $u - v$ path P passes through v'_{i-1} or v''_{i-1} then this path must pass through v''_i and as v''_i is adjacent to v'_{i+1} and v''_{i+1}, P must pass through at least one of v'_{i+1} and v''_{i+1}.

In the first case by deleting all vertices in P with suffices $\leq i$, and by adding the single edge $v'_i v'_{i+1}$ and in the second case by deleting all vertices in G with suffices $\leq i$ and by adding the single edge $v'_i v''_{i+1}$ we get another $u - v$ path of strength $\geq$ that of P and length $<$ that of P, a contradiction.

Similarly we can prove that P does not pass through v'_{j+1} or v''_{j+1}. Thus any extra strong $u - v$ path lie in G'.

From the adjacency relation in $S(G)$ every $u-v$ path traverses at least once through v'_k or $v''_k, i \leq k \leq j$. As $\mu(v'_k) = \mu(v''_k)$ each extra strong $u-v$ path contains exactly one v'_k or v''_k for $i < k < j$. Thus any such path is given by $uu_{i+1} \dots u_{j-1}v$ where $u_k = v'_k$ or v''_k for $i < k < j$. Therefore the length of such $u-v$ path is $j-i$. If $u = v'_1$ and $v = v'_n$ then the length of the extra strong $u-v$ path is equal to $n-1$. Also the case is same when $u, v \in \{v''_1, v''_2, \dots, v''_n\}$.

Case 2. $u \in \{v'_1, v'_2, \dots, v'_n\}$ and $v \in \{v''_1, v''_2, \dots, v''_n\}$ in $S(G)$.

Let $u = v'_i$ and $v = v''_j, i < j$. Here also we can conclude that every extra strong $u-v$ path lies in the maximal partial fuzzy subgraph G with vertex set $\{v'_k : k = i-1, \dots, j\} \cup \{v''_k : k = i, \dots, j\}$. But all the $u-v$ paths in G'' must pass either through v'_k or through v''_k or through both v'_k and v''_k, where $i < k < j$. As $\mu_S(v'_k) = \mu_S(v''_k)$, all the $u-v$ paths in G'' have strength $\leq \mu_S(v'_i) \wedge \mu_S(v'_{i+1}) \wedge \dots \wedge \mu_S(v'_{j-1}) \wedge \mu_S(v''_j)$. Thus the path $P = uu_2u_3 \dots u_{j-1}v$ has strength equal to $\mu_S(v'_i) \wedge \mu_S(v'_{i+1}) \wedge \dots \wedge \mu_S(v'_{j-1}) \wedge \mu_S(v''_j)$, where $u_k = v'_k$ or v''_k for $2 \leq k \leq j-1$, and no other $u-v$ path in G having length less than that of P have strength greater than P. So P is an extra strong $u-v$ path and is of length equal to $j-i$. When $u = v'_1$ and $v = v''_n$ the length of the extra strong $u-v$ path is equal to $n-1$.

□

Theorem 3.5.5. *Let $G(V, \mu, \sigma)$ be a strong fuzzy complete graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. Then the strength of the shadow graph $S(G)$ of G is 2 for*

$n \geq 2$.

Proof. Let $S(G)(W, \mu_S, \sigma_S)$ be the shadow graph of G with the underlying crisp graph has vertex set $W = V(G') \cup V(G'')$, where G' and G'' are two copies of G with vertex set $V(G') = \{v'_1, v'_2, \ldots, v'_n\}$, $V(G'') = \{v''_1, v''_2, \ldots, v''_n\}$. For $n = 2$ the shadow graph of G is a fuzzy cycle on 4 vertices. Hence the result is true by Theorem 1.4.2. So assume that $n \geq 3$. Let u, v be two non-adjacent vertices of $S(G)$ (See Figure 3.23). Then $u = v'_i$, for $1 \leq i \leq n$ and $v = v''_i, 1 \leq i \leq n$. Note that both v'_i and v''_i are adjacent to all the other vertices of $S(G)$. So uwv where $w \in W \setminus \{v'_i, v''_i\}$ such that $\mu_s(w) = \bigvee_{j \neq i} \mu_s(v'_j)$ is an extra strong $u - v$ path. Hence the theorem.

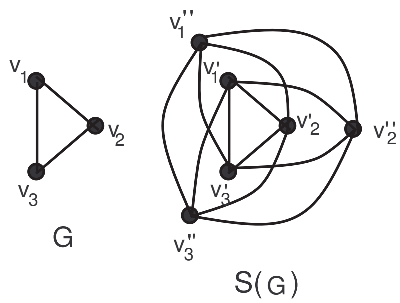


Figure 3.23: A fuzzy complete graph on 3 vertices and its shadow graph.

□

Chapter 4

Products of fuzzy graphs

In this chapter we discuss the strength of Cartesian product, tensor product, composition and normal product of certain strong fuzzy graphs.

4.1 Cartesian product

First of all we consider the Cartesian product of two strong fuzzy paths G_1 and G_2 on 2 vertices. Also here we discuss the strength of Cartesian product of two fuzzy paths, a fuzzy path on two vertices and a fuzzy cycle on n vertices, a fuzzy path on two vertices and a strong fuzzy star graph.

Definition 4.1.1. [26] For $i = 1, 2$, let $G_i(V_i, \mu_i, \sigma_i)$ be two fuzzy graphs with underlying crisp graphs $G_i(V_i, E_i)$. Their Cartesian product G, denoted by $G_1 \square G_2$ is the fuzzy graph $G(V, \mu, \sigma)$ with the underlying crisp graph $G(V, E)$, the Cartesian product of the crisp graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ with vertex set $V =$

$V_1 \times V_2$ and edge set $E = \{(u, u_2)(u, v_2) | u \in V_1, u_2v_2 \in E_2\} \cup \{(u_1, w)(v_1, w) | w \in V_2, u_1v_1 \in E_1\}$ and whose membership functions μ and σ are defined as

$\mu(u_1, u_2) = \mu_1(u_1) \wedge \mu_2(u_2); (u_1, u_2) \in V,$

$$\sigma((u_1, u_2)(v_1, v_2)) = \begin{cases} \mu_1(u_1) \wedge \sigma_2(u_2v_2) & \text{if } u_1 = v_1 \text{ and } u_2v_2 \in E_2, \\ \mu_2(u_2) \wedge \sigma_1(u_1v_1) & \text{if } u_2 = v_2 \text{ and } u_1v_1 \in E_1, \\ 0 & \text{otherwise.} \end{cases}$$

Notation 4.1.1. Unless otherwise specified for $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ the notation w_{ij} is used to denote the vertex $(u_i, v_j) \in V_1 \times V_2$.

Lemma 4.1.1. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy paths, each has P_2 as its underlying crisp graph. Then the Cartesian product $G_1 \square G_2$ of G_1 and G_2 is a fuzzy cycle.

Proof. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_, \sigma_2)$ be two fuzzy graphs with P_2 as their underlying crisp graph. The fuzzy graph $G_1 \square G_2$ is depicted in Figure 4.1.

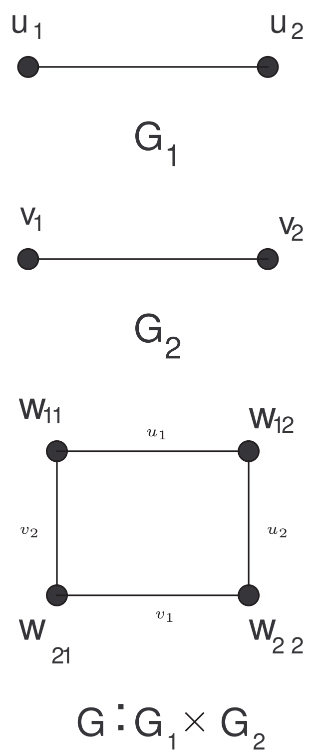


Figure 4.1: The fuzzy paths G_1, G_2 and their Cartesian product $G_1\Box G_2$.

Suppose that $\sigma_1(u_1u_2) \leq \sigma_2(v_1v_2)$. Then $\sigma(w_{11}w_{21}) = \sigma(w_{12}w_{22}) = \sigma_1(u_1u_2)$ and $\sigma(w_{11}w_{21}) = \sigma(w_{12}w_{22}) = \sigma_1(u_1u_2)$. $\sigma(w_{11}w_{12}) = \mu_1(u_1) \wedge \sigma_2(v_1v_2)$ and $\sigma(w_{21}w_{22}) = \mu_1(u_2) \wedge \sigma_2(v_1v_2)$. Clearly $w_{11}w_{12}$ and $w_{12}w_{22}$ are weakest edges of $G_1\Box G_2$. Therefore $G_1\Box G_2$ has at least two weakest edges. Hence $G_1\Box G_2$ is a fuzzy cycle.

Note 4.1.1. If $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ are two strong fuzzy paths then $\sigma(u_1u_2) = \mu_1(u_1) \wedge \mu_1(u_2)$ and $\sigma_2(v_1v_2) = \mu_2(v_1) \wedge \mu_2(v_2)$. If let us suppose that $\mu_1(u_1) = min\{\mu_1(u_1), \mu_1(u_2), \mu_2(v_1), \mu_2(v_2)\}$. Then $\sigma(w_{11}w_{12}) = \sigma(w_{11}w_{21}) =$

$\sigma(w_{12}w_{22}) = a$ say and $\sigma(w_{21}w_{22})$ is greater than or equal to this common value a. Thus if G_1 and G_2 are strong fuzzy graphs then at least three edges of $G_1 \Box G_2$ are weakest edges.

$\square$

The following lemma holds by Lemma 4.1.1 and Note 4.1.1.

Lemma 4.1.2. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs. Suppose both the graphs have underlying crisp graphs P_2 on two vertices. Then the strength of the Cartesian product of G_1 and G_2 is two.

Lemma 4.1.3. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with crisp graphs P_2 and P_3 respectively. Then the strength of $G_1 \Box G_2$ is 3.

Proof. Let the fuzzy graphs $G_1(V_1, \mu_1, \sigma_1)$, $G_2(V_2, \mu_2, \sigma_2)$ and their Cartesian product $G_1 \Box G_2$ be as depicted in Figure 4.2. We denote the weights of the edges $w_{11}w_{12}, w_{11}w_{21}, w_{21}w_{22}$ and $w_{12}w_{22}$ by a, b, c and d respectively.

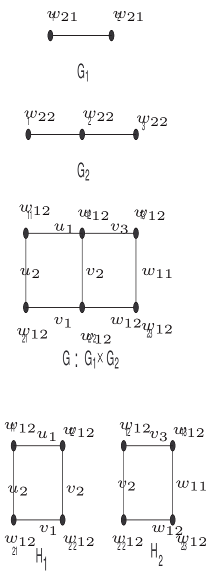


Figure 4.2: The fuzzy subgraphs G_1 and G_2, their Cartesian product $G_1 \Box G_2$ and two partial fuzzy subgraphs H_1 and H_2 of $G_1 \Box G_2$.

The two partial fuzzy subgraphs H_1 and H_2 of $G_1 \Box G_2$ shown in Figure 4.2 are fuzzy cycles by Lemma 4.1.1. Theorem 4.1.2 shows that both H_1 and H_2 have strength 2. Suppose the weakest edge of H_1 has weight α and those of H_2 have weight β.

Case 1. $\alpha \geq \beta$.

In this case $d \geq \alpha$.

Subcase 1. $d > \beta$. Then $e = g = f = \beta \longrightarrow (1)$. Let u and v be two vertices of G. If u and v are in $V(H_1)$, then the length of the extra strong path joining u and v is $\leq$ the strength of H_1, ie 2. Because, if a $u - v$ path P passes through a vertex in $V(G) \setminus V(H_1)$ then it has strength $\leq$ any $u - v$ path in H_1 and its length must be greater than or equal to any $u - v$ path in H_1.

If u and v are in $V(G \backslash H_1)$ then $u, v \in \{w_{13}, w_{23}\}$ and hence adjacent. Therefore, the extra strong path joining u and v is $w_{13}w_{23}$, which is of length one.

If u is in $V(G \backslash H_2)$ and v is in$V(G \backslash H_1)$. Then all the paths joining u and v must pass through an edge having weight β. Therefore, all the paths joining u and v have same strength. So, length of the extra strong path joining u and v is ≤ 3.

In particular if $u = w_{11}$ and $v = w_{23}$ or $u = w_{21}$ and $v = w_{13}$ then the length of extra strong path is equal to 3.

Subcase 2. $d = \beta$.

Then $\mu_1(u_1) = \beta$ or $\mu_1(u_2) = \beta$ or $\mu_2(v_2) = \beta$. In the first case $d = f = e = a = b = \beta$. In the second case $d = e = g = b = c = \beta$. In the third case $d = e = g = a = c = \beta$. In these cases the strength of any path connected by any two nonadjacent vertices are the same.

Case 2. $\alpha < \beta$.

The proof follows by interchanging the roles of H_1 and H_2.

$\square$

Theorem 4.1.1. *Let G_1 and G_2 be two strong fuzzy graphs with respective underlying crisp graphs P_2 and P_n. Then the strength of Cartesian product $G_1 \square G_2$*

of G_1 and G_2 is n.

Proof. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with underlying crisp graphs P_2 with vertex set $\{u_1, u_2\}$ and P_n with vertex set $\{v_1, v_2, \ldots, v_n\}$ respectively.

Let $G(V, \mu, \sigma)$ be the Cartesian product $G_1 \Box G_2$ of G_1 and G_2 with underlying crisp graph $G(V, E)$ where the vertex set $V = \{(u_i, v_j) = w_{ij} : u_i \in V_1, v_j \in V_2, i = 1, 2, j = 1, 2, \ldots, n\}$ and edge set $E = \{w_{ij}w_{ij+1} : 1 \leq j \leq n-1, i = 1, 2\} \cup \{w_{1j}w_{2j} : j = 1, 2, \ldots, n\}$.

We prove the theorem by induction on n. The result is trivial when $n = 1$ and the result is true for $n = 2$, and $n = 3$ by Lemmas 4.1.2 and 4.1.3. When $n = 2$, ie, when G_1 and G_2 are two fuzzy graphs with respective crisp graphs P_2, we proved that, the strength of the graph is 2, by showing that if $u = w_{11}$ and $v = w_{22}$ (or $u = w_{21}$ and $v = w_{12}$) then length of the extra strong $u - v$ path is 2 and for any other u and v, it is 1. Also in the case, G_1 is a fuzzy graph with the underlying crisp graph P_2 and G_2 a fuzzy graph with underlying crisp graph P_3, we proved that the length of any extra strong $u - v$ path is 3, when $u = w_{11}$ and $v = w_{23}$ or $u = w_{21}$ and $v = w_{13}$. For all other choices of u and v the length of the extra strong $u - v$ path is < 3 and the extra strong $w_{11} - w_{13}$ path is $w_{11}w_{12}w_{13}$.

We assume that the result is true for $n = m$, where $m \geq 3$. That is if G_1 is the fuzzy path P_2 with vertex set $\{u_1, u_2\}$ and G_2 is a fuzzy path P_m with vertex set $\{v_1, v_2, \ldots, v_m\}$ then assume that length of the extra strong path joining the vertices w_{11} and $w_{2\,m}$ or the vertices w_{21} and $w_{1\,m}$ in $G_1 \Box G_2$ is m and if $u = w_{11}$ and $v = w_{1\,m}$ or if $u = w_{21}$ and $v = w_{2m}$ then the length of the extra strong $u - v$ path is $m - 1$, and in fact $w_{11}w_{12}\ldots w_{1m}$ is the extra strong $w_{11} - w_{1m}$ path. u and v are any other vertices of $G_1 \Box G_2$ then the length of the extra strong $u - v$ path is $< m - 1$.

Let us suppose that G_1 be the fuzzy path on the vertex set $\{u_1, u_2\}$ and G_2 be the fuzzy path on the vertex set $\{v_1, v_2, \ldots, v_{m+1}\}$. For $1 \leq p < q \leq m + 1$, H_{pq} denotes the maximal partial fuzzy subgraph of G with vertex set $\{w_{ij}; i = 1, 2, p \leq j \leq q\}$. (See Figure 4.3).

Clearly, $H_{1\,m+1} = G_1 \Box\, G_2$.

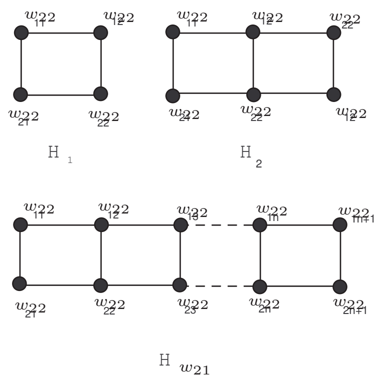


Figure 4.3: Partial fuzzy subgraphs H_{12} , H_{13} and $H_{1\,n+1}$ of $G = G_1 \Box G_2$.

Let u and v be two non -adjacent vertices of $G_1 \Box G_2$. We assert that if $u = w_{ij}$ and $v = w_{kl} \in H_{2\,m+1}$ then any extra strong $u - v$ path of G lie in H_{2m+1} and the length of any extra strong $u-v$ path in $G_1 \Box G_2$ is $\leq m+1$, by the induction hypothesis when $u = w_{21}$ and when $v = w_{1\,m+1}$ then the length of the extra strong $u-v$ path is $m+1$.

Case 1. Suppose that u and v are in $\{w_{ij} : i = 1, 2; j = 2, 3, \ldots, m\}$.

Then any path joining u and v in G can be viewed either as a path in the maximal partial fuzzy graph H_{1m} with vertex set $\{w_{ij} : i = 1, 2, 1 \leq j \leq m\}$ or as a path in the maximal partial fuzzy graph H_{2m+1} with vertex set $\{w_{ij} : i = 1, 2; 2 \leq j \leq (m+1)\}$. Note that both these graphs have underlying crisp graphs isomorphic to $P_2 \Box P_m$. Therefore by induction hypothesis the length of the extra strong $u-v$ path is $\leq m < (m+1)$.

Case 2. $u, v \in \{w_{11}, w_{21}, w_{1\,m+1}, w_{2\,m+1}\}$.

Suppose $u \in \{w_{11}, w_{21}\}$ and $v \in \{w_{1\ m+1}, w_{2\ m+1}\}$. Then we can prove the result in two steps.

(i) If $u = w_{11}$ and $v = w_{1m+1}$ (or $u = w_{21}$ and $v = w_{2m+1}$). Any path P_m in $H_{1\,m+1}$ joining w_{11} and w_{1m+1} can be considered as sum of two paths P^1 and P^2 where P^1 is a path in H_{1m} joining w_{11} and w_{1m} or it is a path joining w_{11} and w_{2m} in H_{1m} and P^2 is $P \cap H_{m\ m+1}$. Note that the strength of the path P is minimum of strength of the paths $P^i : i = 1, 2$. By induction hypothesis if P^1 is a path joining w_{11} and w_{1m} then it has maximum strength if $P^1 = w_{11}w_{12\ldots w_{1m}}$. Since w_{1m} and $w_{1\ m+1}$ are adjacent, the path $w_{1m}w_{1m+1}$ is the extra strong path joining w_{1m} and $w_{1\ m+1}$. In the second case, that is P^1

is a path from w_{11} to w_{2m} in H_{1m} and $P^2 = P \cap H_{m\ m+1}$ then by induction hypothesis P^1 has length m when P^1 is an extra strong path. Therefore in this case length of the path P is $m+2$ and it has strength $\leq$ the strength of the path $w_{11}w_{12}\ldots w_{1\ m+1}$. Therefore, we can conclude that the path P has maximum strength if $P^1 = w_{11}w_{12}\ldots w_{1m}$ and $P^2 = w_{1m}w_{1\ m+1}$. Also the length of P^1 is minimum among all paths in H_{1m} between w_{11} and $w_{1\ m}$.

(ii) If $u = w_{11}$ and $v = w_{2m+1}$ (or $u = w_{21}$ and $v = w_{1m+1}$).

In this case as in the proof of (i) we can prove that the strength of $u - v$ path is $m+1$ in $H_{1\ m+1}$.

Hence the theorem. □

Theorem 4.1.2. *Let G_1 and G_2 be two strong fuzzy graphs with the underlying crisp graphs the path P_m and the path P_n on m and n vertices respectively. Then the strength of the Cartesian product $G = G_1 \Box G_2$ of G_1 and G_2 is $m+n-2$.*

Proof. For a fixed n, we prove this theorem by induction on m. If $m = 1$ then G_1 is a fuzzy trivial graph. Thus when $m = 1$, $G = G_1 \Box G_2$ is a copy of P_n, a fuzzy path on n vertices. If $n = 1$, its strength is zero. If $n > 1$ then its strength is $n-1$. In either case we have the strength is $m+n-2$. Assume that the result is true for $m = k > 1$. To prove the result for $m = k+1$, let G_1 and G_2 be strong fuzzy graphs with underlying crisp graphs P_{k+1} and P_n respectively and let G be their Cartesian product. If $n = 1$ then G is a copy of G_1. Therefore strength of G is $k = m + n - 2$ thus in this case the theorem holds. So assume that $n > 1$. Also let $u, v \in V(G)$.

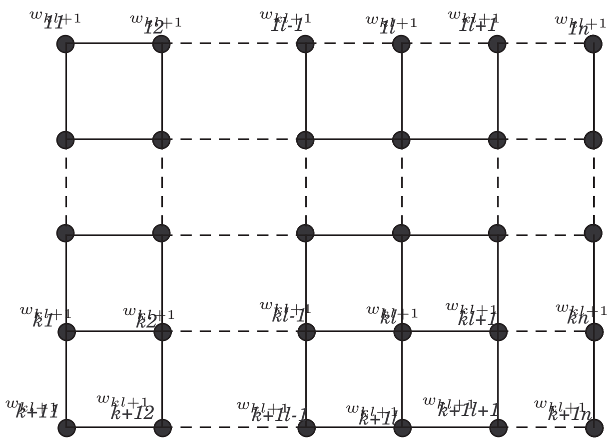


Figure 4.4: Cartesian product of two fuzzy graphs with underlying graphs P_{k+1} and P_n.

Case 1. $u, v \in \{w_{ij} : 1 \leq i \leq k, 1 \leq j \leq n\}$ or $u, v \in \{w_{ij} : 2 \leq i \leq k+1, 1 \leq j \leq n\}$. Let H_1 and H_2 be the two maximal partial fuzzy subgraphs of G with vertex set $\{w_{ij} : 1 \leq i \leq k, 1 \leq j \leq n\}$, $\{w_{ij} : 2 \leq i \leq k+1, 1 \leq j \leq n\}$ respectively. Then any extra strong path joining u and v in G can be either a path in H_1 or in H_2 of G.

To prove this assertion we proceed as follows. Let us suppose that $u, v \in V(H_1)$. Suppose P is an extra strong $u - v$ path in G, which passes through at least one of the vertices $w_{11}, w_{12}, \ldots, w_{1n}$. Then, we claim that P does not pass through any of the vertices $w_{k+1\,1}, w_{k+1\,2}, \ldots, w_{k+1\,n}$. If so, it contains a subpath $w_{kl}w_{k+1\,l}w_{k+1\,l+1}\ldots w_{k+1\,j}w_{kj}$ of G, which can be viewed as a path of the maximal partial fuzzy subgraph with vertex set $\{w_{k1}, w_{k2}, \ldots, w_{kn}, w_{k+1\,1} \ldots, w_{k+1\,n-1}, w_{k+1\,n}\}$ of G which is of the form $P_2 \square P_n$. Therefore the extra strong path joining w_{kl} and w_{kj} is $w_{kl}w_{k\,l+1} \ldots w_{kj}$ by the proof of Theorem 4.1.1. Therefore we can conclude that every path like P is contained in H_1. Hence its length by induction $\leq k+n-2$. Similar is the case when $u, v \in V(H_2)$.

Case 2. $u \in \{w_{1l} : l = 1, 2, \ldots, n\}$ and $v \in \{w_{k+1l} : l = 1, 2, \ldots, n\}$.

Let us suppose that $u = w_{1j}$ and $v = w_{k+1l}$. For $i = 1, 2, \ldots, k+1$ we denote the path $w_{i1}w_{i2} \ldots w_{in}$ with vertices $w_{i1}, w_{i2}, \ldots, w_{in}$ in G by L_i. We claim that for a fixed $l, l = 1, 2, \ldots, n$ the edge $w_{k+1l}w_{kl}$ has strength greater than or equal to the strength of any path from v to any vertex w of L_k. Suppose a path P_1 from v to a vertex of L_k contains a subpath $Q_1 = w_{k+1\,j}w_{k+1\,j-1} \ldots w_{k+1\,l}$ of L_{k+1} where $j > l$, then the path P_1 has strength less than or equal to that of the edge $w_{k+1l}w_{kl}$. For if the edge $w_{k+1\,l}w_{k\,l}$ is not a weakest edge of the cycle $C : w_{k\,l+1}w_{k+1\,l+1}w_{k+1\,l}w_{k\,l}w_{k\,l+1}$ then weight of $w_{k+1\,l}w_{k+1\,l+1} <$ weight of $w_{k+1\,l}w_{k\,l}$. Therefore the strength of $P_1 <$ strength of $w_{k+1\,l}w_{k\,l}$.

If $w_{k+1l}w_{kl}$ is a weakest edge of C then the subpath Q_1 of P_1 which belongs to L_{k+1} has strength $\leq$ strength of $w_{k+1l}w_{kl}$. If Q_1 has strength greater than that of $w_{k+1\,l}w_{k\,l}$ then all the edges $w_{k+1\,l}w_{k\,l}, \ldots, w_{k+1\,j}w_{kj}$ have weight equal to that of $w_{k+1l}w_{kl}$. Therefore we can conclude that in this case the path P_1 has strength $\leq$ that of $w_{k+1l}w_{kl}$. If P_1 contains no subpath of L_{k+1} then any path from v to a vertex of L_k pass through the edge vw_{kl}. Hence its strength must be less than or equal to the strength of the edge vw_{kl}. Hence the path having minimum length and with maximum strength from w_{k+1l} to a vertex of L_k is just the edge $w_{k+1l}w_{kl}$.

By the same argument, the edge $w_{kl}w_{k-1l}$ has the maximum strength and minimum length from w_{kl} to any vertex in L_{k-1}. Therefore the path $w_{k+1l}w_{kl}w_{k-1l}$ is the path from w_{k+1l} to L_{k-1}. Proceeding similarly we get the path $w_{k+1l}\ldots w_{1l}$ is the path with maximum strength and minimum length from w_{k+1l} to any vertex of L_1. Proceeding similarly $w_{1l}\ldots w_{1j}$ is the path with maximum strength and minimum length path joining w_{1j} and w_{1l}. Therefore the strength of the $u-v$ path is $\leq (n-1)+k = k+n-1$.

When $u = w_{11}$ and $v = w_{k+1n}$, the strength of the $u-v$ path is equal to $k+n-1$. Thus the theorem is true for $m = k+1$. Therefore the theorem follows by induction. □

Next we consider the Cartesian product of the fuzzy graphs P_2 and a fuzzy cycle C_n. Suppose $V_1 = \{u_1, u_2\}$, and $V_2 = \{v_1, v_2, \ldots, v_n\}$ are the vertex set of G_1 and G_2 respectively. Then the Cartesian product of G_1 and G_2 is the fuzzy graph $G(V, \mu, \sigma)$ where the underlying crisp graph is $G(V, E)$ with vertex set $V = \{w_{ij}, i = 1, 2, j = 1, 2, \ldots, n\}$ and edge set $E = \{w_{ij}w_{ij+1}, 1 \leq j < n, i = 1, 2\} \cup \{w_{1j}w_{2j}, 1 \leq j < n\} \cup \{w_{i1}w_{in}, i = 1, 2\}$ where $\mu(w_{ij}) = \mu_1(u_i) \wedge \mu_2(v_j), \forall w_{ij} \in V$
$\sigma(w_{ij}w_{ij+1}) = \mu_1(u_i) \wedge \sigma_2(v_jv_{j+1}), u_i \in V_1, (v_j, v_{j+1}) \in E_2$;
$\sigma(w_{1j}w_{2j}) = \sigma_1(u_1u_2) \wedge \mu_2(v_j)$; $\sigma(w_{i1}w_{in}) = \mu_1(u_i) \wedge \sigma_2(v_1v_n)$.

For example

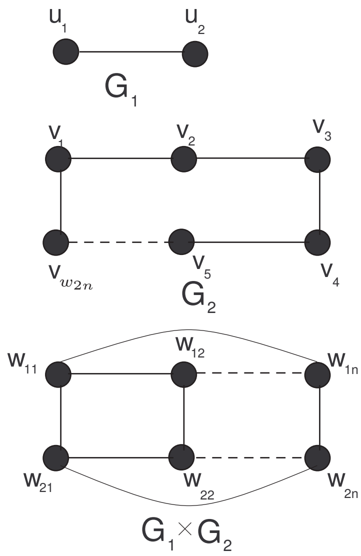


Figure 4.5: Cartesian product of the fuzzy graphs G_1 with underlying crisp graph P_2 and G_2 with underlying crisp graph C_n.

Theorem 4.1.3. *Let G_1 and G_2 be two strong fuzzy graphs with underlying crisp graphs the path P_2 with vertex set $V_1 = \{u_1, u_2\}$ and the cycle C_n with vertex set $V_2 = \{v_1, v_2, \ldots, v_n\}$ respectively and the weight of the weakest vertices of G_1 is greater than the weight of the weakest vertices of G_2. If the weakest vertices of G_2 altogether form a subpath of length l in G_2 then the strength of the Cartesian product of G_1 and G_2 is $(n - l + 1)$ if $l < [\frac{n+1}{2}]$ and $[\frac{n}{2}]$ if $l \geq [\frac{n+1}{2}]$.*

Proof. Let u and v be two non-adjacent vertices of G. Without loss of generality assume that $v_1, v_2, \ldots, v_{l-1}$ are the weakest vertices of G_2. Also assume that the weight of each $v_i, i = 1, 2, \ldots, l-1$ is w and these vertices altogether form a subpath in G_2. Then in G, the vertices $w_{11}, w_{12}, \ldots, w_{1l-1}$ and $w_{21}, w_{22}, \ldots, w_{2l-1}$ have the same weight w (See Figure 4.6).

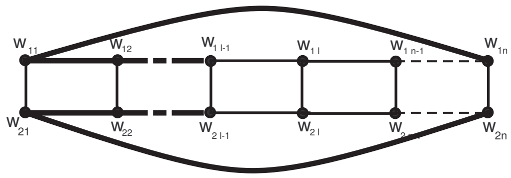


Figure 4.6: The Cartesian product of G_1 and $G_2 - \{v_1, \ldots, v_{l-1}\}$.

Case 1. $l < [\frac{n+1}{2}]$.

If $u, v \in V(G) - \{w_{11}, \ldots, w_{1l-1}, w_{21}, \ldots, w_{2l-1}\}$ then the length of the extra strong $u-v$ path in G is $\leq n-l+1$, since the extra strong paths joining u and v lie completely in the maximal partial fuzzy subgraph $G_1 \Box (G_2 - \{v_1, v_2, \ldots, v_{l-1}\})$ of G with underlying crisp graph is of the form $P_2 \Box P_{n-(l-1)}$. Therefore by Theorem 4.1.2 the length of the extra strong $u - v$ path in $G \leq n - l + 1$.

If $u, v \in \{w_{11}, \ldots, w_{1l-1}, w_{21}, \ldots, w_{2\,l-1}\}$ then all the $u - v$ paths have same strength in G. So all the extra strong paths joining u and v lie in the maximal partial subgraph $G_1 \Box G_2'$ of G, where G_2' is the maximal partial fuzzy graph of G_2 with vertex set $\{v_1, v_2, \ldots, v_{l-1}\}$. Also since $l \leq [\frac{n+1}{2}]$ the length of the extra strong $u - v$ path is $\leq l - 1 \leq n - l + 1$.

If $u \in \{w_{11}, \ldots, w_{1\,l-1}, w_{2\,1}, \ldots, w_{2\,l-1}\}$ and $v \in V(G) - \{w_{11}, \ldots, w_{1\,l-2}, w_{2\,1}, \ldots, w_{2\,l-2}\}$ or vice versa then all the paths joining u and v have same strength. So the length of the extra strong $u - v$ path is the minimum distance between u

and v in the underlying crisp graph of G, $P_2 \square C_n$ which is $\leq (n-l+1)$.

If $u = w_{2\,l}$ and $v = w_{1\,n}$ then the length of the extra strong $u-v$ path is equal to $n-l+1$.

Case 2. $l \geq [\frac{n+1}{2}]$.

If $u, v \in V(G) \setminus \{w_{11}, \ldots, w_{1l-1}, w_{21}, \ldots, w_{2l-1}\}$ then as in Case 1 strength of $u-v$ path in G is $n-l+1 \leq [\frac{n}{2}]$. If $u, v \in \{w_{11}, \ldots, w_{1l-1}, w_{21}, \ldots, w_{2l-1}\}$ or $u \in G - \{w_{11}, \ldots, w_{1l-1}, w_{21}, \ldots, w_{2\,l-1}\}$ and $v \in \{w_{11}, \ldots, w_{1\,l-1}, w_{2\,1}, \ldots, w_{2\,l-1}\}$ then all the $u-v$ paths must have same strength in G, and therefore the length of the extra strong path joining u and v is $\leq [\frac{n}{2}]$, since $l > [\frac{n+1}{2}]$. When $u = w_{11}$ and $v = w_{1k}$ where $k = [\frac{n}{2}]$ then strength of the $u-v$ path in G is exactly equal to $[\frac{n}{2}]$. Hence the Theorem. □

Theorem 4.1.4. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs $K_1 =< u >$ and the cycle $C_n = v_1 v_2 \ldots v_n v_1$ respectively. Let $G(V, \mu, \sigma)$ be the Cartesian product of G_1 and G_2. If v be a weakest vertex of G_2 then*

$$\mathscr{S}(G) = \begin{cases} [\frac{n}{2}] & \text{if } \mu_1(u) \leq \mu_2(v), \\ \mathscr{S}(G_2) & \text{otherwise.} \end{cases}$$

Proof. If $\mu_1(u) \leq \mu_2(v)$ then all the vertices of $G_1 \square G_2$ have the same weight $\mu_1(u)$. Therefore it is a regular fuzzy cycle. Hence by Theorem 1.4.1, strength of $G_1 \square G_2$ is $[\frac{n}{2}]$.

If $\mu_1(u) > \mu_2(v)$, then,

$$\mu(u, v_i) = \begin{cases} \mu_2(v_i) & \text{if } \mu_2(v_i) \leq \mu_1(u), \\ \mu_1(u) & \text{otherwise.} \end{cases}$$

Thus a vertex (u, v_i) of G is a weakest vertex of G if and only if v_i is a weakest vertex of G_2. Therefore, the strength $\mathscr{S}(G)$ of G is that of G_2. □

Theorem 4.1.5. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs the path $P_2 = u_1u_2$ and $C_n = v_1v_2 \ldots v_nv_1$ respectively. Suppose that $\mu_1(u_1) \leq \mu_1(u_2) \wedge \mu_2(v_1) \wedge \mu_2(v_2) \wedge \ldots \wedge \mu_2(v_n)$. Let $G = G_1 \Box G_2$ be the Cartesian product of G_1 and G_2. Then the strength $\mathscr{S}(G)$ of the Cartesian product G of G_1 and G_2 is,*

$$\mathscr{S}(G) = max\left\{\mathscr{S}(G_2 \Box G_3), \left\lceil \frac{n+1}{2} \right\rceil \right\};$$

where G_3 is the null graph with vertex set $\{u_2\}$.

Proof. Let u and v be two distinct vertices of G.

Case 1. $\mu_1(u_2) > \mu_2(v_1) \wedge \mu_2(v_2) \ldots \wedge \mu_2(v_n)$.

Subcase 1. Let $u, v \in \{w_{1j}, 1 \leq j \leq n\}$. Since $\mu(w_{1j}) = \mu_1(u_1); 1 \leq j \leq n$, all the edges having w_{1j} as one of the end vertices, $1 \leq j \leq n$ have weight equal to $\mu_1(u_1)$. Therefore, the length of the extra strong path joining u and v is the minimum length of the path joining u and v in G. That is less than or equal to $\left[\frac{n}{2}\right]$.

Subcase 2. Let $u, v \in \{w_{2j}, 1 \leq j \leq n\}$.

Since $\mu(w_{1j}) \leq \mu(w_{2j})$, the extra strong path joining u and v lies in the maximal partial fuzzy subgraph $G_3 \Box G_2$ of G. So we have by Theorem 4.1.4, the length of the extra strong $u - v$ path is the strength of G_2.

Subcase 3. Let $u \in \{w_{1j} : 1 \leq j \leq n\}$ and $v \in \{w_{2j} : 1 \leq j \leq n\}$.

Since $\mu_1(u_1) \leq \mu_1(u_2) \wedge \mu_2(v_1) \wedge \ldots \wedge \mu_2(v_n)$, all the $u - v$ paths in G have strength $\mu_1(u_1)$. So length of the extra strong $u - v$ path in G is the length of the shortest $u - v$ path in G which is $\leq \left\lceil \frac{n}{2} \right\rceil$.

Case 2. $\mu_1(u_2) \leq \mu_2(v_1) \wedge \mu_2(v_2) \ldots \wedge \mu_2(v_n)$.

Subcase 1. $\mu_1(u_1) = \mu_1(u_2)$. Then $\mu(w_{ij}) = \mu_1(u_1)\ \forall i, j$. Therefore, the length of the extra strong path joining u and v in G is the minimum length of the path joining u and v in G, which is less than or equal to $[\frac{n+1}{2}]$.

Subcase 2. $\mu_1(u_1) < \mu_1(u_2)$. Then $\mu(w_{1j}) = \mu_1(u_1)$ and $\mu(w_{2j}) = \mu_1(u_2)\ \forall i, j$. If u or $v \in \{w_{1j}, 1 \leq j \leq n\}$, then all the paths joining u and v have weight $\mu_1(u_1)$. Therefore, the length of the extra strong path joining u and v is the minimum length of the path joining u and v in G which is $[\frac{n}{2}]$.

If u and $v \in \{w_{2j}, 1 \leq j \leq n\}$, then the extra strong path joining u and v lie in the subgraph $G_3 \Box G_2$. So by Theorem 4.1.4 the length of the extra strong $u - v$ path in G is $[\frac{n}{2}]$.

□

Note 4.1.2. Let $G(V, \mu, \sigma)$ be a fuzzy graph. If W is a subset of V then $< W >$ denotes the maximal partial fuzzy subgraph of G on W.

Definition 4.1.2. The fuzzy book is defined as the Cartesian product of graphs G_1 with underlying crisp graph P_2 and fuzzy star graph S_n, where $n > 2$. Let $V(P_2) = \{u_1, u_2\}$ and $V(S_n) = \{v_1, v_2, \ldots, v_n\}$ with v_1 as the central vertex. For $i = 2, 3, \ldots, n$, the maximal partial fuzzy subgraph $< \{w_{11}, w_{21}, w_{1i}, w_{2i}\} >$ with vertex set $< \{w_{11}, w_{21}, w_{1i}, w_{2i}\} >$ is called a fuzzy page of the fuzzy book, whose underlying crisp graph is isomorphic to $P_2 \Box P_2$. The crisp graph of the union of two fuzzy pages $< \{w_{11}, w_{21}, w_{1i}, w_{2i}\} >$ and $< \{w_{11}, w_{21}, w_{1j}, w_{2j}\} >$ is isomorphic to $P_2 \Box P_3, 2 \leq i \neq j \leq n$. It is called a fuzzy Domino graph.

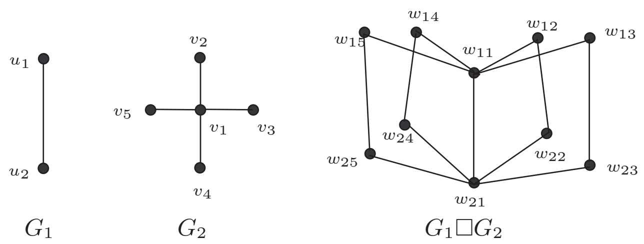


Figure 4.7: Cartesian product $G_1 \Box G_2$ of a fuzzy path G_1 and a fuzzy star graph G_2.

Theorem 4.1.6. *Let G_1 and G_2 be two strong fuzzy graphs with underlying crisp graphs the path P_2 and the star graph S_n respectively. Let $V(P_2) = \{u_1, u_2\}$ and*

$V(S_n) = \{v_1, v_2, \ldots, v_n\}$ with v_1 as the central vertex. Then the strength of the Cartesian product $G = G_1 \Box G_2$ is 3.

Proof. Let $\{w_{11}, w_{12}, \ldots, w_{1n}, w_{21}, w_{22}, \ldots, w_{2n}\}$, where $n \geq 3$, be the vertex set of G. Clearly $w_{11}w_{21}$ is the common edge of the pages of $G_1 \Box G_2$. Let u and v be two non-adjacent vertices of G (See Figure 4.7). Then u and v lie on the same page or different pages of G. For $2 \leq i \neq j \leq n$, denote the partial fuzzy subgraph $< \{w_{11}, w_{21}, w_{1i}, w_{2i}\} > \cup < \{w_{11}, w_{21}, w_{1j}, w_{2j}\} >$ of $P_2 \Box S_n$ by H_{ij}. Therefore any extra strong path joining u and v can be considered as a path in H_{ij} for some i and j. Since the underlying crisp graph of H_{ij} is $P_2 \Box P_3$, the length of any extra strong path joining u and v in G is less than or equal to 3, by Theorem 4.1.3.

In particular if $u = w_{12}$ and $v = w_{23}$, then any extra strong path joining u and v lie completely in H_{23} and hence has length exactly 3. Hence the theorem.

□

Now we are going to find the strength of the Cartesian product of fuzzy path and a fuzzy butterfly graph.

Theorem 4.1.7. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with crisp graphs the path P_2 with vertex set $\{u_1, u_2\}$ and the butterfly graph with vertex set $\{v_1, v_2, \ldots, v_5\}$ respectively. Then the strength of the Cartesian product $G(V, \mu, \sigma)$ of G_1 and G_2 is 3.*

Proof. First of all assume that the degree of the vertex v_1 of G_2 is 4 and $\mu_1(u_1) \leq \mu_1(u_2)$.

Let u and v be any two non-adjacent vertices of $G = G_1 \square G_2$ with vertex set $\{w_{11}, w_{12}, \ldots, w_{15}, w_{21}, w_{22}, \ldots, w_{25}\}$.

Case 1. $\mu_1(u_1)$ or $\mu_1(u_2) \leq \mu_2(v_1) \wedge \mu_2(v_2) \wedge \ldots \wedge \mu_2(v_5)$.

Then all the $u - v$ paths passing through any of $w_{1j}, j = 1, 2, \ldots, 5$ have strength $\mu_1(u_1)$, because every edge incident with w_{1j} has weight $\mu_1(u_1)$. Therefore if at least one of u and v belongs to $\{w_{11}, w_{12}, \ldots, w_{15}\}$ then the extra strong $u - v$ paths are the shortest $u - v$ paths in the underlying crisp graph of G and therefore has length less than or equal to 3.

If $u, v \in \{w_{21}, w_{22}, \ldots, w_{25}\}$ then any extra strong $u - v$ path lie in the maximal partial fuzzy subgraph with vertex set $\{w_{21}, w_{22}, \ldots, w_{25}\}$ which is a strong fuzzy butterfly graph. Therefore, by Corollary 2.3.1 the length of any extra strong $u - v$ path in G is 2.

Case 2. $\mu_2(v_j)$ less than $\mu_1(u_1)$ for at least one j. Let us suppose that $\mu_2(v_j) \leq \mu_2(v_1) \wedge \mu_2(v_2) \ldots \wedge \mu_2(v_5)$.

Subcase 1. $v_j = v_1$.

Then all the paths passing through $w_{i1}, i = 1, 2$ have strength $\mu_2(v_1)$. The fuzzy graph of G can be viewed as the union of two fuzzy subgraphs H_1 and H_2, as shown in Figure 4.8. Note that $P_2 \square C_2$ is the underlying crisp graph of both H_1 and H_2.

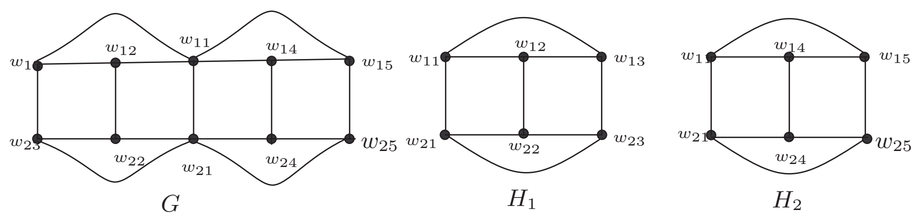


Figure 4.8: Cartesian product $G = G_1 \Box G_2$ of a fuzzy path G_1 on 2 vertices and G_2, a fuzzy butterfly graph and the fuzzy subgraphs H_1 and H_2 of G.

Suppose u and v belong to $V(H_1)$. Then any extra strong $u - v$ path lie in H_1, since $\mu(w_{11}) = \mu(w_{21}) = \mu_2(v_1)$, all the $u - v$ paths through w_{11} and w_{21} have the same strength. Therefore the length of the extra strong $u - v$ path is ≤ 2. Similarly if u and $v \in V(H_2)$ the length of any extra strong $u - v$ path is ≤ 2.

Let $u \in V(H_1)$ and $v \in V(H_2) \setminus V(H_1)$. In this case all the $u - v$ paths pass through w_{11} or w_{21} or both. Therefore all the $u - v$ paths have same strength. Hence the length of the extra strong path joining u and v is less than or equal to the minimum distance between u and v in G which is 3.

Subcase 2. $v_j \neq v_1$.

Without loss of generality assume that $v_j = v_2$. Then by our assumption, $\mu_2(v_2) \leq \mu_2(v_1) \wedge \mu_2(v_2) \wedge \ldots \wedge \mu_2(v_5)$. Let u or $v \in V(H_1)$. If at least one of the vertices u and $v \in \{w_{12}, w_{22}\}$, then all the $u - v$ paths have strength $\mu_2(v_2)$. So the length of any extra strong $u - v$ path in G is ≤ 3. If u and $v \notin \{w_{12}, w_{22}\}$

then all the extra strong $u - v$ paths lie in the graph H in Figure 4.9, which is obtained by deleting the vertices w_{12}, w_{22} from G.

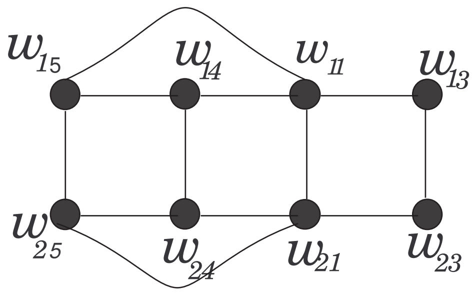


Figure 4.9: A fuzzy subgraph H of G.

In this case if u and $v \in V(H_1)$ then either $u = w_{13}$, and $v = w_{21}$ or $u = w_{11}$ and $v = w_{23}$. In both these cases if a path joining u and v pass through a vertex of H_2 then it must pass through w_{11} and w_{21} and any such path have strength $\leq \mu(w_{11}) \wedge \mu(w_{21})$. Thus each extra strong path lies in the maximal partial fuzzy subgraph with vertex set $\{w_{11}, w_{21}, w_{13}, w_{23}\}$. Hence the length of the extra strong $u-v$ path is 2 by Theorem 4.1.2. Now suppose u and $v \in V(H_2)$, if any of the $u - v$ path through w_{13} (or w_{23}), definitely will pass through w_{23} (or w_{13}), w_{11} and w_{21}. Any such path has strength $\leq \mu(w_{11}) \wedge \mu(w_{21})$. So every extra strong path lies in H_2. Therefore, the length of any extra strong $u-v$ path is 2.

If $u = w_{13}$ and $v = w_{25}$ then any $u - v$ path in H has length ≥ 3. Also any $u - v$ path through the vertices w_{14} or w_{24} has length > 3 and strength $\leq$ any other $u - v$ path in H. Therefore the length of extra strong $u - v$ path is the minimum distance between u and v, which is 3. Hence we can conclude that $\mathscr{S}(G) = 3$. □

4.2 Tensor product

This section discusses strength of tensor product of certain graphs.

Definition 4.2.1. [12] Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with underlying crisp graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ respectively. Then the tensor product G, denoted by $G_1 \otimes G_2$, of G_1 and G_2 is the fuzzy graph $G(V, \mu_1 \otimes \mu_2, \sigma_1 \otimes \sigma_2)$ with the underlying crisp graph $G(V, E_1 \otimes E_2)$ is the tensor product of $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, where $V = V_1 \times V_2$ and $E_1 \otimes E_2 = \{(u_1, u_2)(v_1, v_2) : u_1v_1 \in E_1, u_2v_2 \in E_2\}$, $(\mu_1 \otimes \mu_2)(u_1, u_2) = \mu_1(u_1) \wedge \mu_2(u_2)$ for $(u_1, u_2) \in V$ and $(\sigma_1 \otimes \sigma_2)((u_1, u_2)(v_1, v_2)) = \sigma_1(u_1, v_1) \wedge \sigma_2(u_2, v_2)$ for $(u_1, u_2) \in E_1$ and $(v_1, v_2) \in E_2$.

Theorem 4.2.1. *Let G_1 and G_2 be two fuzzy graphs with underlying crisp graphs P_2 and P_n respectively. Then the strength $\mathscr{S}(G_1 \otimes G_2)$ of the tensor product of G_1 and G_2 is $n - 1$.*

Proof. If $n = 1$ then $G_1 \otimes G_2$ is a null fuzzy graph. Therefore $\mathscr{S}(G_1 \otimes G_2) = 0 = n - 1$. If $n > 1$ then it is the disjoint union of two fuzzy paths on n vertices (See Figure 4.10). So by Theorem 1.4.1 $\mathscr{S}(G) = n - 1$.

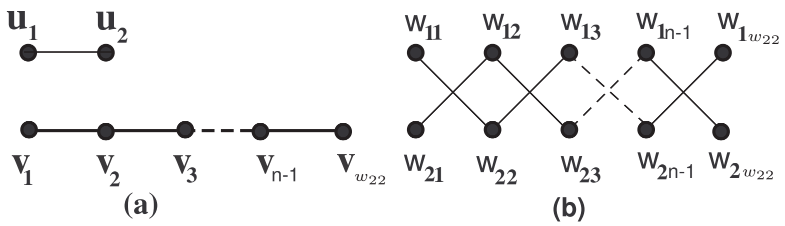


Figure 4.10: Tensor product of two fuzzy paths.

□

If we replace the fuzzy graph G_2 of Theorem 4.2.1 by an another fuzzy graph, having star graph as the underlying crisp graph on n vertices and keeping G_1 as it is then, their tensor product G is a null fuzzy graph, if $n = 1$. It is a disjoint union of two fuzzy paths if $n = 2$ and if $n > 2$ it is a disjoint union of two fuzzy star graphs on n vertices. Therefore in the first case, that is if $n = 1$ then $\mathscr{S}(G) = 0$ and in the second case that is if $n = 2$, $\mathscr{S}(G) = 1$ and when $n \geq 3$, $\mathscr{S}(G) = 2$ by Theorem 3.1.4. We can summarize these results as follows.

Theorem 4.2.2. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with underlying crisp graph the path P_2 and the star graph S_n respectively. Then the strength of the tensor product G is*

$$\mathscr{S}(G) = \begin{cases} 0 & \textit{if } n = 1, \\ 1 & \textit{if } n = 2, \\ 2 & \textit{if } n \geq 3. \end{cases}$$

Theorem 4.2.3. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with the underlying crisp graphs the path P_2 with vertex set $V_1 = \{u_1, u_2\}$ and the cycle C_n with vertex set $V_2 = \{v_1, v_2, \ldots, v_n\}$. Let $\mu_\circ = \mu_1(u_1) \wedge \mu_1(u_2) \wedge \mu_2(v_1) \wedge \mu_2(v_2) \ldots \wedge \mu_2(v_n)$. Then the strength of the tensor product of $G_1 \otimes G_2(V, \mu, \sigma)$ with vertex set $V = \{w_{ij} : i = 1, 2; j = 1, 2, \ldots, n\}$ is*

$$\mathscr{S}(G) = \begin{cases} [\frac{n}{2}] & \text{if } |V(G_2)| \text{ is even and} \\ & \text{there exist } w \in V(G_1), \text{such that } \mu_1(w) = \mu_\circ, \\ \mathscr{S}(G_2) & \text{if } |V(G_2)| \text{ is even and} \\ & \text{there exist no } w \in V(G_1), \text{such that } \mu_1(w) = \mu_\circ, \\ n & \text{if } |V(G_2)| \text{ is odd.} \end{cases}$$

Proof.

Case 1. $|V(G_2)|$ is even.

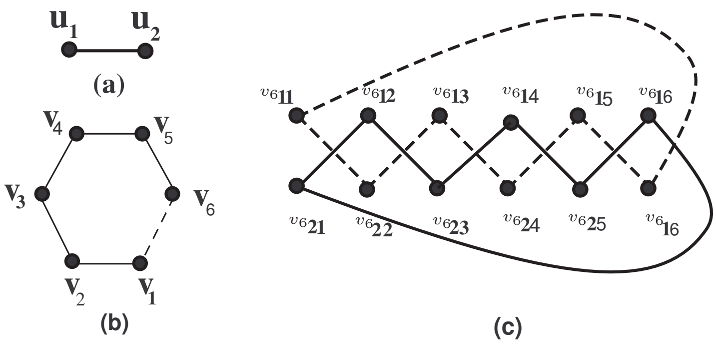


Figure 4.11: Tensor product of two strong fuzzy graphs.

Then $G = G_1 \otimes G_2$ is a disjoint union of two fuzzy cycles H_1 with vertex set $\{w_{11}, w_{22}, w_{13}, w_{24}, \ldots, w_{1n-1}, w_{2n}\}$, and H_2 with vertex set $\{w_{12}, w_{23}, w_{14}, w_{25}, \ldots, w_{2n-1}, w_{1n}, w_{21}\}$. (See Figure 4.11).

Subcase 1. There exist $w \in V_1$ such that $\mu_1(w) = \mu_\circ$.

In this case, all the edges of G have the same weight. So, the strength of $G =$ strength of $H_1 =$ strength of $H_2 = [\frac{n}{2}]$.

Subcase 2. There exist no $w \in V_1$ such that $\mu_1(w) = \mu_\circ$.

In this case, there exists a $w \in V_2$ such that $\mu_2(w) = \mu_\circ$. Without loss of generality assume that $w = v_1$. Then w_{11} and w_{21} are two weakest vertices of G. In fact each weakest vertex of G_2 determines exactly one weakest vertex in H_1 as well as in H_2. So the number of weakest vertices of H_1 and that of H_2 are equal and equal to that of G_2. Note only that if G_2 has m consecutive weakest vertices then both H_1 and H_2 have the same number of consecutive weakest vertices. From this we can conclude that the strength of G is equal to that of G_2.

Case 2. $|V(G_2)|$ is odd.

In this case $G = G_1 \otimes G_2$ is a strong fuzzy cycle with vertex set $\{w_{11}, w_{22}, w_{13}, w_{24}, \ldots, w_{2n-1}, w_{1n}, w_{21}, w_{12}, w_{23}, \ldots, w_{1n-1}, w_{2n}\}$. (See Figure 4.12).

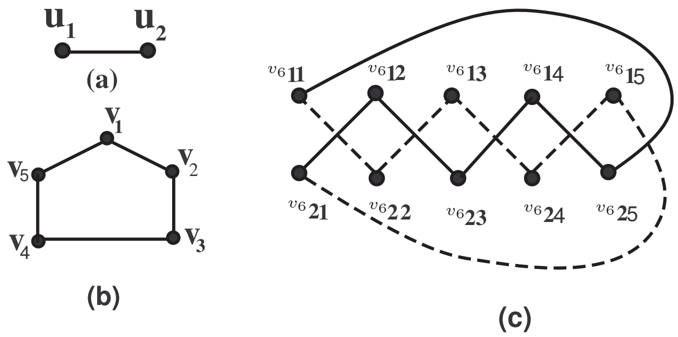


Figure 4.12: (a) A fuzzy path on two vertices G_1, (b) a strong fuzzy cycle G_2 and (c) their tensor product of G.

Subcase 1.

Then all the edges of G have the same weight. Therefore by Theorem 1.4.1 ; $\mathscr{S}(G) = [\frac{2n}{2}] = n$.

Subcase 2. There exist no $w \in V_1$ such that $\mu_1(w) = \mu_\circ$.

By our assumption there exists a vertex $w \in V_2$ such that $\mu_2(w) = \mu_\circ$. Assume that $w = v_1$. Then w_{11} and w_{21} are weakest vertices of the partial fuzzy subgraph $P =< \{w_{11}, w_{22}, w_{13}, w_{24}, \ldots, w_{2n-1}, w_{1n}\} >$ and $Q =< \{w_{21}w_{12}w_{23} \ldots w_{1n-1}$ $w_{2n}\} >$ of G. Also corresponding to each weakest path of length m in G_2 there exist weakest paths of the same length in P and in Q. Let u and v be any two vertices of G. Then the path joining u and v having length $\geq n$ passes through at least one weakest edge of G. So the length of the extra strong $u - v$ path in G is $\leq n$. If $u = w_{11}$ and $v = w_{21}$ then the length of the extra strong $u - v$ path is exactly n. Hence the proof. □

Theorem 4.2.4. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs K_n and K_m respectively. Let $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ be the set of all vertices of K_n and K_m. Then the strength of the tensor product $G_1 \otimes G_2(V, \mu, \sigma)$ of G_1 and G_2 is,*

$$\mathscr{S}(G_1 \otimes G_2) = \begin{cases} 0 & \text{for} \quad n = 1, m \geq 1 \quad or\ n \geq 1, m = 1, \\ 1 & \text{for} \quad n = m = 2, \\ 2 & \text{for} \quad n > 2 \quad and \quad m > 2, \\ 3 & n = 2, m > 2\ or\ n > 2, m = 2. \end{cases}$$

Proof. Let u and v be two non-adjacent vertices of $G = G_1 \otimes G_2$, say $u = w_{ij}$ and $v = w_{kl}$. Then u_i is not adjacent to u_k in G_1 or v_j is not adjacent to v_l in G_2.

Case 1. $n = 1, m \geq 1$ or $m = 1, n \geq 1$.

In this case $G = G_1 \otimes G_2$ is a null fuzzy graph on m (or n) vertices. Therefore $\mathscr{S}(G)$ is 0.

Case 2. $n = m = 2$.

In this case the tensor product is the disjoint union of two fuzzy paths with P_2 as the underlying crisp graphs. So strength of G is 1 by Theorem 1.4.1.

Case 3. $n > 2$ and $m > 2$.

Since G_1 and G_2 are complete fuzzy graphs of order > 2 there exist at least one vertex in $G_1 \otimes G_2$ which is adjacent to both u and v in $G_1 \otimes G_2$.

Whether $i = k$ or not, since n and $m > 2$, we can find a $u_r \in V(G_1)$ different from u_i and u_k such that $\mu_1(u_r) = \vee\{\mu_1(u_p) : 1 \leq p \neq i, k \leq n\}$ and a $v_s \in V(G_2)$ such that $\mu_2(v_s) = \vee\{\mu_2(v_q) : 1 \leq q \neq l, j \leq m\}$, so that w_{rs} is adjacent to both u and v in G. By the choice of w_{rs} the path $uw_{rs}v$ is an extra strong path joining u and v in G of length 2.

Case 4. $m > 2$ and $n = 2$ (or $n > 2$ and $m = 2$).

First of all suppose that $n = 2$ and $m > 2$. The case $m > 2$ and $n = 2$ can be dealt as in the same way. We have the following cases,

i $u = w_{1j}, v = w_{1l}, 1 \leq j \neq l \leq m$,

ii $u = w_{2j}, v = w_{2l}, 1 \leq j \neq l \leq m$,

iii $u = w_{1j}$ and $v = w_{2j}$ for some j.

In the first two cases we can proceed as in the proof of Case 3 and prove that the length of the extra strong path joining u and v is 2.

When $u = w_{1j}$ and $v = w_{2j}$, there is no vertex in G which is adjacent to both u and v. Since w_{1j} is adjacent to w_{2k}, for $k \neq j$ and w_{2j} is adjacent to w_{1l}, for $l \neq j$, the extra strong path joining u and v is $uw_{2r}w_{1s}u$, where $(v_r), r \neq j$ is chosen so that $\mu_2(v_r) \geq \vee\{\mu_2(v_p); r \neq j\}$ and $v_s, s \neq j, r$, is chosen such that $\mu_2(v_s) \geq \vee\{\mu_2(v_q); q \neq j, r\}$. Hence the length of the extra strong path joining u and v is 3. $\square$

4.3 Composition

Another product we consider is the composition.

Definition 4.3.1. [47] Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with underlying crisp graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ respectively. Then the composition $G(V, \mu, \sigma)$, denoted by $G_1[G_2]$, of G_1 and G_2 is the fuzzy graph with the underlying crisp graph $G(V, E)$ is the composition of the crisp graphs of G_1 and G_2 where $V = V_1 \times V_2$ and $E = \{((u_1, u_2)(v_1, v_2)) : u_1 = v_1, (u_2, v_2) \in E_2$ or $(u_1, v_1) \in E_1\}$ are the vertex set and edge set of $G(V, E)$ respectively and μ and σ are defined as

$\mu(u_1, u_2) = \mu_1(u_1) \wedge \mu_2(u_2), (u_1, u_2) \in V,$

$$\sigma((u_1, u_2)(v_1, v_2)) = \begin{cases} \mu_1(u_1) \wedge \sigma_2(u_2, v_2) & \text{if } u_1 = v_1 \text{ and } (u_2, v_2) \in E_2, \\ \mu_2(u_2) \wedge \mu_2(v_2) \wedge \sigma_1(u_1, v_1) & \text{if } (u_1, v_1) \in E_1, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the vertex (u_i, v_j) of $V_1 \times V_2$ is denoted by w_{ij}.

For $m = 1$ and $n = 2$ or $m = 2$ and $n = 1$ the composition of paths P_m and P_n is a path on two vertices and for $m = n = 2$ their composition is a complete graph on 4 vertices. Hence in both these cases the strength of composition of P_m and P_n is one.

Lemma 4.3.1. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy paths with underlying crisp graphs $P_2 = u_1u_2$ and $P_n = v_1v_2 \ldots v_n$. Let $G(V, \mu, \sigma) = G_1[G_2]$ be the composition of G_1 and G_2. Then

$$\mathscr{S}(G) = \begin{cases} 2 & \text{if } \mu_1(u_1) = \mu_1(u_2) \text{ or } l \leq 1, \\ l & \textit{otherwise.} \end{cases}$$

where l is the maximum length of subpaths of G_2 having strength $> \mu_1(u_1) \wedge \mu_1(u_2)$ if such a path exists, zero otherwise.

Proof. Let $u = w_{ij}$ and $v = w_{km}$ be two nonadjacent vertices of G. Then $i = k$ and v_j and v_m are not adjacent. Assume that $\mu_1(u_1) \geq \mu_1(u_2)$. If $u = w_{2j}$ and $v = w_{2m}$ then any $u - v$ path has strength $\leq \mu_1(u_2) \wedge \mu_2(v_j) \wedge \mu_2(v_m)$. As the path $uw_{1j}v$ has strength $\mu_1(u_2) \wedge \mu_2(v_j) \wedge \mu_2(v_m)$, it is an extra strong $u - v$ path in G.

Now suppose that $u = w_{1j}$ and $v = w_{1m}$. Also suppose that $\mu_1(u_1) = \mu_1(u_2)$. In this case by interchanging the values of u_1 and u_2 in the discussion above we get $uw_{2j}v$ is an extra strong $u - v$ path in G. If $\mu_1(u_2) < \mu_1(u_1)$ and if $l \leq 1$ then any subpath of P_2 of length ≥ 2 has strength $\leq \mu_1(u_2)$. Thus any $u-v$ path which lies in the maximal partial subgraph of G with vertex set $\{w_{11}, w_{12}, \ldots, w_{1n}\}$ has strength $\leq \mu_1(u_2)$. Therefore in this case $uw_{2j}v$ is an extra strong $u - v$ path in G.

Now suppose that $l > 1$. If we choose v_j and v_m as the ends of a subpath of

P_n of length l and strength $> \mu_1(u_2)$ then $w_{1j}w_{1j+1}w_{1m}$ or $w_{1m}w_{1m+1}\ldots w_{1j}$ is an extra strong $u-v$ path according as $m > j$ or $j > m$ respectively. Therefore the length of extra strong $u-v$ path is $\leq l$. Thus if we choose v_j and v_m as the ends of the maximal subpath we get the length of the extra strong $w_{1j} - w_{1m}$ path is l. Hence the lemma. □

In general, for two graphs G_1 and G_2 , $G_1[G_2] \neq G_2[G_1]$. Therefore if G_1 and G_2 are two fuzzy graphs then also $G_1[G_2] \neq G_2[G_1]$. For example if G_1 and G_2 are fuzzy graphs with the underlying crisp graphs P_n and P_2 respectively then $G_1[G_2]$ is a $2-$ linked fuzzy graph with $n-1$ parts, each part is a complete fuzzy graphs on 4 vertices (See Figure 4.13(b)). On the other hand $G_2[G_1]$ is as shown in Figure 4.13(a).

Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy paths with crisp graphs P_n and P_2 respectively and $G(V, \mu, \sigma)$ be their composition. Then $G(V, \mu, \sigma)$ is a properly linked fuzzy graphs with $n-1$ parts, each is complete. Then by Theorem 2.3.2 we have the following result.

Theorem 4.3.1. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with crisp graphs P_n and P_2 respectively and $G(V, \mu, \sigma)$ be their composition. Then the strength $\mathscr{S}(G)$ of $G = G_1[G_2]$ is* 1 *for* $n = 1$ *and* $(n-1)$ *for* $n > 1$.

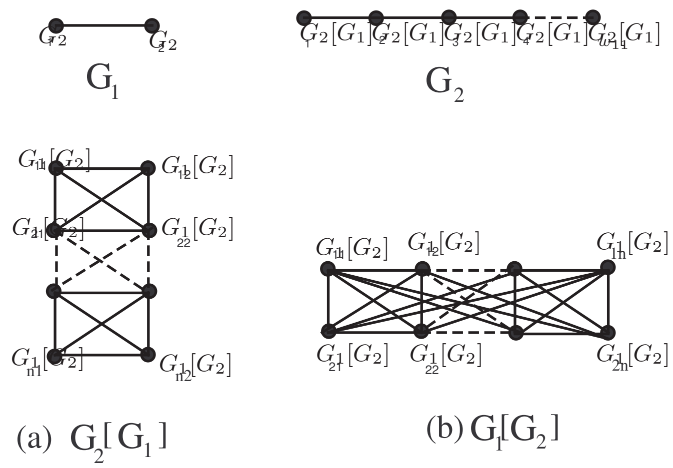


Figure 4.13: Composition of fuzzy graphs.

Theorem 4.3.2. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy paths with underlying crisp graphs $P_2 = u_1u_2$ and $P_n = v_1v_2 \ldots v_n$ respectively. Also let $G(V, \mu, \sigma)$ be their composition. If $\mu_1(u_1) \vee \mu_1(u_2) < \mu_2(v_1) \wedge \mu_2(v_2) \wedge \ldots \wedge \mu_2(v_n)$, then the strength $\mathscr{S}(G)$ of the composition $G = G_1[G_2]$ of G_1 and G_2 is as follows.*

$$\mathscr{S}(G) = \begin{cases} 1 & if\, n = 1 \;\; or \;\; n = 2, \\ (n-1) & if \;\; \mu_1(u_1) \neq \mu_1(u_2) \;\; and \;\; n > 2, \\ 2 & if \;\; \mu_1(u_1) = \mu_1(u_2) \;\; and \;\; n > 2. \end{cases}$$

Proof. For $n = 1$, G is a strong fuzzy path on two vertices and for $n = 2$, G is a strong fuzzy complete graph on 4 vertices. Therefore in these cases $\mathscr{S}(G) = 1$ by Theorems 1.4.1.

Now suppose that $n > 2$.

Case 1. $\mu_1(u_1) \neq \mu_1(u_2)$.

Without loss of generality assume that $\mu_1(u_1) < \mu_1(u_2)$ in G_1. Let u and v be any two non - adjacent vertices of G. If u or v or both belong to the set $\{w_{1j} : 1 \leq j \leq n\}$ then all the paths joining u and v have strength $\mu_1(u_1)$. Therefore $uw_{2j}v, j = 1, 2, \ldots, n$ are all extra strong $u - v$ paths.

If u and $v \in \{w_{2j} : 1 \leq j \leq n\}$. Let us suppose that $u = w_{2i}$ and $v = w_{2j}$ with $i < j$. If a path joining u and v contains a vertex $w_{1k}; 1 \leq k \leq n$ then its strength is $\mu_1(u_1)$. Therefore the extra strong path joining u and v is $uw_{2i+1} \ldots w_{2j-1}v$. Its length is clearly less than or equal to $n - 1$. If $u = w_{21}$ and $v = w_{2n}$ then the length of the extra strong $u - v$ path is equal to $n - 1$.

Similarly we can prove that if $\mu_1(u_1) > \mu_2(u_2)$ then $\mathscr{S}(G) = n - 1$.

Case 2. $\mu_1(u_1) = \mu_1(u_2)$.

Then all the edges of G have the same weight $\mu_1(u_1)$. For $u = w_{1i}, v = w_{1j}, uw_{2k}v$ and for $u = w_{2i}, v = w_{2j}$, $uw_{1k}v$ are extra strong paths. Therefore in this case strength of G is 2. □

Theorem 4.3.3. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with crisp graphs the path $P_2 = u_1u_2$ and the path $P_n = v_1v_2 \ldots v_n$ respectively and $G(V, \mu, \sigma)$ be their composition. Let l = maximum length of all subpaths of the path $w_{11}w_{12} \ldots w_{1n}$ of G of strength $> \mu_1(u_2) \vee$ maximum length of all subpaths of the path $w_{21}w_{22} \ldots w_{2n}$ of G of strength $> \mu_1(u_1)$ if such subpaths exist, otherwise let $l = 0$. Let $\mu_1(u_1) \vee \mu_1(u_2) \geq \mu_1(v_1) \wedge \mu_2(v_2) \wedge \ldots \wedge \mu_2(v_n)$.*

Then the strength $\mathscr{S}(G)$ *of the composition of* G_1 *and* G_2 *is* 2 *if* $\mu_1(u_1) = \mu_1(u_2)$. *Otherwise, it is* $l \vee 2$.

Proof. Let u and v be two non - adjacent vertices of G. Without loss of generality assume that $\mu_1(u_1) \leq \mu_1(u_2)$. If $u, v \in \{w_{1j} : 1 \leq j \leq n\}$. Then $u = w_{1i}$ and $v = w_{1k}$ for some $1 \leq i \neq k \leq n$. Then $w_{1i}w_{2i}w_{1k}$ has strength $\mu(w_{1i}) \wedge \mu(w_{1k})$. Therefore $w_{1i}w_{2i}w_{1k}$ is an extra strong $u - v$ path.

Suppose u and $v \in \{w_{2j} : 1 \leq j \leq n\}$. Let $u = w_{2i}$ and $v = w_{2j}$ with $i < j$. If all the vertices $v_k, i \leq k \leq j$ have weight $> \mu_1(u_1)$ then the extra strong path joining u and v is the path $w_{2i}w_{2i+1} \ldots w_{2j-1}w_{2j}$ of G_2 joining w_{2i} and w_{2j}. Otherwise $w_{2i}w_{1k}w_{2j}$, for some k for which $\mu_2(v_k) = \underset{i=1,2,\ldots n}{max} \{\mu_2(v_i)\}$ is an extra strong path joining u and v. Therefore $\mathscr{S}(G) = \max\{2, \text{length of the maximal subpath of} G_2 \text{ having strength } > \mu_1(u_1)\}$. □

Theorem 4.3.4. *Let* $G_1(V_1, \mu_1, \sigma_1)$ *and* $G_2(V_2, \mu_2, \sigma_2)$ *be two strong fuzzy graphs with crisp graphs* P_m *and* P_n *respectively where* $P_m = u_1u_2 \ldots u_m$ *and* $P_n = v_1v_2 \ldots v_n$ *where* $m, n > 2$. *Let the paths* $P_m = u_1u_2 \ldots u_m$ *and* $P_n = v_1v_2 \ldots v_n$ *be their respective underlying crisp graphs, where,* $m, n > 2$. *Let* $G(V, \mu, \sigma)$ *be the composition of* G_1 *and* G_2. *Then the strength* $\mathscr{S}(G)$ *of* G *is* $(m-1) \vee (n-1)$.

Proof. Let u, v be two non- adjacent vertices of G.

Case 1. $u, v \in \{w_{ij} : j = 1, 2, \ldots, n\}$.

Without loss of generality assume that $u = w_{ik}$ and $v = w_{iq}$ with $k < q$. Suppose there exist a vertex which is adjacent to both u and v in G such that $\mu(w) \geq \mu(w_{ik}) \wedge \mu(w_{ik+1}) \wedge \ldots \wedge \mu(w_{iq})$ such a vertex may exist if $\mu_1(u_{i-1})$ or $\mu_1(u_{i+1})$ is greater than or equal to $\mu_1(u_i)$. Then uwv is an extra strong path, which is of length 2.

Otherwise, the path $P_{kq} = w_{ik}w_{ik+1}\ldots w_{iq}$ is an extra strong $u-v$ path in G. The length of P_{kq} is $q-k \leq n-1$.

Case 2. $u, v \in \{w_{ij} : i = 1, 2, \ldots, m\}$ for some j, $1 \leq j \leq n$.

For $1 \leq j \leq n$, let H_j be the path $w_{1j}w_{2j}\ldots w_{mj}$ of G and for $1 \leq i \leq m$, L_i be the path $w_{i1}w_{i2}\ldots w_{in}$. Let $u = w_{kj}$ and $v = w_{pj}$, $k \neq p, 1 \leq j \leq n$. Then, all the $u-v$ paths pass through at least one vertex of each L_i; $k \leq i \leq p$. So $w_{kj}w_{k+1j}\ldots w_{k+mj}$ is an extra strong $u-v$ path. Every such path has length $|p-k|$. Therefore, if $u = w_{11}$ and $v = w_{m1}$ then the length of the extra strong path is $m-1$.

Case 3. $u = w_{ij}$ and $v = w_{kl}$, where $i \neq k$ and $j \neq l$.

Without loss of generality assume that $i < k$ and $j < l$. Then all the $u-v$ paths pass through at least one vertex of each $L_{i+1}, L_{i+2}, \ldots, L_{k-1}$. So the strength of the $u-v$ path in G must be $\leq \mu_1(u_i) \wedge \mu_1(u_{i+1}) \wedge \ldots \wedge \mu_1(u_{k-1}) \wedge \mu_1(u_k) \wedge \mu_2(v_j) \wedge \mu_2(v_l)$. Here $w_{ij}w_{i+1l}\ldots w_{k-1l}w_{kl}$ is an extra strong path in G and is of length equal to $|k-i|$, which is $= n-1$ when $k = 1$ and $i = n$.

Hence the theorem. ☐

Theorem 4.3.5. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs the path P_2 with vertex set $\{u_1, u_2\}$ and the star graph $S_n, n \geq 3$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ having v_n as the central vertex respectively. If $G(V, \mu, \sigma)$ is their composition, then the strength of G is 2.*

Proof. Let $u = w_{ij}$ and $v = w_{kl}$ be two non - adjacent vertices of G. Then either $i = k = 1$ or $i = k = 2$ and j and l are distinct from n. Let us suppose that $i = k = 1$. In this case any $u - v$ path has strength $\leq \mu_1(u_1) \wedge \mu_2(v_j) \wedge \mu_2(v_l)$. If $\mu_1(u_1) \leq \mu_1(u_2)$ then $uw_{2j}v$ is an extra strong path in G.

Now consider the case $\mu_1(u_1) > \mu_1(u_2)$. In this case we have the following subcases. If $\bigvee\limits_{i \neq j,l} \mu_2(v_i) \leq \mu_1(u_2)$ then again $uw_{2j}v$ is an extra strong $u-v$ path in G. Otherwise, let $\alpha = \bigvee\limits_{i \neq j,l} \mu_2(v_i) > \mu_1(u_2)$. If $\mu_2(v_m) = \alpha$ then $uw_{1n}v$ is an extra strong $u - v$ path in G. If $\mu_2(v_m) = \alpha$ for some $m \neq j, l, n$ and $\mu_2(v_n) \leq \mu_1(u_2)$ then $uw_{2m}v$ is an extra strong $u - v$ path in G. Thus in the case $i = k = 1$ the length of the extra strong $u - v$ path is 2.

If $i = k = 2$, as above, we can prove that the length of extra strong $u - v$ path is 2. Hence $\mathscr{S}(G) = 2$.

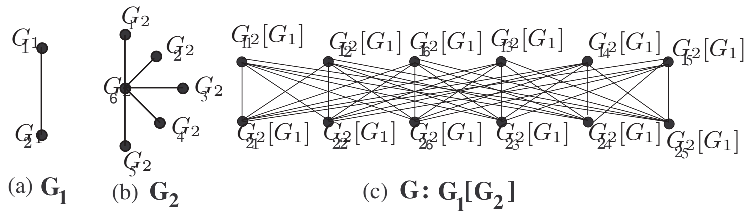


Figure 4.14: (a) A strong fuzzy path G_1, (b) a strong fuzzy star graph G_2 and (c) their composition G.

□

Theorem 4.3.6. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs, the path P_2 with vertex set $\{u_1, u_2\}$ and the Bull graph with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ respectively and $G(V, \mu, \sigma)$ be their composition. Then the strength $\mathscr{S}(G)$ of G is 2 if $\mu_1(u_1) = \mu_1(u_2)$.*

Proof. Let G_1, G_2 and G be as shown in Figure 4.15. Let u and v be two non-adjacent vertices of G. Then $u, v \in \{w_{11}, w_{13}, w_{15}\}$ or $u, v \in \{w_{11}, w_{14}\}$ or $u, v \in \{w_{12}, w_{15}\}$ or $u, v \in \{w_{21}, w_{23}, w_{25}\}$ or $u, v \in \{w_{21}, w_{24}\}$ or $u, v \in \{w_{22}, w_{25}\}$.

First of all we suppose that u, v belong to $\{w_{11}, w_{13}, w_{15}\}$ or belong to $\{w_{11}, w_{14}\}$ or belong to $\{w_{12}, w_{15}\}$. In these cases let us write $u = w_{1i}$ and $v = w_{1j}$ for suitable i and j. Then since $\mu_1(u_1) = \mu_1(u_2)$ and strength of any path joining u and v is $\leq \mu_1(u_1) \wedge \mu_2(v_i) \wedge \mu_2(v_j)$ we have $uw_{2i}v$ is an extra strong $u - v$ path in G, which is of length 2.

Similarly if u, v belong to $\{w_{21}, w_{23}, w_{25}\}$ or belong to $\{w_{21}, w_{24}\}$ or belong to

$\{w_{22}, w_{25}\}$ then also every extra strong path joining them has length 2. Hence $\mathscr{S}(G) = 2$.

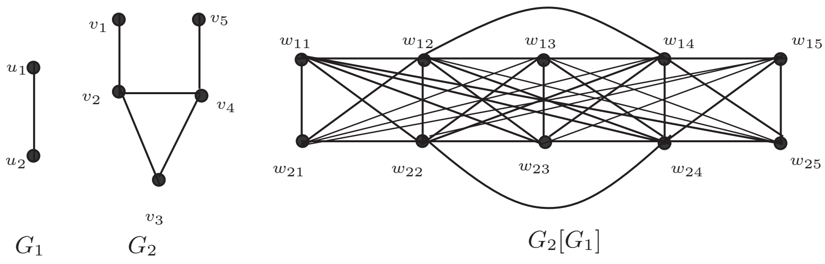


Figure 4.15: Composition of a fuzzy path and a fuzzy bull graph.

□

Theorem 4.3.7. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs, the path P_2 with vertex set $\{u_1, u_2\}$ and the Bull graph with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$. Let $G(V, \mu, \sigma)$ be their composition. If $\mu_1(u_1) > \mu_2(v_2) \wedge \mu_2(v_4) > \mu_1(u_2)$ or $\mu_1(u_1) < \mu_2(v_2) \wedge \mu_2(v_4) < \mu_1(u_2)$ then the strength of G is* 3.

Proof. Let G_1, G_2 and G as shown in Figure 4.15. First of all suppose that $\mu_1(u_1) > \mu_2(v_2) \wedge \mu_2(v_4) > \mu_1(u_2)$. Let u and v be two nonadjacent vertices of G. Then $u, v \in \{w_{11}, w_{13}, w_{15}\}$ or $u, v \in \{w_{11}, w_{14}\}$ or $u, v \in \{w_{12}, w_{15}\}$ or $u, v \in \{w_{21}, w_{23}, w_{25}\}$ or $u, v \in \{w_{21}, w_{24}\}$ or $u, v \in \{w_{21}, w_{25}\}$. If $u = w_{11}$ and $v = w_{15}$ or vice versa then there is only one extra strong path P, which is $w_{11}w_{12}w_{14}w_{15}$. All other $u - v$ paths have strength either strictly less than that of P or length $\geq$ that of P and strength $\leq$ that of P. Clearly length of P is 3.

In all other cases the length of extra strong $u - v$ paths are of length 2. Therefore strength of G is 3.

Similarly we can prove that strength of G is 3 if $\mu_1(u_1) < \mu_2(v_2) \wedge \mu_2(v_4) < \mu_1(u_2)$. □

Theorem 4.3.8. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs, the path P_2 with vertex set $\{u_1, u_2\}$ and the Bull graph with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$. Let $G(V, \mu, \sigma)$ be their composition. If $\mu_1(u_1) \neq \mu_1(u_2)$ and $\mu_2(v_2) \wedge \mu_2(v_4) = \mu_1(u_1) \wedge \mu_1(u_2)$. Then strength of G is 2.*

Proof. Suppose that $\mu_1(u_1) > \mu_1(u_2)$ and $\mu_2(v_2) \leq \mu_2(v_4)$. The other case can be dealt in the same fashion. Then the given condition becomes $\mu_1(u_2) = \mu_2(v_2)$. In this case if $u = w_{11}$ and $v = w_{15}$ (or $u = w_{21}$ and $v = w_{25}$) then $uw_{24}v$ (respectively $uw_{14}v$) is an extra strong $u - v$ path in G of length 2. In all other cases clearly extra strong $u - v$ paths have length 2. Therefore strength of G is 2. □

Theorem 4.3.9. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs the path $P_2 = u_1u_2$ and $C_n = v_1v_2 \ldots v_n$ respectively. Let $v \in V(G_2)$ be such that $\mu_2(v) = \mu_2(v_1) \wedge \mu_2(v_2) \ldots \wedge \mu_2(v_n)$. Then the strength of composition of G_1 and G_2 is*

$$\mathscr{S}(G) = \begin{cases} [\frac{n}{2}] & \textit{if} \quad \mu_1(u_1) \ \textit{and} \quad \mu_1(u_2) \leq \mu_2(v) \quad \textit{and} \ \mu_1(u_1) \neq \mu_1(u_2), \\ 2 & \textit{if} \ \mu_1(u_1) \ \textit{and} \quad \mu_1(u_2) \leq \mu_2(v) \ \textit{and} \quad \mu_1(u_1) = \mu_1(u_2) \\ & \textit{or if} \quad \mu_1(u_1) \ \textit{and} \quad \mu_1(u_2) > \bigvee_{i=1}^{n} \mu_2(v_i), \\ \mathscr{S}(G_2) & \textit{if} \quad \mu_1(u_1) > \bigvee_{i=1}^{n} \mu_2(v_i) \quad \textit{and} \ \mu_1(u_2) < \mu_2(v) \\ & \textit{or if} \ \ \mu_1(u_2) > \bigvee_{i=1}^{n} \mu_2(v_i) \quad \textit{and} \ \mu_1(u_1) < \mu_2(v). \end{cases}$$

Proof. Let u and v be two nonadjacent vertices of $G = G_1[G_2]$. Then either $u, v \in \{w_{1j} : j = 1, 2, \ldots, n\}$ or $u, v \in \{w_{2j} : j = 1, 2, \ldots, n\}$(See Figure 4.16).

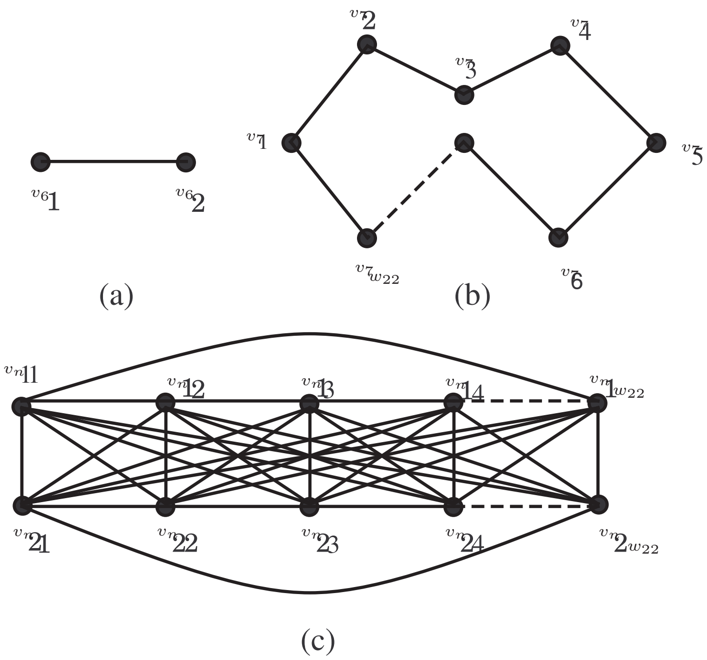


Figure 4.16: (a) A strong fuzzy path G_1, (b) a strong fuzzy cycle G_2 and (c) their composition G.

Case 1. $\mu_1(u_1)$ and $\mu_1(u_2) \leq \mu_2(v)$ and $\mu_1(u_1) \neq \mu_1(u_2)$.

In this case $\mu(w_{ij}) = \mu_1(u_i)$ for $i = 1, 2$. Without loss of generality assume that $\mu_1(u_1) < \mu_1(u_2)$. Then for the first choice of u and v ie, for $u, v \in \{w_{1j} : j = 1, 2, \ldots, n\}$ all the $u-v$ paths have same strength in G. So uwv is an extra strong $u - v$ path of length 2 where w is any vertex in the set $\{w_{2j} : j = 1, 2, \ldots, n\}$.

For the second choice of u and v ie, for $u, v \in \{w_{2j} : j = 1, 2, \ldots, n\}$, the vertices of the extra strong path joining them lie completely in the set of $\{w_{2j} : j = 1, 2, \ldots, n\}$. In G this set of vertices forms a fuzzy cycle and each vertex has same strength as $\mu_1(u_2)$. So the strength of the $u - v$ path in G is $[\frac{n}{2}]$.

Case 2. $\mu_1(u_1)$ and $\mu_1(u_2) \leq \mu_2(v)$ and $\mu_1(u_1) = \mu_1(u_2)$.

Then all the vertices of G have same weight $\mu_1(u_1)$. So in both choices for u and v ie, for $u, v \in \{w_{1j} : j = 1, 2, \ldots, n\}$ or $u, v \in \{w_{2j} : j = 1, 2, \ldots, n\}$, every extra strong path is of length 2.

Case 3. $\mu_1(u_1) \wedge \mu_1(u_2) >$ the weight of every vertex of G_2.

In this case the vertices in $\{w_{1j} : j = 1, 2, \ldots, n\}$ and in $\{w_{2j} : j = 1, 2, \ldots, n\}$ form two fuzzy cycles both are copies of G_2. Therefore, if $u, v \in \{w_{1j} : j = 1, 2, \ldots, n\}$ and by choosing a vertex w of $\{w_{2j} : j = 1, 2, \ldots, n\}$ of maximum weight, we get an extra strong $u - v$ path namely uwv of length 2. Similarly if $u, v \in \{w_{2j} : j = 1, 2, \ldots, n\}$ we get an extra strong $u - v$ path of length 2. Therefore in this case strength of G is 2.

Case 4. $\mu_1(u_1) > \bigvee_{i=1}^{n} \mu_2(v_i)$ and $\mu_1(u_2) < \mu_2(v)$ or $\mu_1(u_2) > \bigvee_{i=1}^{n} \mu_2(v_i)$ and $\mu_1(u_1) < \mu_2(v)$.

Without loss of generality assume $\mu_1(u_1) > \bigvee_{i=1}^{n} \mu_2(v_i)$ and $\mu_1(u_2) < \mu_2(v)$. Then $\mu(w_{2j}) = \mu_2(u_2) \forall j$.

If u and v are in the first choice, the vertices of the extra strong paths joining them lie completely in the set of $\{w_{1j} : j = 1, 2, \ldots, n\}$. In G this set of vertices forms a fuzzy cycle, which is a copy of G_2. So the strength of the $u - v$ path in G is $\mathscr{S}(G_2)$.

If u, v are as in the second choice then all the $u - v$ paths have same strength $\mu_1(u_2)$. So the length of the extra strong path joining u and v is 2.

□

4.4 Normal products

In this section Normal products of strong fuzzy graphs and their strength are discussed.

Definition 4.4.1. [38]

For $i = 1, 2$, let $G_i(V_i, \mu_i, \sigma_i)$ be two fuzzy graphs with underlying crisp graphs $G_i(V_i, E_i)$. Their normal product, denoted by $G_1 \boxtimes G_2$, of G_1 and G_2 is the fuzzy graph $G(V, \mu, \sigma)$ with the underlying crisp graph the normal product of the crisp graph $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ with vertex set $V = V_1 \times V_2$ and the edge set $E = \{(u, u_2)(u, v_2) | u \in V_1, (u_2, v_2) \in E_2\} \cup \{(u_1, w)(v_1, w) | (u_1, v_1) \in E_1, w \in V_2\} \cup \{(u_1, u_2)(v_1, v_2) | (u_1, v_1) \in E_1, (u_2, v_2) \in E_2\}$ and whose membership functions μ and σ are defined as $\mu(u_1, u_2) = \mu_1(u_1) \wedge \mu_2(u_2)$ if $(u_1, u_2) \in V$ and

$$\sigma((u_1, u_2)(v_1, v_2)) = \begin{cases} \mu_1(u_1) \wedge \sigma_2(u_2, v_2) & \text{if } u_1 = v_1 \text{ and } (u_2, v_2) \in E_2, \\ \sigma_1(u_1, v_1) \wedge \mu_2(u_2) & \text{if } u_2 = v_2 \text{ and } (u_1, v_1) \in E_1, \\ \sigma_1(u_1, v_1) \wedge \sigma_2(u_2, v_2) & \text{if } (u_1, u_2) \in E_1 \text{ and } (v_1, v_2) \in E_2, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.4.1. *Let $G(V, \mu, \sigma)$ be the normal product of two strong fuzzy graphs $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ with their respective underlying crisp graphs*

1. the paths P_2 and $P_n, n > 1$. Then $\mathscr{S}(G) = n - 1$.

2. the complete graphs K_n and K_m. Then $\mathscr{S}(G) = 1$.

3. the paths P_2 and the star graph S_n. Then $\mathscr{S}(G) = 2$.

4. the star graphs S_m and S_n. Then $\mathscr{S}(G) = 2$.

Proof. □

1. In this case the normal product of G_1 and G_2 is a $2-$ connected fuzzy graph with n parts. Each part of which is a complete fuzzy graph on 4 vertices. Hence the proof follows by Theorem 2.3.2.

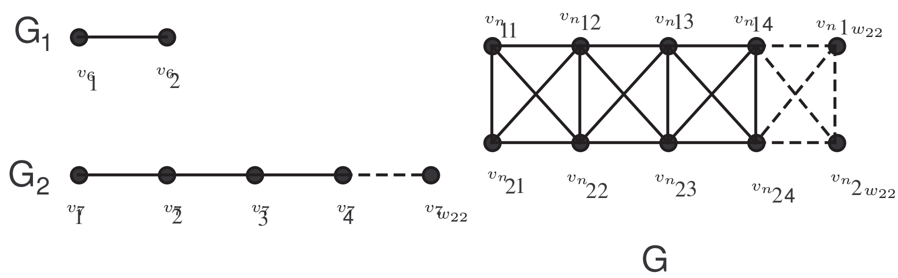


Figure 4.17: Normal product of a strong fuzzy path on two vertices and a strong fuzzy path on n vertices.

2. In this case the normal product of G_1 and G_2 is a complete fuzzy graph. So $\mathscr{S}(G) = 1$ by Theorem 1.4.1.

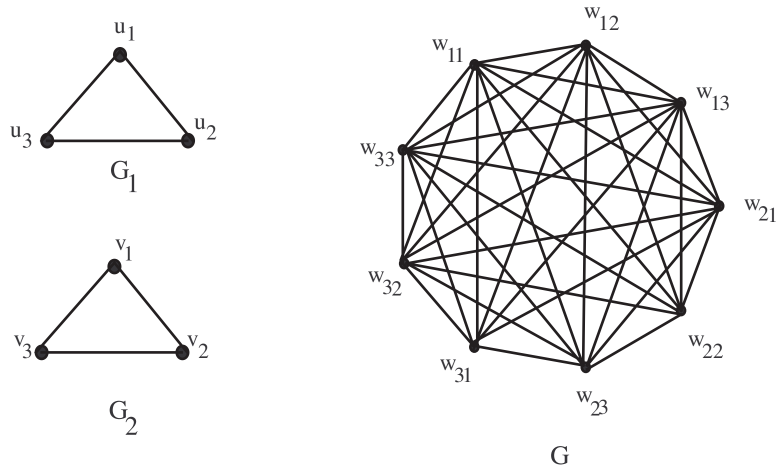


Figure 4.18: Normal product of two complete fuzzy graphs G_1 and G_2.

3. Let G_1 be a strong fuzzy path on the vertex set $V_1 = \{u_1, u_2\}$ and G_2 be the strong fuzzy star graph with vertex set $V_2 = \{v_1, \ldots, v_n\}$ with v_n as the central vertex of G_2. Let u, v be two non-adjacent vertices of $G = G_1 \boxtimes G_2$. (See Figure 4.19). As all the $u - v$ paths contain the vertex w_{1n} or the vertex w_{2n} or both w_{1n} and w_{2n}, the strength of any $u - v$ path is $\leq (\mu(w_{1n}) \vee \mu(w_{2n})) \wedge \mu(u) \wedge \mu(v)$. From this it is clear that uwv is an extra strong $u - v$ path in G, where $w = w_{1n}$ or w_{2n} according as $\mu(w_{1n}) \geq \mu(w_{2n})$ or $\mu(w_{2n}) \geq \mu(w_1 n)$. Therefore $\mathscr{S}(G) = 2$.

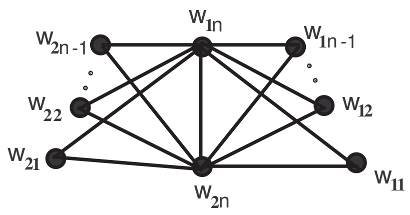


Figure 4.19: Normal product of a strong fuzzy path on two vertices and a strong fuzzy star graph.

4. Let G_1 and G_2 be two strong fuzzy star graphs with their underlying crisp graphs S_m and S_n respectively. Let $V(G_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_n\}$. Let u_m be the central vertex of G_1 and v_n be the central vertex of G_2. Let u, v be two non-adjacent vertices of G. Then $u, v \neq w_{mn}$, because w_{mn} is adjacent to all the other vertices of G. If one of them is $w_{ij}, j = 1, 2, \ldots, n-1$, then the other is different from w_{in} and one of them is $w_{ij}, i = 1, 2, \ldots, m-1$ then the other is different from w_{mj}.

   Let $u, v \in \{w_{ij} : j = 1, 2, \ldots, n-1\}$ for some $i, 1 \leq i < m$. Then all the $u - v$ paths pass through either w_{in} or through w_{mj} or through w_{mn}. Therefore uwv is an extra strong path where $w \in \{w_{in}, w_{mj}, w_{mn}\}$ such that $\mu(w) = max\{\mu(w_{mn}), \mu(w_{in}), \mu(w_{mj})\}$. Therefore the length of the extra strong $u - v$ path is 2. Similarly if $u, v \in \{w_{ij} : i = 1, 2, \ldots, m-1\}$ for some $j, 1 \leq j \leq n$, the length of extra strong $u - v$ path is 2.

   Let $u = w_{ij}$ and $v = w_{kl}$ where $i \neq k$ and $j \neq l$ and $1 \leq i, k \leq m$,

$1 \leq j, l \leq n$. Then all the paths must pass through w_{mn}. Hence we have only one extra strong $u - v$ path in G, that is $uw_{mn}v$.

Theorem 4.4.2. *Let $G(V, \mu, \sigma)$ be the normal product of a strong fuzzy path $G_1(V_1, \mu_1, \sigma_1)$ on two vertices and a strong fuzzy butterfly graph $G_2(V_2, \mu_2, \sigma_2)$. Then $\mathscr{S}(G) = 2$.*

Proof. □

Let H_1 be the strong fuzzy path with vertex set $V_1 = \{u_1, u_2\}$ and H_2 be the strong fuzzy butterfly graph with vertex set $V_2 = \{v_1, v_2, v_3, v_4, v_5\}$ as shown in Figure 4.20.

The merger graph of the normal product G of G_1 and G_2 is a $1-$ linked graph with two parts. Therefore $\mathscr{S}(G) = 2$.

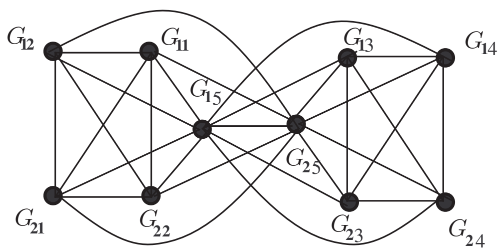


Figure 4.20: Normal product of a strong fuzzy path on two vertices and a strong fuzzy butterfly graph and their merger graph.

Conjecture 4.4.1. Let $G(V, \mu, \sigma)$ be the normal product of two strong fuzzy graphs $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ with their respective underlying crisp graphs are the paths P_n and P_m with $n \geq m, m, n > 1$. Then $\mathscr{S}(G) = n - 1$.

Conjecture 4.4.2. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with their underlying crisp graphs $P_2 = u_1u_2$ and $C_n = v_1v_2 \ldots v_n$ respectively and the weight of the weakest vertices of G_1 is greater than the weight of the weakest vertices of G_2. If the weakest vertices of G_2 altogether form a subpath of length l in G_2 then the strength of normal product $G(V, \mu, \sigma)$ of G_1 and G_2 is $n - l$ if $l \leq \left[\frac{n}{2}\right]$ and $[\frac{n}{2}]$ if $l > [\frac{n}{2}]$.

Conjecture 4.4.3. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with their underlying crisp graphs $P_2 = u_1u_2$ and $C_n = v_1v_2 \ldots v_n$ respectively. Suppose that $\mu_1(u_1) \leq \mu_1(u_2) \wedge \mu_2(v_1) \wedge \mu_2(v_2) \wedge \ldots \wedge \mu_2(v_n)$. Let $G(V, \mu, \sigma)$ be the normal product of G_1 and G_2. Then the strength $\mathscr{S}(G)$ is

$$\mathscr{S}(G) = \begin{cases} max\{[\frac{n}{2}], \mathscr{S}(G_2)\} & \text{if } \mu_1(u_2) > \wedge\mu_2(v_i), \\ [\frac{n}{2}] & \text{if } \mu_1(u_2) \leq \wedge\mu_2(v_i). \end{cases}$$

Chapter 5

Relation between some fuzzy graphs and their line graphs

The line graph of a graph $G(V, E)$ represents the adjacencies between edges of G. Whitney and Krausz (1943) constructed the line graph in their papers 'Congruent graphs and the connectivity of graphs' and and the name line graph was given by Harary and Norman [19]. John N. Mordeson [33] defined and gave some results of fuzzy line graph in his paper 'Fuzzy line graphs'.

In this chapter we find the strength of the line graphs of strong fuzzy butterfly graph, strong fuzzy star graph, strong fuzzy bull graph and strong fuzzy diamond graph, strong fuzzy path, strong fuzzy cycle in terms of the respective graphs.

Definition 5.0.1. [33] Let $G(V, \mu, \sigma)$ be a fuzzy graph with its underlying crisp

Some results of this chapter are included in the following paper Chithra K. P., Raji Pilakkat, Annals of Fuzzy Mathematics and Informatics, Volume 30, No:2, 2017, 107-115

graph $G(V, E)$. The fuzzy line graph $L(G)(V_L, \mu_L, \sigma_L)$ of $G(V, \mu, \sigma)$ is the fuzzy graph with its underlying crisp graph $L(G)(V_L, E_L)$ is the line graph of $G(V, E)$ where the vertex set $V_L = E$ and edge set
$E_L = \{uv : u$ and v are edges in G, which have a common vertex in $G\}$, $\mu_L(u) = \sigma(u)$ if $u \in V_L$ and for $u, v \in E_L$

$$\sigma_L(uv) = \begin{cases} \sigma(u) \wedge \sigma(v) & \text{if } u \text{ and } v \text{ have a vertex in common,} \\ 0 & \text{otherwise.} \end{cases}$$

5.1 Line graph of some strong fuzzy graphs

5.1.1 Strong fuzzy butterfly graph

Theorem 5.1.1. *The strength of the line graph of a strong fuzzy butterfly graph is three.*

Proof. The line graph $L(G)$ of a strong fuzzy butterfly graph $G(V, \mu, \sigma)$ is a $2-$linked fuzzy graph with parts $G_1(V_1, \mu_1, \sigma_1), G_2(V_2, \mu_2, \sigma_2)$, and $G_3(V_3, \mu_3, \sigma_3)$, where $G_1(V_1, \mu_1, \sigma_1)$ and $G_3(V_3, \mu_3, \sigma_3)$ are fuzzy triangles and $G_2(V_2, \mu_2, \sigma_2)$ is fuzzy complete graph on 4 vertices (A butterfly graph and its line graph are shown in figure 1). So by Theorem 2.3.2 strength of $L(G)$ is 3.

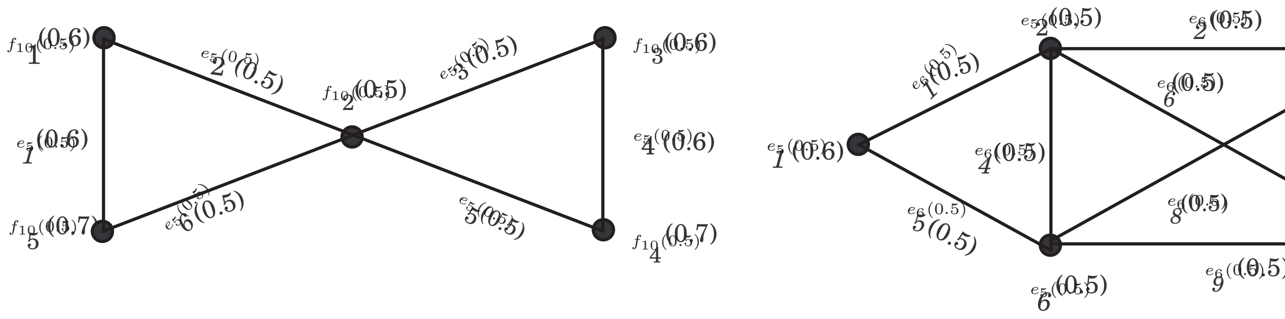


Figure 5.1: (a) A strong fuzzy Butterfly graph G and (b) its line graph L(G).

□

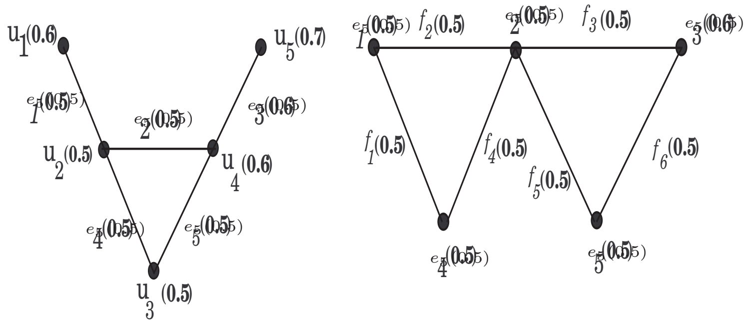
5.1.2 Strong fuzzy star graph

Theorem 5.1.2. *The strength of the line graph of a strong fuzzy star graph is one.*

Proof. In a strong fuzzy star graph S_n all the edges are adjacent. So the line graph of the strong fuzzy star graph is a strong fuzzy complete graph. Therefore by Theorem 1.4.1 the strength of the line graph of a strong fuzzy star graph is one. □

5.1.3 Strong fuzzy bull graph

Theorem 5.1.3. *The strength of the line graph of a strong fuzzy bull graph is* 2.



Proof.

Figure 5.2: (a) A strong fuzzy Bull graph G and (b) its line graph L(G).

The line graph of a strong fuzzy bull graph is a strong fuzzy butterfly graph (A bull graph $G(V, \mu, \sigma)$ and its line graph are shown in Figure 2). Therefore by Theorem 2.3.2 the strength of the line graph of a strong fuzzy bull graph is 2.

□

5.1.4 Strong fuzzy diamond graph

Theorem 5.1.4. *The strength of line graph of a strong fuzzy diamond graph is 2.*

Proof. The line graph of a strong fuzzy diamond graph is a strong fuzzy wheel graph on 5 vertices as shown in Figure 5.3. Therefore by Theorems 3.1.4, 3.1.5, 3.1.6 strength of line graph of a strong fuzzy diamond graph is 2.

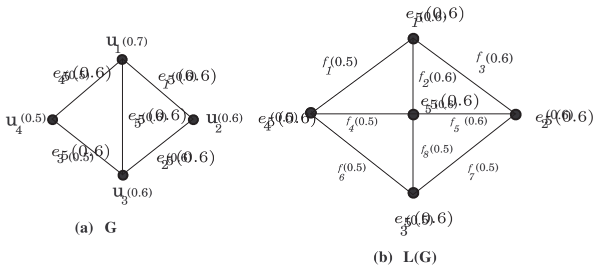


Figure 5.3: A strong fuzzy diamond graph G and its line graph L(G).

□

5.2 Line graph of strong fuzzy cycle

Proposition 5.2.1. In a strong fuzzy cycle of length n suppose there are l weakest edges which do not altogether form a subpath. Let s denote the maximum length of a subpath which does not contain any weakest edge. Then

$$\mathscr{S}(G) = \begin{cases} [\frac{n}{2}] & if \;\; s \leq [\frac{n}{2}], \\ s & if \;\; s > [\frac{n}{2}]. \end{cases}$$

Proof. Let u, v be two non-adjacent vertices of G. Then in G there are two paths joining u and v. If both the paths contain a weakest vertex then the extra strong path joining u and v is the shortest path joining u and v in its underlying crisp graph, which is of length $\leq [\frac{n}{2}]$. If u and v are the end vertices of a path having length $[\frac{n}{2}]$ then the extra strong path joining u and v is of length $= [\frac{n}{2}]$.

Otherwise, there is a $u-v$ path P having no weakest vertices. Then P is an extra strong path joining u and v. The length of P, by hypothesis, is $\leq s$. If u and v are the end vertices of the maximal subpath which does not contain any weakest edge in G then the length of P is s. Hence the theorem. $\square$

Now we consider the case of the strength of line graph of a fuzzy cycle. To determine this we introduce the following definitions:

Definition 5.2.1. Two paths P_1 and P_2 of a fuzzy cycle C are said to be vertex disjoint or simply disjoint if $V(P_1) \cap V(P_2) = \phi$ and edge disjoint if $E(P_1) \cap E(P_2) = \phi$ where $V(P_2)$ denotes the vertices of P_2 and $E(P_2)$ denotes the edges of $P_i, i = 1, 2$.

Definition 5.2.2. Suppose P_1 and P_2 are two disjoint paths of a fuzzy cycle C with respective end points u_1, v_1 and u_2, v_2. Then, $< (V(C) \setminus (V(P_1 \cup P_2)) \cup \{u_1, u_2, v_1, v_2\}) >$ is a union of two disjoint paths of C, called complementary paths relative to the paths P_1 and P_2.

Definition 5.2.3. Let $G(V, \mu, \sigma)$ be a fuzzy graph. A path P in G with all its edges have weight equal to w where $w = \min \{\sigma(uv) : \sigma(uv) > 0 \text{ in } G\}$ is called a weakest path. A weakest path which is not a proper subpath of any other weakest path in the fuzzy graph G is called a maximal weakest path in G.

Here after in this chapter we denote the weight of weakest paths of any fuzzy graph G by w.

Note 5.2.1. A graph may have more than one maximal weakest paths. For example, in the strong fuzzy cycle G in Figure 5.4 $u_2u_3u_4u_5u_6u_7$ and $u_8u_9u_{10}u_{11}u_{12}$ are maximal weakest paths of G.

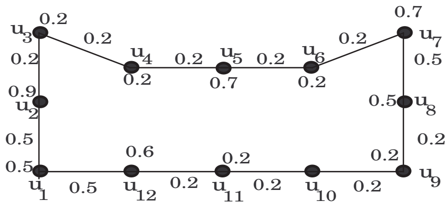


Figure 5.4: A strong fuzzy cycle G.

Definition 5.2.4. Two paths of the collection P of pairwise disjoint paths in a fuzzy cycle C are said to be consecutive if one of the complementary paths relative to them contains all other paths of P.

Definition 5.2.5. A collection P of pairwise disjoint paths in a fuzzy cycle C is said to form a chain if its members can be arranged in a sequence $P_1, P_2, \ldots, P_n$ such that $(P_1, P_2), (P_2, P_3), \ldots, (P_{n-1}, P_n)$ and (P_1, P_n) are consecutive.

Proposition 5.2.2. Let G be a strong fuzzy path (or a strong fuzzy cycle), then its fuzzy line graph $L(G)$ is also a strong fuzzy path (strong fuzzy cycle).

Proof. Let G be a strong fuzzy path. Let underlying crisp graph be the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, e_2, \ldots, e_{n-1}\}$ where $e_i = v_iv_{i+1}, i = 1, 2, \ldots, n-1$. Since for $1 < i < n-1$ the edge e_i in the underlying crisp graph is adjacent only to the edge e_{i-1} and e_{i+1}, the vertex e_i of the crisp graph $L^*(G)$ of

$L(G)$ is adjacent only to the vertices e_{i-1} and e_{i+1} of $L^*(G)$. Since the edge e_1 of underlying crisp graph is adjacent only to the edge e_2 of underlying crisp graph and the edge e_n of underlying crisp graph is adjacent only to the edge e_{n-1} of underlying crisp graph, the vertices e_1 and e_n of $L^*(G)$ are adjacent only to its vertices e_2 and e_{n-1} respectively. Thus $L^*(G)$ is a path with vertices $e_1, e_2, \ldots, e_n$ and edges $e_1e_2, e_2e_3, \ldots, e_{n-1}e_n$. The lemma now follows from the definition of $L(G)$. □

Similar is the case of a fuzzy cycle.

Proposition 5.2.3. If P is a weakest path of length k in a strong fuzzy graph G then in the fuzzy line graph $L(G)$ of G the path P' corresponding to the path P of G with vertex set as edge set of P is a weakest path in $L(G)$ of length $k-1$.

Theorem 5.2.1. *Let G be a strong fuzzy cycle of length n. Suppose there are l weakest edges which form m maximal weakest paths in G. Then for $n > 3$ and $m < [\frac{n}{2}]$ the line graph $L(G)$ of G has $l + m$ weakest edges.*

Proof. By Proposition 5.2.3, for a weakest path P of G with strength w and length l, the path P' of $L(G)$ with vertex set as edge set of P is a path of length $(l-1)$ with strength w. Note that the end vertices of u and v of P' are also have weight w. So the edges incident with u and v in $L(G)$ are also have weight w. So each maximal weakest path P in G of length p gives a weakest path in $L(G)$ of length $p+1$. Therefore m weakest paths, give rise to $(l+m)$ weakest edges in $L(G)$.

Also if P_1' and P_2' are two paths of $L(G)$ corresponding to two distinct maximal paths P_1 and P_2 of G, then they are edge disjoint. [Note that the path P' of L(G) thus obtained need not be maximal. See Figure 5.5]. $\square$

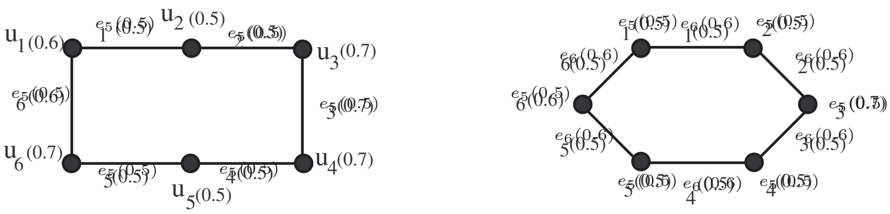


Figure 5.5: A fuzzy cycle G of length 6 with 4 weakest edges and its line graph $L(G)$.

Proposition 5.2.4. Suppose P_1 and P_2 are two disjoint weakest paths of lengths n_1 and n_2 respectively in the fuzzy cycle C. Suppose one of the complementary paths P relative to these paths is of length one, then there exists a weakest path of length $(n_1 + n_2)$ in $L(G)$ with edges of P_1, P_2 and P as vertex set.

Theorem 5.2.2. *Let G be a strong fuzzy cycle of length n. Suppose G contains exactly one maximal weakest path P. Let its length be l. Then the strength $\mathscr{S}(L(G))$ of the line graph $L(G)$ of G is*

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) - 1 & \text{if } l \le [\frac{n-1}{2}], \\ \mathscr{S}(G) & \text{if } l > [\frac{n-1}{2}]. \end{cases}$$

Proof. Since P is a path of length l in G by Proposition 5.2.3 the path P' of $L(G)$ with vertex set as edge set of P is a weakest path of $L(G)$ of length $l - 1$.

If $l = n-1$, then all edges in G but one is weakest. In this case all the edges of $L(G)$ are weakest. Hence by 1.4.1, $\mathscr{S}(L(G)) = [\frac{n}{2}] = \mathscr{S}(G)$. Let us suppose that $l < n-1$. Since all the vertices of P' are weakest the edges incident to the vertices of P' are also weakest edges and all the other edges are non-weakest. There are $l+1$ edges incident with the vertices of P' by Theorem 5.2.1. In this case there is only one weakest path $L(G)$ which is of length $l+1$.

Now by Theorem 1.4.2.

$$\mathscr{S}(L(G)) = \begin{cases} n-(l+1) & \textit{if } \ l+1 \leq [\frac{n+1}{2}], \\ [\frac{n}{2}] & \textit{if } \ l+1 > [\frac{n+1}{2}]. \end{cases}$$

$$= \begin{cases} \mathscr{S}(G)-1 & \textit{if } \ l \leq [\frac{n-1}{2}], \\ \mathscr{S}(G) & \textit{if } \ l > [\frac{n-1}{2}]. \end{cases}$$

□

Theorem 5.2.3. *Let G be a strong fuzzy cycle of length n with l weakest edges. Let there be m maximal weakest paths $P_1, P_2, \ldots, P_m$ in G, where $m \geq 1$. If for $i = 1, 2, \ldots, m-1$, one of the complementary paths Q_i between P_i and P_{i+1} is of length one such that $P_1Q_1P_2Q_2 \ldots P_{m-1}Q_{m-1}P_m$ is a path of length $l+m-2$ and the complementary paths between P_1 and P_m which does not contain any P_i*

is of length ≥ 2. *Then the strength* $\mathscr{S}(L(G))$ *of* $L(G)$ *is*

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) - m & \text{if } \ l \leq [\frac{n+1}{2}] - m, \\ [\frac{n}{2}] & \text{if } \ l > [\frac{n+1}{2}] - m. \end{cases}$$

Proof. Let Q be the complementary path between P_m and P_1. Then Q does not contain any of the paths $P_1, P_2, P_3, \ldots, P_{m-1}, P_m$.

If Q is of length one then in $L(G)$ either both ends of each edge is weakest vertices or one of the ends is a weakest vertex. Thus every edge in $L(G)$ in this case is a weakest edge. Hence $\mathscr{S}(L(G)) = [\frac{n}{2}] = \mathscr{S}(G)$ by Theorem 1.4.1.

Now suppose that the length of Q is not one. Note that the vertices of $L(G)$ corresponding to the edges of $P_1, P_2, \ldots, P_m$ are weakest. Though the vertices of $L(G)$ corresponding to the edges $Q_1, Q_2, \ldots, Q_{m-1}$ are not weakest, the edges incident with them have weakest vertices on the other end. Thus the path P in $L(G)$ with vertex set as edge set of the path $P_1Q_1P_2Q_2 \ldots P_{m-1}Q_{m-1}P_m$ of G together forms a weakest path of length $l + m - 2$. Since there are more than one edge in Q, the edge e_1 of Q incident with P_1 and the edge e_2 of Q incident with the path P_m are different in $L(G)$. The vertex of $L(G)$ corresponding to the edge e_1 of G is adjacent to one end vertex of P by a weakest edge and the vertex of $L(G)$ corresponding to the edge e_2 is adjacent to the other end of P by a weakest edge. All other edges of $L(G)$ are of non weakest. Hence $L(G)$ contains only one maximal weakest path of length $l+m$. Therefore the strength

$\mathscr{S}(L(G))$ of $L(G)$ is

$$\mathscr{S}(L(G)) = \begin{cases} n-(l+m) & \text{if } \ l \leq [\frac{n+1}{2}]-m, \\ [\frac{n}{2}] & \text{if } \ l > [\frac{n+1}{2}]-m. \end{cases}$$

$$= \begin{cases} (n-l)-m & \text{if } \ l \leq [\frac{n+1}{2}]-m, \\ [\frac{n}{2}] & \text{if } \ l > [\frac{n+1}{2}]-m. \end{cases}$$

$$= \begin{cases} \mathscr{S}(G)-m & \text{if } \ l \leq [\frac{n+1}{2}]-m, \\ [\frac{n}{2}] & \text{if } \ l > [\frac{n+1}{2}]-m. \end{cases}$$

Hence the proof. □

Theorem 5.2.4. *Let G be a strong fuzzy cycle of length n. Suppose there are l weakest edges in G which do not altogether form a subpath in G. Let $P_1, P_2, \ldots, P_m$ be the chain of all m maximal weakest paths in G. If for every P_i, P_{i+1} which do not contain any of the P_k's are of length greater than one (when $j = m, P_{j+1} = P_1$). Let s denote the maximum length of the subpaths which do not contain any weakest edge of G. Then if $l < [\frac{n}{2}] - (m+1)$, the strength $\mathscr{S}(L(G))$ of the line graph $L(G)$ of G is*

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G)-1 & \text{if } \ s = [\frac{n}{2}]+1 \quad \text{and } n \text{ odd}, \\ \mathscr{S}(G) & \text{otherwise}. \end{cases}$$

Proof. Since l weakest edges of G are distributed to form m maximal weakest paths in G, there are $l+m$ weakest edges in $L(G)$. Also the maximum length of paths in $L(G)$ which do not contain any weakest edge is clearly $s-1$. By Theorem 1.4.4, the strength $\mathscr{S}(L(G))$ of $L(G)$, when $l+m<[\frac{n}{2}]-1$ is

$$\mathscr{S}(L(G))=\begin{cases}[\frac{n}{2}] & \textit{if } \ s\leq[\frac{n}{2}]+1,\\ s-1 & \textit{if } \ s>[\frac{n}{2}]+1.\end{cases}$$

Consider the case $s\leq[\frac{n}{2}]+1$. Then either $s\leq[\frac{n}{2}]$ or $s=[\frac{n}{2}]+1$. Also $l+m<[\frac{n}{2}]-1$ implies that $l<[\frac{n}{2}]-1$. So when $s\leq[\frac{n}{2}]$, $\mathscr{S}(G)=[\frac{n}{2}]=\mathscr{S}(L(G))$ by Theorem 1.4.4.

When $s=[\frac{n}{2}]+1$,

$$\mathscr{S}(G)=[\frac{n+1}{2}]=\begin{cases}[\frac{n}{2}] & \textit{if } n \textit{ even},\\ [\frac{n}{2}]+1 & \textit{if } n \textit{ odd}.\end{cases}$$

where as $\mathscr{S}(L(G))=[\frac{n}{2}]$ which implies

$$\mathscr{S}(L(G))=\begin{cases}\mathscr{S}(G) & \textit{if } n \textit{ even},\\ \mathscr{S}(G)-1 & \textit{if } n \textit{ odd}.\end{cases}$$

When $s > [\frac{n}{2}] + 1$, $s > \frac{n}{2}$. Therefore $\mathscr{S}(G) = \mathscr{S}(L(G))$. Therefore

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) - 1 & \textit{if } s = [\frac{n}{2}] + 1 \textit{ and n odd,} \\ \mathscr{S}(G) & \text{otherwise.} \end{cases}$$

Hence the proof. □

Theorem 5.2.5. *Let G be a strong fuzzy cycle of length n. Let there be l weakest edges in G which do not altogether form a subpath in G and form a chain of paths $P_1, P_2, \ldots, P_n$. Also there exist at least two indices $i < j$ such that the complementary paths between P_i, P_{i+1} and P_j, P_{j+1} which do not contain any one of the P_ks are of length greater than one (when $j = m, P_{j+1} = P_1$ in G). Let s denote the maximum length of the subpaths which do not contain any weakest edge in G. Then if $l > [\frac{n}{2}] - (m+1)$ the strength $\mathscr{S}(L(G))$ of the line graph $L(G)$ of G is*

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) & \textit{if } l > [\frac{n}{2}] - 1, \textit{ or if } l \leq [\frac{n}{2}] - 1 \textit{ and } s \leq [\frac{n}{2}], \\ \mathscr{S}(G) - 1 & \textit{if } l \leq [\frac{n}{2}] - 1, \textit{ and } s > [\frac{n}{2}]. \end{cases}$$

If $l = [\frac{n}{2}] - (m+1)$ then

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) & \textit{if } l < [\frac{n}{2}] - 1, \; s \leq [\frac{n}{2}] \textit{ and n is odd,} \\ \mathscr{S}(G) + 1 & \textit{if } l < [\frac{n}{2}] - 1, \; s \leq [\frac{n}{2}] \textit{ and n even,} \\ \mathscr{S}(G) - 1 & \textit{if } l < [\frac{n}{2}] - 1, \; s > [\frac{n}{2}]. \end{cases}$$

Proof. For $l > [\frac{n}{2}] - (m+1)$, consider the following cases.

Case 1. $l > [\frac{n}{2}] - 1$.

Here, by applying Theorem 1.4.3 we get $\mathscr{S}(L(G)) = [\frac{n}{2}]$ which is equal to $\mathscr{S}(G)$.

Case 2. $l \leq [\frac{n}{2}] - 1 < l + m$

Then by Lemma 5.2.1 and by Theorem 1.4.4

$$\mathscr{S}(G) = \begin{cases} [\frac{n}{2}] & \text{if } s \leq [\frac{n}{2}], \\ s & \text{if } s > [\frac{n}{2}]. \end{cases}$$

That is if $s \leq [\frac{n}{2}]$ then $s - 1 \leq [\frac{n}{2}]$ which gives $\mathscr{S}(L(G)) = [\frac{n}{2}] = \mathscr{S}(G)$. (See Figure 5.6 with $n = 12, l = 4, m = 2$ and Figure 5.7 with $n = 13, l = 5, m = 2$). *If* $s > [\frac{n}{2}]$ then $s - 1 = [\frac{n}{2}]$. So $\mathscr{S}(L(G)) = [\frac{n}{2}] = s - 1 = \mathscr{S}(G) - 1$. (See Figure 5.8 with $n = 13, l = 4, m = 2$).

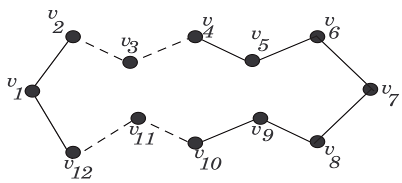


Figure 5.6: A fuzzy graph G with 12 vertices and 4 nonconsecutive weakest edges.

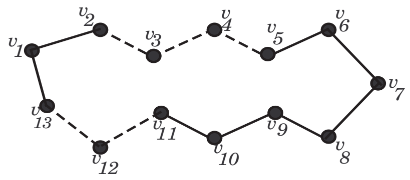


Figure 5.7: A fuzzy graph G with 13 vertices and 5 nonconsecutive weakest edges.

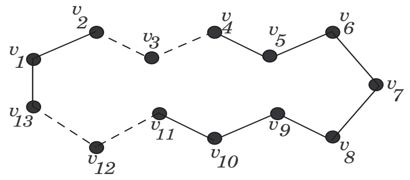


Figure 5.8: A fuzzy graph G with 13 vertices and 4 nonconsecutive weakest edges.

Consider the case $l = [\frac{n}{2}] - (m+1)$ then $\mathscr{S}(L(G)) = [\frac{n+1}{2}]$. Since $m \geq 2$, $l < [\frac{n}{2}] - 1$. By applying Theorem 1.4.3 $\mathscr{S}(G) = [\frac{n}{2}]$ if $s \leq [\frac{n}{2}]$. So

$$\mathscr{S}(L(G)) = [\frac{n+1}{2}] = \begin{cases} [\frac{n}{2}] & \textit{if } n \textit{ even,} \\ [\frac{n}{2}] + 1 & \textit{if } n \textit{ odd.} \end{cases}$$

Therefore

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) & \textit{if } n \textit{ even,} \\ \mathscr{S}(G) + 1 & \textit{if } n \textit{ odd.} \end{cases}$$

If $l < [\frac{n}{2}] - 1$ then if $s > [\frac{n}{2}]$, $\mathscr{S}(G) = s$. So $\mathscr{S}(L(G)) = s - 1 = \mathscr{S}(G) - 1$. Hence the proof. □

Chapter 6

Fuzzy extra strong $k-$ path domination in strong fuzzy graphs

Domination in fuzzy graphs is discussed by A.Somasundram and S.Somasundram [50], by using effective edges [50] in fuzzy graphs. Using strong edges, Nagoor Gani and Chandrasekaran [15] are introduced in fuzzy graphs - the domination, the independent domination and the irredundance. C.Natarajan and S.K.Ayyaswamy [37] introduced strong(weak) domination in fuzzy graphs. The concept of Strong (Weak) domination [45] in graphs was introduced by Sampathkumar and Pushpalatha. This chapter introduces fuzzy extra strong $k-$ path domination in strong fuzzy graphs and discusses some of its properties.

Definition 6.0.1. Let $G(V, \mu, \sigma)$ be a fuzzy graph. Let $u, v \in V$. For a positive integer k, v is said to be an extra strong $k-$ path neighbour of u if there exists an extra strong $u - v$ path of length $\leq k$ in G.

We denote the set of all extra strong $k-$ path neighbours of u by $N_k(u)$. That is $N_k(u) = \{v \in V : \exists$ an extra strong $u - v$ path of length $\leq k\}$.

Definition 6.0.2. Let $G(V, \mu, \sigma)$ be a fuzzy graph. For a subset S of V, the open extra strong $k-$ path neighbourhood of S is defined to be $N_k(S) = \bigcup_{u \in S} N_k(u)$ and the closed extra strong $k-$ path neighbourhood of S is $N_k[S] = N_k(S) \cup S$. If $S = \{u\}$, a singleton subset of V, then instead of $N_k[S]$ we write $N_k[u]$, and call a closed extra strong neighbourhood of u.

Remark 6.0.1. A vertex $v \in N_k(u)$ if and only if $u \in N_k(v)$.

Definition 6.0.3. Let $G(V, \mu, \sigma)$ be a fuzzy graph on V. Let $u, v \in V$. If there does not exist an extra strong $u - v$ path joining u and v of length $\leq k$ in G then v is called an extra strong $k-$ path isolated vertex of u and vice versa.

Remark 6.0.2. If G is of strength k then for any $n \geq k$, then

i $N_n(u) = V \setminus \{u\}$ for any vertex u of V and

ii $N_n[S] = V$, for any subset S of V.

Example 6.0.1.

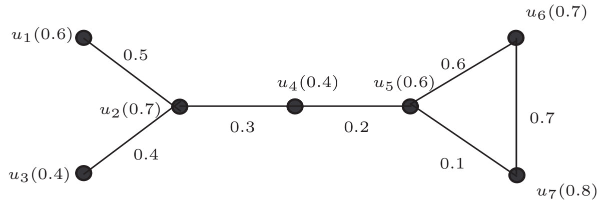


Figure 6.1: A fuzzy graph G having Strength 5.

For the fuzzy graph G in Figure 6.1, there is only one extra strong path P of length 5 which is $u_1u_2u_4u_5u_6u_7$. Therefore $N_5(u_1) = \{u_2, u_3, u_4, u_5, u_6, u_7\}$.

It is to be noted that $N_1(u_5) = \{u_4, u_6\}$, $N_2(u_5) = \{u_2, u_4, u_6, u_7\}$ and for any $k \geq 3, N_k(u_5) = \{u_1, u_2, u_3, u_4, u_6, u_7\}$.

By considering the extra strong path joining two vertices of a fuzzy graph we define two types of degree for each vertex v of a fuzzy graph.

Definition 6.0.4. For a fuzzy graph $G(V, \mu, \sigma)$ on the vertex set V and for a positive integer k, the extra strong $k-$ path degree $dS_k(v)$ of a vertex v in G, is defined as the sum of the strength of all the extra strong paths joining v and vertices in $N_k(v)$. The extra strong $k-$ path neighbourhood degree $dN_k(v)$ of a vertex v of a fuzzy graph is $\sum\limits_{u \in N_k(v)} \mu(u)$.

From Figure 6.1, $dS_1(u_5) = 0.8$, $dN_1(u_5) = 1.1$, $dS_2(u_5) = 1.6$ and $dN_2(u_5) = 2.6$.

Notation 6.0.1. For a fuzzy graph G on the vertex set V and for an integer k, $min\{dS_k(u) : u \in V(G)\}$ is denoted by $\delta_{S_k}(G)$ or simply by δ_{S_k} and $max\{dS_k(u) : u \in V(G)\}$ is denoted by $\Delta_{S_k}(G)$ or by Δ_{S_k}. Similarly minimum extra strong $k-$ path neighbourhood degree of a fuzzy graph and maximum extra strong $k-$ path neighbourhood degree of a fuzzy graph are denoted by $\delta_{N_k}(G)$ and $\Delta_{N_k}(G)$ respectively.

From Figure 6.1, $\delta_{S_1}(G) = 0.4$, $\Delta_{S_1}(G) = 1.3$, $\delta_{N_1}(G) = 0.7$, $\Delta_{N_1}(G) = 1.4$.

Definition 6.0.5. Let $G(V, \mu, \sigma)$ be a fuzzy graph and $S \subseteq V(G)$ and $v \in V - S$, $d_k(v, S)$ is defined to be the minimum length of the extra strong paths from v to u, $u \in S$.

Note 6.0.1. For every vertex $v \in V - S$, $d_k(v, S) \leq$ strength of the graph G.

Definition 6.0.6. Let $G(V, \mu, \sigma)$ be a fuzzy graph. For a positive integer k, a subset $S \subseteq V$ is said to be fuzzy extra strong $k-$ path dominating set of G if for every $v \in V$ either $v \in S$ or there exist an extra strong path of length $\leq k$ from v to a vertex of S in G.

Note 6.0.2. Let $G(V, \mu, \sigma)$ be a fuzzy graph. A subset S of V is said to be fuzzy extra strong $k-$ path dominating set of G, if for every vertex $v \in V - S$, $\exists$ an extra strong path of length $\leq k$ from v to a vertex u of S then we simply say that v extra strong $k-$ path dominates u.

Remark 6.0.3. If S is a fuzzy extra strong $k-$ path dominating set of a fuzzy graph G then every superset $S' \supseteq S$ is also a fuzzy extra strong $k-$ path dominating set.

Definition 6.0.7. A fuzzy extra strong $k-$ path dominating set S is a minimal fuzzy extra strong $k-$ path dominating set if no proper subset $S'' \subseteq S$ is a fuzzy extra strong $k-$ path dominating set.

Note 6.0.3. The set of all minimal fuzzy extra strong $k-$ path dominating sets of a fuzzy graph G is denoted by $ESmk - DS(G)$.

Definition 6.0.8. A fuzzy extra strong $k-$ path dominating set of a fuzzy graph with minimum number of vertices is called a minimum extra strong $k-$ path dominating set.

Definition 6.0.9. The fuzzy extra strong $k-$ path domination number $\gamma_{S_k}(G)$ of a fuzzy graph G is the minimum cardinality of a $ESmk - DS(G)$ set.

The fuzzy extra strong $k-$ path upper domination number $\Gamma_{S_k}(G)$ is the maximum cardinality of sets in $ESmk - DS(G)$.

Example 6.0.2.

From Figure 6.1, for $k = 1$, the sets $\{u_2, u_6\}, \{u_1, u_3, u_5, u_6\}$, $\{u_1, u_3, u_4, u_6\}$, $\{u_2, u_5, u_7\}$ are minimal extra strong $k-$ path dominating sets. For $k = 2$, $\{u_1, u_5\}$, $\{u_2, u_6\}, \{u_1, u_4\}, \{u_1, u_7\}$, for $k = 3$, $\{u_4\}, \{u_1, u_7\}, \{u_1, u_6\}, \{u_3, u_6\}, \{u_3, u_7\}$, $\{u_2, u_7\}, \{u_5\}$ are minimal extra strong $k-$ path dominating sets. Also for any $k > 3$, all singletons are minimal dominating for G.

So, ES $\gamma_{S_1}(G) = 2$, ES $\Gamma_{S_1}(G) = 4$.

Remark 6.0.4. Let $G(V, \mu, \sigma)$ be a fuzzy graph. Note that for any $u, v \in V$, if u extra strong $k-$ path dominates v then v extra strong $k-$ path dominates u. Hence extra strong $k-$ path domination is a symmetric relation on V.

Definition 6.0.10. Let $G(V, \mu, \sigma)$ be a fuzzy graph and $V_1 \subset V$. $G \backslash V_1$ is defined to be the fuzzy graph $G(V_2, \mu_1, \sigma_1)$, $V_2 = V \setminus V_1, \mu_1 = \mu/_{V_2}, \sigma_1 = \sigma/_{V_1 \times V_2}$.

Algorithm 6.0.1. Algorithm for finding an extra strong $k-$ path minimal dominating set D of a fuzzy graph G.

Step 1. Find the length of the extra strong path joining every pair of vertices of G using Algorithm (2.2.2).

Step 2. List out all pairs of vertices of G so that the length of extra strong paths between them is less than or equal to k, as U.

Step 3. Select a vertex which appears most number of times in the pairs of U. If there are more than one, select one among them (say u) and put it in the set D. Now group the vertices paired to u in U as V_1.

Step 4. From the fuzzy graph $G_1 = G - (V_1 \cup \{u\})$.

Step 5. Add the isolated vertices I_1 of G_1 to the set D and denote $G_2 = G_1 - I_1$.

Step 6. Repeat Steps 3, 4 and 5 successively for each component of G_2.

Step 7. Stop the process when the union of D and the deleted vertices of G is $V(G)$.

The subset D of V thus obtained will be a minimal ES $k-$ path dominating set.

Illustration:

Let $G(V, \mu, \sigma)$ be a fuzzy graph with vertex set $V = \{u_1, u_2, \ldots, u_{10}\}$. For $u_i, u_j \in V$, denote the length of an extra strong $u_i - u_j$ path of G by $k_{u_i u_j}$.

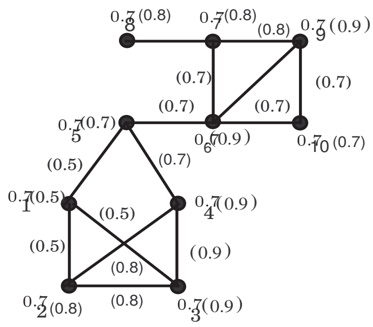


Figure 6.2: A fuzzy graph G.

$$
\begin{array}{lllll}
k_{u_1u_2}=2, & k_{u_2u_3}=2, & k_{u_3u_4}=1, & k_{u_4u_5}=1, & k_{u_5u_6}=1,\\
k_{u_1u_3}=1, & k_{u_2u_4}=1, & k_{u_3u_5}=5, & k_{u_4u_6}=2, & k_{u_5u_7}=2,\\
k_{u_1u_4}=2, & k_{u_2u_5}=2, & k_{u_3u_6}=3, & k_{u_4u_7}=3, & k_{u_5u_8}=3,\\
k_{u_1u_5}=3, & k_{u_2u_6}=4, & k_{u_3u_7}=4, & k_{u_4u_8}=4, & k_{u_5u_9}=2,\\
k_{u_1u_6}=4, & k_{u_2u_7}=5, & k_{u_3u_8}=5, & k_{u_4u_9}=3, & k_{u_5u_{10}}=3,\\
k_{u_1u_7}=5, & k_{u_2u_8}=6, & k_{u_3u_9}=4, & k_{u_4u_{10}}=4, & k_{u_6u_7}=1,\\
k_{u_1u_8}=6, & k_{u_2u_9}=5, & k_{u_3u_{10}}=4, & k_{u_6u_8}=2, & k_{u_6u_9}=1,\\
k_{u_1u_9}=5, & k_{u_2u_{10}}=6, & k_{u_6u_{10}}=2, & k_{u_7u_8}=1, & k_{u_7u_9}=2,\\
k_{u_1u_{10}}=6, & k_{u_7u_{10}}=3, & k_{u_8u_9}=3, & k_{u_8u_{10}}=4, & k_{u_9u_{10}}=1.
\end{array}
$$

For finding a minimal extra strong $1-$ path dominating set D, the vertex pairs to be considered are $(u_1,u_3),(u_2,u_4),(u_3,u_4),(u_4,u_5),(u_5,u_6),(u_6,u_7),(u_6,u_9),(u_7,u_8)$. Here u_6 repeats maximum number of times. Therefore $u_6 \in D$. Here the vertices paired to u_6 are u_5,u_7,u_9. Now form the graph G_1 by deleting the vertices

u_5, u_6, u_7, u_9 from G. Thus $V(G_1) = \{u_1, u_2, u_3, u_4, u_8, u_{10}\}$.

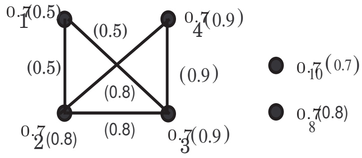


Figure 6.3: The fuzzy subgraph G_1.

Form the graph G_2 by deleting the isolated vertices u_8 and u_{10} from G_1 ie, $G_2 = G_1 \setminus \{u_8, u_{10}\}$. Now add the isolated vertices u_8 and u_{10} of G_2 to D.

For the graph G_2, $k_{u_1u_2} = 3$ $k_{u_1u_3} = 1$ $k_{u_1u_4} = 2$ $k_{u_2u_3} = 2$ $k_{u_2u_4} = 1$ $k_{u_3u_4} = 1$. Consider the vertex pairs $(u_1, u_3), (u_2, u_4)$, and (u_3, u_4). The length of the extra strong path joining the vertices in each pair is one. Here two vertices u_3 and u_4 repeat maximum number of times. Choose one among them, say u_3 and add it to D and delete the vertices paired to u_3 and the vertex u_3 from G_2 and form G_3. The resulting graph G_3 is the trivial fuzzy graph on the vertex $\{u_2\}$. Add u_2 to D. The subset D thus obtained is an an ES $k-$ path minimal dominating set, where $D = \{u_2, u_3, u_6, u_8, u_{10}\}$.

Note 6.0.4.

1. For any strong fuzzy graph G, the length of an extra strong path joining adjacent vertices is 1. Therefore extra strong $1-$ path dominating sets are dominating sets of the underlying crisp graphs.

2. For a positive integer k, if S is a fuzzy extra strong $k-$ path dominating set then it is also a fuzzy extra strong $k+1$ dominating set. In general an extra strong $(k+1)-$ path dominating set will not be an extra strong $k-$ path dominating set. But if k is the strength of the graph, every fuzzy extra strong $(k+1)-$ path dominating set is a fuzzy extra strong $k-$ path dominating set. More generally any fuzzy extra strong $l-$ path dominating set where $l \geq k$ is fuzzy extra strong $k-$ path dominating set.

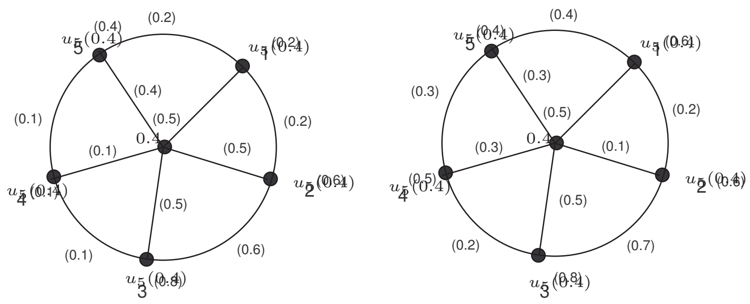
3. For a fuzzy graph G and for $k = 1$ if there exist an extra strong $k-$ path dominating set S consisting of a single vertex v of G then S is a the minimal extra strong $k-$ path dominating set for all values of k. Thus if for a fuzzy graph G if ES $\gamma_{S_1}(G) = 1$ then ES $\gamma_{S_k}(G) = 1, \forall k$.

Note 6.0.5.

1. From Note 6.0.4 if the given fuzzy graph G is a complete fuzzy graph or a strong fuzzy wheel graph or a strong fuzzy butterfly graph or a strong fuzzy star graph, ES $\gamma_{S_k}(G) = 1$ for all values of k.

   Example 6.0.3.

   Figure 6.4 (a) shows that for a strong fuzzy wheel graph G, with fuzzy hub v, $\{v\}$ is an ES $k-$ path minimal dominating set for all values of k. But from Figure 6.4(b) it is clear that $\{v\}$ is not a minimal extra strong $1-$ path dominating set.



(a) (b)

Figure 6.4: A strong fuzzy wheel graph and a fuzzy wheel graph.

Theorem 6.0.1. *For a fuzzy path G on n vertices, ES* $\gamma_{S_k}(G) = \left\lceil \frac{n}{2k+1} \right\rceil, \ \forall k.$

Proof. For a fuzzy path, there is only one path joining any two vertices of G. So for each value k,

ES $\gamma_{S_k}(G) \leq \left\lceil \frac{n}{2k+1} \right\rceil$.

To see the reverse inequality, let D be a fuzzy ES $k-$ path dominating set with $|D| = r$. If possible, let $r \leq \left\lceil \frac{n}{2k+1} \right\rceil - 1$. The r vertices of D dominate at the most $r(2k+1)$ vertices of G including the vertices of D. But $r(2k+1) < (\frac{n}{2k+1} + 1 - 1)(2k+1) < n$. Thus D can dominates only $< |G|$ vertices, a contradiction. Thus $r \geq \left\lceil \frac{n}{2k+1} \right\rceil$. Hence the result. □

Corollary 6.0.1. The ES $k-$ path domination number of the line graph of a strong fuzzy butterfly graph $G(V, \mu, \sigma)$ is $\gamma_{S_k}(G) = \begin{cases} 2 & \text{if } k = 1, \\ 1 & \text{if } k \geq 2. \end{cases}$

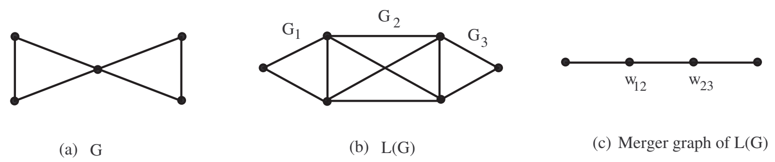


Figure 6.5: (a) A strong fuzzy Butterfly graph G, (b) its line graph L(G) and (c) merger graph of L(G).

Corollary 6.0.2. Let G be a fuzzy graph with its underlying crisp graph is a path on n vertices. Suppose $L(G)$ is the line graph of G. Then ES $\gamma_{S_k}(L(G)) = \lceil \frac{n-1}{2k+1} \rceil$.

Corollary 6.0.3. Let G be a fuzzy graph with its underlying crisp graph is a path on n vertices. If $sd(G)$ is the subdivision graph of G, then ES $\gamma_{S_k}(sd(G)) = \lceil \frac{2n-1}{2k+1} \rceil$.

Theorem 6.0.2. *For a fuzzy graph $G(V, \mu, \sigma)$, with underlying crisp graph a cycle of length n, ES $\gamma_{S_k}(G) = \lceil \frac{n}{2k+1} \rceil \ \forall k$.*

Proof. Let $V = \{u_1, u_2, \ldots, u_n\}$. Also let u be a vertex in V such that $\mu(u) = \bigwedge_{i=1}^{n} \mu(u_i)$. We have by Theorem 1.4.2, 1.4.3, 1.4.4, $\mathscr{S}(G) \geq [\frac{n}{2}]$. It is obvious that for $k \geq [\frac{n}{2}]$, u fuzzy extra strong $k-$ path dominates all the vertices of G. So ES $\gamma_{S_k}(G) = 1$, $\forall k \geq [\frac{n}{2}]$. Now we want to prove the result for $k < [\frac{n}{2}]$. Let G^* be the underlying crisp graph of G on n vertices $u_1, u_2, \ldots, u_n$. Suppose there are l weakest edges which altogether form a subpath, say, $P' = u_1 u_2 \ldots u_l$ where $l > \frac{n+1}{2}$ in G. Then strength of the graph is $[\frac{n}{2}]$. In this case the vertex u_1 dominates $2k + 1$ vertices of G including u_1.

The remaining $n-(2k+1)$ vertices are extra strong $k-$ path dominated by $\left\lceil \frac{n-(2k+1)}{2k+1} \right\rceil$ vertices of G. So for each value of k,

$$ES\ \gamma_{S_k}(G) \le \left\lceil \frac{n-(2k+1)}{2k+1} \right\rceil + 1 = \left\lceil \frac{n}{2k+1} \right\rceil$$

.

Suppose the weakest edges of G do not altogether form a subpath. Then $\exists$ at most one extra strong path joining any two vertices of G of length $\le k$. So

$$ES\ \gamma_{S_k}(G) \le \left\lceil \frac{n}{2k+1} \right\rceil$$

.

That is in both the cases the r vertices of G dominates $r(2k+1)$ vertices including these r vertices of G. So the converse part follows as in the case of a fuzzy path.

□

Theorem 6.0.3. *Let G be a strong fuzzy complete bipartite graph with K_{mn} as its underlying crisp graph. Then the ES $k-$ path domination number of G is*

$$ES\gamma_{S_k}(G) = \begin{cases} 1 & \text{if } k \ge 2 \text{ or } k = 1 \text{ and } m \text{ or } n \text{ is equal to } 1, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the bipartite sets of G.

Case 1. $k \geq 2$, or $k = 1$ and $m = 1$ or $k = 1$ and $n = 1$.

If m and n are not simultaneously one then $\mathscr{S}(G) = 2$ and if $m = n = 1$ then $\mathscr{S}(G) = 1$. Therefore in these cases length of extra strong path joining any two vertices of G is at most 2. Clearly a single vertex of G can extra strong $k-$ path dominate all the other vertices of G. Clearly $ES\gamma_{S_k}(G) = 1$.

Case 2. $k = 1$ and m, n are greater than 1.

It is obvious that any vertex in U extra strong $1-$ path dominate all the vertices of V, and any vertex in V can extra strong 1-path dominate all the vertices of U. So if $u \in U$ and $v \in V$ then $\{u, v\}$ extra strong $1-$ path dominates all the vertices of G. So ES$\gamma_{S_1}(G) \leq 2$. Also for $k = 1$, no vertex in U can dominate any other vertices of U, so ES $\gamma_{S_1}(G) \neq 1$. Therefore ES $\gamma_{S_1}(G) = 2$.

□

Theorem 6.0.4. *Let G be a properly linked fuzzy graph with the complete fuzzy graphs $G_1, G_2, \ldots, G_m$ as its parts. Suppose for $i = 1, 2, \ldots, m-1$, $V(G_i) \cap V(G_{i+1}) = K_{n_i}$, a complete graph on n_i vertices. Then the ES $k-$ path domination number of G, ES $\gamma_{S_k}(G) = \lceil \frac{m}{2k} \rceil$.*

Proof. As each G_i is complete, each vertex of G_i dominates all the vertices of G_i. If a vertex belongs to $V(G_i) \cap V(G_{i+1})$ then it dominates all the vertices of both G_i and G_{i+1}. In the case of $k > 1$, a vertex in $V(G_i) \cap V(G_{i+1})$, dominates k parts to the left and k parts to the right of that vertex, (if they exist). So as far as the domination is concerned, instead of taking the whole chain we take its merger graph G'. The merger graph G' is a $1-$ liked fuzzy graph with m parts where each part is complete. The strength of a strong fuzzy complete graph is 1 by Theorem 1.4.1. As there is only one extra strong path joining any two vertices of G', we have for each value of k, ES $\gamma_{S_k}(G) \leq \lceil \frac{m}{2k} \rceil$.

Conversely, let S be an arbitrary extra strong $k-$ path dominating set of G' with $|S| < \lceil \frac{m}{2k} \rceil$. Let u be any vertex of S. If $u = w_{i\ i+1}$ with the notation of Definition 2.3.6 for some i, j then it extra strong $k-$ path dominates $2k$ parts and itself. If $u \neq w_{ij}$ then it dominates at most $2k-1$ parts. Therefore if $|S| < \lceil \frac{m}{2k} \rceil$ it will not $ES\,k-$ path dominate all the vertices of G. Hence the proof. □

The middle graph of a strong fuzzy path on n vertices is a $1-$ linked graph with $n-1$ fuzzy complete graphs as its parts.

Corollary 6.0.4. Let G be a fuzzy graph with its underlying crisp graph is a path on n vertices and $M(G)$ its middle graph of G. Then $\gamma_{S_k}(M(G)) = \lceil \frac{n-1}{2k} \rceil$, for $k \geq 1$.

Corollary 6.0.5. The ES $k-$ path domination number of a strong fuzzy Bull graph G is $\gamma_{S_k}(G) = \begin{cases} 2 & \text{if } k = 1, \\ 1 & \text{if } k \geq 2. \end{cases}$

Proof. A strong fuzzy Bull graph is a strong fuzzy graph with three parts each of which is complete and from Theorem 6.0.4 the proof follows. □

Corollary 6.0.6. The ES $k-$ path domination number of a strong fuzzy diamond graph G is $1, \forall k$.

Proof. The line graph G of a strong fuzzy diamond graph is a strong fuzzy wheel graph on 5 vertices. Hence by Note 1 extra strong $k-$ path domination number of a strong fuzzy diamond graph is 1. □

Theorem 6.0.5. *The ES $k-$ path domination number of line graph of strong fuzzy diamond graph G is $1, \forall k$.*

Proof. The line graph of a strong fuzzy diamond graph G is a strong fuzzy wheel graph on 5 vertices. See Figure 5.3. So the fuzzy hub can dominate all the other 4 vertices of the line graph of G. Hence the theorem. □

Definition 6.0.11. For $S \subseteq V$, a vertex $v \in S$ is called an extra strong $k-$ path enclave of S if $N_k[v] \subseteq S$, and $v \in S$ is an extra strong $k-$ path isolate of S if $N_k(v) \subseteq V \setminus S$. A set is said to be extra strong $k-$ path enclaveless if it does not contain any extra strong $k-$ path enclaves.

Property 1. The following statements are equivalent for a strong fuzzy graph $G(V, \mu, \sigma)$. Let $S \subset V$ be an extra strong $k-$ path dominating set.

i For every vertex $v \in V \setminus S$, $\exists$ a vertex $u \in S$ such that the length of the extra strong path joining u to $v \leq k$.

ii For every vertex $v \in V \setminus S, d_k(v, S) \leq k$.

iii $N_k[S] = V$.

iv For every vertex $v \in V \setminus S, |N_k[v] \cap S| \geq 1$, that is for every vertex $v \in V \setminus S$, there exists $u \in S$ and extra strong path joining v to u of length $\leq k$.

v For every vertex $v \in V$, $|N_k[v] \cap S| \geq 1$.

vi The set $V \setminus S$ is extra strong $k-$ path enclaveless.

Theorem 6.0.6. *Let $G(V, \mu, \sigma)$ be a fuzzy graph. An extra strong $k-$ path dominating set S is an extra strong minimal $k-$ path dominating set if and only if for each vertex $u \in S$ any one of the following conditions holds:*

(a) u is an extra strong $k-$ path isolate of S.

(b) there exist a vertex $v \in V \setminus S$ for which $N_k(v) \cap S = \{u\}$.

Proof. Suppose S is an extra strong $k-$ path dominating set and for each vertex $u \in S$ one of the conditions (a) and (b) holds. Suppose that S is not an extra strong minimal $k-$ path dominating set. That is there exists a vertex $u \in S$ such that $S \setminus \{u\}$ is an extra strong $k-$ path dominating set. Hence there exists an extra strong path joining u to at least one vertex in $S \setminus \{u\}$ having length $\leq k$ that is, (a) does not hold for S. Since $S \setminus \{u\}$ is an extra strong $k-$ path dominating set, for every vertex in $V \setminus S$ there exist an extra strong path having length $\leq k$ to at least one vertex in $S \setminus \{u\}$, that is (b) does not hold.

Conversely, assume that S is an extra strong minimal $k-$ path dominating set of G. Then for every vertex $u \in S, S \setminus \{u\}$ is not an extra strong $k-$ path dominating set. This means for some $v \in (V \setminus S) \cup \{u\}$, there does not exist an extra strong path joining u to v having length $\leq k$. Now either $v = u$ or $v \in V \setminus S$ in the first case u is an extra strong $k-$ path isolate of S. In the second case, since v is not extra strong $k-$ path dominated by $S \setminus \{u\}$, but is extra strong $k-$ path dominated by S, $N_k(v) \cap S = \{u\}$. □

Definition 6.0.12. Let $G(V, \mu, \sigma)$ be a fuzzy graph and S be a set of vertices of G and let $u \in S$. A vertex $v \in V$ is said to be an ES $k-$ path private neighbour of u with respect to S if $N_k[v] \cap S = \{u\}$. The set of all ES $k-$ path private neighbours of u is called the $ES\, k-$ path private neighbour set of u and is denoted by $ES\ PN_k[u, S]$.

In other words, $ES\ PN_k[u, S] = N_k[u] - N_k[S - \{u\}]$. Also notice that, if $u \in ES\, PN_k[u, S]$ then u is an extra strong $k-$ path isolated vertex in S.

Example 6.0.4.

Let S be the subset $\{u_2, u_6\}$ of the vertex set of the graph in Figure 6.1

$ES\, PN_1[u_2, S] = \{u_1, u_2, u_3, u_4\}$,

$ES\, PN_1[u_6, S] = \{u_5, u_6, u_7\}$,

$ES\, PN_2[u_2, S] = \{u_1, u_2, u_3, u_4, u_5\}$,

$ES\, PN_2[u_6, S] = \{u_4, u_5, u_7\}$.

Remark 6.0.5. A subset S of the vertex set of a fuzzy graph $G(V, \mu, \sigma)$ is a minimal fuzzy $ESk-$ path dominating set if and only if for every vertex $v \in S$ there exists a vertex $w \in V - (S - \{v\})$ which is not dominated by $S - \{v\}$.

Which is equivalent to, S is a minimal fuzzy $ESk-$ path dominating set if and only if $ES\ PN_K[u, S] \neq \phi$ for every vertex $u \in S$, that is every vertex $u \in S$ has at least one $ESk-$ path private neighbour with respect to S.

Definition 6.0.13. If there is no extra strong path of length $\leq k$ between u and v, two vertices of a fuzzy graph G then in G, u and v are said to be fuzzy extra strong $k-$ path independent. If any two vertices of D, a subset of V, are fuzzy extra strong $k-$ path independent and are extra strong $k-$ path dominating then D is said to be an extra strong $k-$ path independent set of G.

In Figure 6.1 $\{u_4, u_7\}$ is a fuzzy extra strong $2-$ path independent set.

Definition 6.0.14. If for every vertex $v \in V - S$, S is a fuzzy extra strong $k-$ path independent set of $G(V, \mu, \sigma)$, the set $S \cup \{v\}$ is not a fuzzy extra strong $k-$ path independent set of G then S is a maximal fuzzy extra strong $k-$ path independent set of $G(V, \mu, \sigma)$.

Proposition 6.0.1. A fuzzy extra strong $k-$ path independent set S in a fuzzy graph $G(V, \mu, \sigma)$ is maximal fuzzy extra strong $k-$ path independent set if and only if it is fuzzy extra strong $k-$ path independent and fuzzy extra strong $k-$ path dominating.

Proof. Let S be a maximal fuzzy extra strong $k-$ path independent set. Then from the definition it is clear that S is both fuzzy extra strong $k-$ path independent and fuzzy extra strong $k-$ path dominating. Conversely, if a set S is both fuzzy extra strong $k-$ path independent and fuzzy extra strong $k-$ path dominating. Suppose S is not maximal fuzzy extra strong $k-$ path independent. Then there exists a vertex $u \in V - S$ for which $S \cup \{u\}$ is fuzzy extra strong $k-$ path independent. Therefore there does not exist an extra strong path of length less than or equal to k joining any vertex in S to u. Hence S cannot be fuzzy extra strong $k-$ path dominating. Hence the proof. □

Theorem 6.0.7. *Every maximal fuzzy extra strong $k-$ path independent set in a fuzzy graph G is a minimal fuzzy extra strong $k-$ path dominating set of G for each value of k.*

Proof. Let S be a maximal fuzzy extra strong $k-$ path independent set in G. Proposition 6.0.1 asserts that S is a fuzzy extra strong $k-$ path dominating set. We must show that S is, in fact, a fuzzy extra strong $k-$ path minimal dominating set. A fuzzy extra strong $k-$ path dominating set S is a minimal fuzzy extra strong $k-$ path dominating set if for every vertex $v \in S$ the set $S - \{v\}$ is not a fuzzy extra strong $k-$ path dominating set. Assume therefore that S is not a minimal fuzzy extra strong $k-$ path dominating set. But if for some $v \in S$, $S - \{v\}$ fuzzy extra strong $k-$ path dominates $V - (S - \{v\})$, then there is an extra strong path of length less than or equal to k joining at least one vertex in $S - \{v\}$ to v. This contradicts our assumption that S is a maximal fuzzy extra strong $k-$ path independent set of G. Therefore, S must be a minimal fuzzy extra strong $k-$ path dominating set. Hence the proof. □

Definition 6.0.15. Let $G(V, \mu, \sigma)$ be a fuzzy graph and $S \subseteq V$. A vertex $u \in S$ is said to be a fuzzy extra strong $k-$ path redundant vertex with respect to S if

$ES\ PN_k[u, S] = \phi$. This means for any $v \in V$, $N_k[v] \cap S = \phi$ or $|N_k[v] \cap S| > 1$ or $N_k[v] \cap S \subset S \setminus \{u\}$. Equivalently u is fuzzy extra strong $k-$ path redundant in S if $N_k[u] \subseteq N_k[S - \{u\}]$. Otherwise u is said to be fuzzy extra strong $k-$ path irredundant vertex.

Definition 6.0.16. Let $G(V, \mu, \sigma)$ be a fuzzy graph and $S \subseteq V$. The set S is said to be fuzzy extra strong $k-$ path irredundant set if $ES\ PN_k[u, S] \neq \phi$ for every vertex u in S. That is, every vertex $u \in S$ has at least one extra strong $k-$ path private neighbour in S. A fuzzy ES $k-$ path irredundant set S is a maximal irredundant set, if for every vertex $u \in V \setminus S$, the set $S \cup \{u\}$ is not fuzzy irredundant set. The minimum cardinality taken over all maximal ES $k-$ path irredundant sets of vertices of G is called lower irredundance number and is denoted by $ES\ ir_{S_k}$. The maximum cardinality taken over all maximal ES $k-$ path irredundant sets of vertices of G is called upper irredundance number and is denoted by $ES\ IR_{S_k}$.

Proposition 6.0.2. A fuzzy extra strong $k-$ path dominating set S is a minimal fuzzy ES $k-$ path dominating set if and only if it is fuzzy extra strong $k-$ path dominating and fuzzy extra strong $k-$ path irredundant.

Proof. The fact that a minimal extra strong $k-$ path dominating set is both fuzzy extra strong $k-$ path dominating and fuzzy extra strong $k-$ path irredundant. Conversely, if a set S is both fuzzy extra strong $k-$ path dominating and fuzzy extra strong $k-$ path irredundant, we must show that it is minimal extra strong $k-$ path dominating. Suppose not, by Remark 6.0.5 it is sufficient to show that there exists a vertex $v \in S$ such that $S - \{v\}$ is a fuzzy extra strong $k-$ path dominating set. But since S is irredundant, $ESPN_k[v, S] \neq \phi$. Let $w \in ESPN_k[v, S]$. By Definition 6.0.15 there does not exist an extra strong path joining w to any vertex in $S - \{v\}$. Therefore $S - \{v\}$ is not a dominating set, a contradiction. □

Theorem 6.0.8. *Let $G(V, \mu, \sigma)$ be a fuzzy graph and $S \subseteq V$. If no vertex of S is an extra strong $k-$ path isolate of S and if S is an extra strong $k-$ path irredundant set then $V - S$ is an extra strong $k-$ path dominating set.*

Proof. Let S be an extra strong $k-$ path irredundant set in a fuzzy graph G which has no extra strong $k-$ path isolated vertex. Suppose $V \setminus S$ is not an extra strong $k-$ path dominating set. Then there exists a vertex v in S such that the length of extra strong paths joining v to any vertex of $V \setminus S$ is $> k$, because no vertex of S is an extra strong isolate of S. Therefore $ES\ PN_k[v, S] = \phi$, a contradiction. Hence the theorem. □

Theorem 6.0.9. *Let $G(V, \mu, \sigma)$ be a fuzzy graph with $S \subseteq V$ be a fuzzy extra strong $k-$ path irredundant set. Then $ES\ \gamma_{S_k}(G)/2 < ES\ ir_{S_k}(G) \leq ES\ \gamma_{S_k}(G) < 2ES\, ir_{S_k}(G) - 1$.*

Proof. Let $ES\ ir_{S_k}(G) = p$ and let $S = \{v_1, v_2, \ldots, v_p\}$ be a fuzzy extra strong $k-$ path irredundant set of G. Therefore ES $PN_k[v_i, S] \neq \phi$, for $1 \leq i \leq p$. Let $S' = \{u_1, u_2, \ldots, u_p\}$ where $u_i \in ES\ PN_k[v_i, S], i = 1, 2, \ldots, p$. Note that possibly $u_i = v_i$ (if v_i is its own ES $k-$ path private neighbour), but in any case the cardinality of $S \cup S'$ is $\leq 2p = 2ES\ ir_{S_k}(G)$.

We claim that the set $S'' = S \cup S'$ is an extra strong $k-$ path dominating set. If not, then there must exist at least one vertex $w \in V - S''$ which is not extra strong $k-$ path dominated by S''. This means that $w \notin N_k[x]$ for any vertex $x \in S''$, and therefore $ES\ PN_k[w, S \cup \{w\}] \neq \phi$.

In particular $v_i \notin N_k[w]$ for any vertex $v_i \in S$. Therefore $ES\ PN_k[v_i, S \cup \{w\}] \neq \phi$. Thus $S \cup \{w\}$is a fuzzy extra strong $k-$ path irredundant set, which contradicts the assumption that S is a maximal fuzzy extra strong $k-$ path irredundant set. Therefore, S'' is a fuzzy extra strong $k-$ path dominating set.

By Theorem 6.0.7, $ES\ ir_{S_k}(G) \leq ES\ \gamma_{S_k}(G)$.

To prove the last inequality, note that although S'' is an extra strong $k-$ path dominating set it cannot be a minimal fuzzy extra strong $k-$ path dominating set unless $|S''| = S$, by Theorem 6.0.7. Therefore $ES\ \gamma_{S_k}(G) \leq 2ES\ ir_{S_k}(G) - 1$ and $ES\ \gamma_{S_k}(G)/2 < ES\ ir_{S_k}(G)$. □

Epilogue

In this research work the strength of various strong fuzzy graphs, derived strong fuzzy graphs, products of strong fuzzy graphs have been determined. The results obtained in this work may be extended to all types of fuzzy graphs. Also by suitable modifications the results obtained here may be extended to directed fuzzy graphs. Much more research remains to be done on fuzzy extra strong $k-$ path domination.

We presume that the above stated problems will be beneficial for research aspirants.

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